Optimal Collusion-Proof Auctions

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ABSTRACT: We study an optimal collusion-proof auction in an environment where subsets of bidders may collude not just on their bids but also on their participation. Despite their ability to collude on participation, informational asymmetry facing the potential colluders can be exploited significantly to weaken their collusive power. The second-best auction — i.e., the optimal auction in a collusion-free environment — can be made collusion-proof, if at least one bidder is not collusive, or there are multiple bidding cartels, or the second-best outcome involves a nontrivial probability of the object not being sold. Regardless, optimal collusion-proof auction prescribes nontrivial exclusion of collusive bidders, i.e., a refusal to sell to any collusive bidder with positive probability.

KEYWORDS: Collusion on participation, subgroup collusion, multiple bidding cartels, an exclusion principle.

1 Introduction

Collusion by participants often poses a serious threat to markets and organizations. Collusion is a special concern in auctions, where bidders can manipulate or simply withdraw their bids to limit competition. Not surprisingly, auctions have provided the volume and prominence to the case law and research on collusion, with its evidence found in the auctions of commercial equipment (*U.S. v. Seville Industrial Machinery*), antiques (*U.S. v. Ronald Pook*), real estate (*District of Columbia v. George Basilinko, et al.*), highway construction contracts (Porter and Zona, 1993), timber (Baldwin et al., 1997), school milk delivery contracts (Pesendorfer, 2000; Porter and Zona, 1999), and stamps (Asker, 2008).[1]

Consistent with these evidences are the theoretical findings that "standard" auctions are vulnerable to bidder collusion, even when the cartel members face mutual asymmetric information. Graham and Marshall (1987) and McAfee and McMillan (1992) demonstrate in their seminal articles that asymmetrically informed cartel members can structure a knock-out auction that enables them to (re)allocate the good among themselves efficiently while limiting the seller's revenue to at most her reserve price. The ability by the cartel members to exchange side payments (without getting detected) is crucial for this result, but they can achieve the same effect via adjusting their bid rotation or market shares, if auctions are repeated.[2]

If standard auctions are vulnerable to collusion, can one find an auction rule that is not? This is the question we address. What makes this question nontrivial is the informational asymmetry facing potential colluders. If colluders have complete information about one another, then they can effectively act like a single agent and maximize their joint payoffs. Then, there would be little room for auction design, for an optimal scheme would simply reduce to textbook monopoly pricing. If bidders face mutual asymmetric information, however, they may not effectively coordinate their behavior, and the seller may exploit this to undermine collusion. Although standard auctions are not capable of this (as has been shown by extant literature), other auction rule may enable the seller to exploit the bidders' mutual asymmetric information more effectively. We seek to identify such an auction rule.

In order to study an optimal response to collusion, one must understand a bidder's incentive to participate in collusion. In particular, one must deal with the question of what happens after a bidder refuses to participate in collusion. What belief would they form about the

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[1] See Cassidy (1967) for case studies of bid rigging and collusion.

[2] Graham and Marshall (1987), McAfee and McMillan (1992), Mailath and Zemsky (1991), Eso and Schummer (2003), and Marshall and Marx (2007) study enforceability of collusion under different auction formats, while Aoyagi (2003), Athey, Bagwell and Sanchirico (2004), Blume and Heidhues (2002), and Skrzypacz and Hopenhayn (2004) study collusion in repeated auctions.

subsequent competition and about the types of bidders they face? Can the remaining cartel members punish the defecting bidder, and if so, to what degree? How one models the (out-of-equilibrium) belief and the cartel members' ability to punish a defector determines a bidder's incentive to participate in a collusive arrangement, which in turn determines the scope and the nature of the seller's response. In this regard, we take an eclectic approach by considering both weak and strong notions of collusion-proof auctions.

The weak notion postulates that collusion arises *only* when it benefits all types of bidders relative to a non-cooperative play *without any updating of beliefs*. An auction rule is said to be *weak collusion-proof* if it admits no such collusion. This notion is reasonable to the extent that cartel members will often find it difficult to punish a defector more severely than bidding noncooperatively. At the same time, the weak notion restricts the bidders' out-of-equilibrium beliefs. Our "strong" notion imposes no such restriction on the cartel members' out-of-equilibrium beliefs or their ability to punish a defector: an auction rule is defined to be *strong collusion-proof* if, under that rule, the seller enjoys in *every* Bayesian Nash equilibrium the same expected revenue as she would absent any collusion. Further, we require the strong collusion-proof auction to be robust to the specifics of the cartel operation.

While the alternative notions matter to some extent, they do not affect the main thrust of our results. We find, largely irrespective of the particular notion used, that a seller can overcome her vulnerability to collusion in a surprisingly broad range of circumstances. Specifically, a seller can attain the highest revenue she would without any collusion, either if a cartel is not all-inclusive or if the object is not sold to any bidder with some probability. This result holds under the weak notion of collusion-proofness but also under the strong notion, given an additional condition (which is satisfied for the case of the all-inclusive cartel). Regardless of the implementability of the second-best, we find that the optimal collusion-proof auction prescribes a positive probability of not selling to any collusive bidder. This *exclusion principle* holds quite generally, regardless of the buyers' support of valuations, thus exhibiting a qualitative departure from the collusion-free auction design.[3] Not only does the principle explain the logic behind how and when the second-best is collusion proof implementable, but it also identifies exclusion as a crucial weapon for curbing collusion in general.

The current paper is related to several recent papers.[4] First, it is related to our companion

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[3]As is well known, the standard optimal auction allocates the good to a buyer with probability one, if there is at least one buyer whose infimum valuation is sufficiently high. See Myerson (1981).

[4]Other authors have studied optimal collusion-proof mechanisms in different contexts. Quesada (2004) finds an optimal collusion-proof mechanism in the LM setting where an (informed) agent proposes a side contract. In fact, she treats collusion on participation, but adopts the strong notion similar to Dequiedt (2007) where the side contract can impose maximum punishment on refusing agents. Jeon and Menicucci (2005) shows

paper, Che and Kim (2006) (henceforth, CK), as well as Laffont and Martimort (1999, 2000) (henceforth, LM). These papers study a collusion-proof contract when, unlike the current setting, agents can collude only after they participate in the contract. This latter assumption may not be appropriate in many auctions where bidders are intimately familiar with their opponents even before participating. In fact, an allegedly predominant form of collusion involves bidders coordinating on their participation decisions: Colluders either refuse to participate or withdraw their bids to allow a designated cartel member to win without facing competition. Further, the idea of "selling the firm" to potential colluders, featured in Che and Kim (2006), relies on the agents' inability to collude on their participation decision. The current paper relaxes the assumption by allowing the bidders to collude on their participation decision.

In this latter respect, the current paper is closely related to Dequiedt (2007) and Pavlov (2008), who also study collusion-proof auctions when bidders can collude on their participation. Dequiedt considers two bidders with binary types (i.e., of either low or high valuation) and finds that, if a cartel can commit to punish a defector to his reservation utility, then the seller can at most collect her reserve price when a bidder's valuation exceeds that price.

Pavlov (2008) independently studied a similar problem as this paper and reached similar conclusions. In particular, his results on symmetric bidders coincide with ours for the same case, although the methods of analysis are different. There are several major differences, however. First, while Pavlov's analysis focuses exclusively on the symmetric bidders case, we treat the general case of ex ante asymmetric bidders. We show that the second-best outcome is collusion-proof implementable, given a somewhat stronger condition than is needed for the symmetric bidders case. Second, while Pavlov's analysis concerns only the case of the all-inclusive cartel, we consider the general case in which arbitrary subset(s) of bidders are collusive. In fact, the most important result concerns the case in which a proper subset of bidders is collusive— i.e., at least one bidder is noncollusive or there are multiple bidding cartels — in which case the second-best outcome is shown to be always collusion-proof implementable. Last, we establish the general exclusion principle, showing that, in a very broad class of environments, an optimal weak-collusion-proof mechanism calls for excluding cartel members with positive probability.

We view the ability to handle collusion by a subset(s) of bidders as practically important and useful. In many circumstances, not all bidders are in a position to collude. Government auctions used in defense procurement, mineral extraction, or spectrum licenses often have

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that the second-best is achievable in the weak collusion-proof sense, much like Che and Kim (2007), in the nonlinear pricing context. Laffont and Martimort (1998), Celik (2004) and Faure-Grimaud et al. (2004) study the optimality of delegation in the presence of collusive agents.

incumbents with long history of operation pitted against relative new comers. Long term interaction and shared experiences among the incumbents will put them in a better position to collude than the new comers. Likewise, in auctions for construction repairs or food service procurement, competition may involve both local and non-local providers, and the former group may be able to collude more effectively, based on their regular contacts and their interaction through trade associations. The problem of only a subset of bidders being collusive introduces a new challenge, since the cartel may prey on noncollusive bidders as much as on the seller. Hence, a collusion-proof design must eliminate incentives for the cartel to engage in such behavior.

The precise relationships with Dequiedt (2007) and Pavlov (2008) are best understood once our results are presented, so we postpone the discussion to Section 7. The rest of the paper is organized as follows. Section 2 introduces an auction model and describes the second-best outcome in a collusion-free environment. Section 3 introduces a model of collusion and the notion of weak collusion-proof auctions. Section 4 identifies the properties of weak collusion-proof auctions. Section 5 obtains a condition necessary and sufficient for implementing the second-best outcome in a weak collusion-proof fashion. Section 6 characterizes strong collusion-proof implementation. Section 8 concludes.

2   Primitives

A risk-neutral seller has an object for sale. The seller's valuation of the object is normalized to zero. There are $n \geq 2$ risk-neutral buyers who each independently draw a value, θ_i, on the object from an interval $\Theta_i := [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}_+$ according to distribution F_i, which has strictly positive density f_i on the support. We assume that both

$$J_i(\theta_i) := \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \text{ and } K_i(\theta_i) := \theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}.$$

are continuous and strictly increasing in θ_i for all $i \in N$. Throughout, we let $\mathbb{E}[\cdot] := \int_{\Theta}[\cdot]d(\prod_{i \in N} F_i(\theta_i))$. and $\mathbb{E}_{\tilde{\theta}_{-i}}[\cdot] := \int_{\Theta_{-i}}[\cdot]d(\prod_{j \neq i} F_j(\theta_j))$ denote expectation operators based on the prior distribution, where $\Theta := \prod_{i \in N} \Theta_i$ and $\Theta_{-i} := \prod_{j \neq i} \Theta_j$.

For a later analysis, it is convenient to augment each bidder's type space to include the "participation decision" as part of his possible type. Specifically, we let $\theta_\emptyset$ denote "non-participation" or "exit" option available to each bidder with the convention that $\theta_\emptyset < \theta_i, \forall i \in N, \forall \theta_i \in \Theta_i$, and define $\overline{\Theta}_i := \{\theta_\emptyset\} \cup \Theta_i$. We then let $\theta := (\theta_1, ..., \theta_n) \in \overline{\Theta} := \prod_{i \in N} \overline{\Theta}_i$ denote a possible profile of types in these enriched type spaces. Since we shall consider randomization in cartel members' reports over their augmented type spaces, it is convenient

to consider arbitrary probability distribution, μ^C, over $\overline{\Theta}_C := \prod_{i \in C} \overline{\Theta}_i$ for any $C \subset N$ and to use $\mathbf{E}_{\mu^C}[\cdot] := \int_{\overline{\Theta}_C} [\cdot] d(\mu^C(\theta_C))$ as an expectation operator relative to μ^C.

We now describe arbitrary auction rules, and we do so in direct mechanisms. An *auction rule*, $M = (q, t)$, consists of *an allocation rule*, $q = (q_1, \cdots, q_n) : \overline{\Theta} \to \mathcal{Q}$, where $\mathcal{Q} := \{x \in [0,1]^n | \sum_{i \in N} x_i \leq 1\}$ and *a payment rule*, $t = (t_1, ..., t_n) : \overline{\Theta} \to \mathbb{R}^n$, such that $q_i(\theta_\emptyset, \theta_{-i}) = t_i(\theta_\emptyset, \theta_{-i}) = 0, \forall i, \theta_{-i} \in \overline{\Theta}_{-i}$. An auction rule determines, for each profile of bidders' reports in $\overline{\Theta}$, a vector of probabilities for the bidders to obtain the object and a vector of expected payments they must pay, subject to the constraint that, if a bidder does not participate, he does not receive the good and collects his reservation utility, normalized to zero. Any equilibrium arising in any game in our environment can be described as a restriction of an auction rule to Θ, so we use the term *"outcome"* to refer to such a restriction.

Fix an auction rule, $M = (q, t)$. Buyer i's interim payoff when his valuation is $\theta_i \in \Theta_i$ but reports $\tilde{\theta}_i \in \overline{\Theta}_i$ is

$$u_i^M(\tilde{\theta}_i, \theta_i) := \theta_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i),$$

where $Q_i(\theta_i) := \mathbb{E}_{\tilde{\theta}_{-i}}[q_i(\theta)]$ and $T_i(\theta_i) := \mathbb{E}_{\tilde{\theta}_{-i}}[t_i(\theta)]$. Given hidden information and the availability of the non-participation option, an auction rule must be incentive compatible and individually rational to be consistent with equilibrium. We say an auction rule M is *feasible* if

$$U_i^M(\theta_i) := u_i^M(\theta_i, \theta_i) \geq u_i^M(\tilde{\theta}_i, \theta_i), \quad \forall i, \theta_i \in \Theta_i, \tilde{\theta}_i \in \overline{\Theta}_i. \qquad (IC^*)$$

Note (IC^*) subsumes both incentive compatibility and individual rationality, since it requires

$$U_i^M(\theta_i) \geq u_i^M(\theta_\emptyset, \theta_i) = 0. \qquad (IR)$$

Let $\mathcal{M}$ denote the set of all feasible auction rules. For later analysis, the following characterization of feasible auction rules proves useful. Its proof, along with most of the others, are relegated to the Appendix.

LEMMA 0. *If $M = (q, t) \in \mathcal{M}$, then, for each $\theta_i \in \Theta_i$,*

$$U_i^M(\theta_i) = \mathbb{E}\left[K_i(\tilde{\theta}_i) q_i(\tilde{\theta}) 1_{\{\tilde{\theta}_i \leq \theta_i\}} + J_i(\tilde{\theta}_i) q_i(\tilde{\theta}) 1_{\{\tilde{\theta}_i \geq \theta_i\}} - t_i(\tilde{\theta}) \right]. \qquad (1)$$

Before proceeding, it is useful to consider a collusion-free environment. It is by now well known that, in such an environment, an optimal auction rule, called *second-best* or *noncollusive optimal outcome*, solves

$$[NC] \qquad\qquad \max_{M \in \mathcal{M}} \mathbb{E}\left[\sum_{i \in N} t_i(\theta) \right],$$

and its associated outcome is characterized as follows:

THEOREM 0. (MYERSON) *An optimal mechanism that solves* $[NC]$ *involves the allocation rule given by* $\forall \theta \in \Theta$,

$$q_i^*(\theta) = \begin{cases} 1 & \text{if } J_i(\theta_i) > \max\{0, \max_{k \neq i} J_k(\theta_k)\}, \\ 0 & \text{otherwise}, \end{cases}$$

and yields revenue of

$$V^* := \mathbb{E}\left[\sum_{i \in N} J_i(\theta_i) q_i^*(\theta)\right]$$

to the seller.

We assume that $\mathbb{E}[q_i^*(\theta)] < 1$ for each $i \in N$, or else the optimal mechanism reduces to bargaining with a single buyer, so there would be no problem of collusion. Letting $\hat{\theta}_i := \min\{\theta_i \in [\underline{\theta}_i, \bar{\theta}_i] | J_i(\theta) \geq 0\}$, the optimal mechanism allocates the good to the bidder with the highest virtual valuation $J_i(\theta_i)$ as long as $\theta_i \geq \hat{\theta}_i$. In particular, the object is sold with probability one if there is a bidder i such that $J_i(\underline{\theta}_i) \geq 0$.

3   A Model of Collusion

We first develop a model of weak cartel. (The strong cartel is introduced later in Section 6.) To this end, we follow LM and CK and suppose that there are subsets of bidders, called *cartels*, that enforce side contracts via uninformed representatives to influence the outcome of the auction game being played.[5] Formally, *a cartel structure* is an arbitrary partition $\mathcal{C}$ on N whose element $C \in \mathcal{C}$ represents a cartel of bidders who "may" collude with one another.[6] This framework encompasses a range of possibilities that allow for the all-inclusive cartel (i.e., $\mathcal{C} = \{N\}$), for the presence of noncollusive bidders (i.e., some elements of $\mathcal{C}$ may be singleton) and for multiple bidding cartels (i.e., $\mathcal{C}$ may include C_j, $j = 1, ..., k$ with $|C_j| \geq 2$). The cartel structure $\mathcal{C}$ is a common knowledge for all bidders in N and for the seller. The assumption that the seller knows the cartel structure, albeit not innocuous, may not be as restrictive as it may appear. For instance, our analysis would still apply if some cartel may not collude

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[5]The idea of an outside third party acting as a representative of cartel is not unrealistic. The cartel of stamp dealers hired an outside agent, a New York taxi driver, to conduct knock-out auctions and act generally as a facilitator (Asker, 2008).

[6]The cartel represents a potential unit of collusion, and may not be active, depending on the incentives created by the seller. More important, even if a cartel is active, we assume that the seller does not detect or have direct evidence of the collusive behavior. If the seller observes active collusion, it may be in her best interest to prosecute it directly, or else she may even forfeit the right to recover damages. We thank a referee for pointing this out.

effectively. Also, the structure of potential bidding cartels (who is likely to collude with whom) can be sometimes discerned from prior auction experiences and other industry observables. Of course, none of these issues arise if there is only one cartel, as has been assumed in all existing papers. In this sense, the current model generalizes many existing models of collusion.[7]

The time line is similar to that of LM and CK, except for one important difference: Cartels are formed prior to the bidders' participation into the mechanism.

☐ TIME LINE:

- At date 0, each bidder learns his type, θ_i, drawn from Θ_i. The realized type is private information of the bidder, unobservable to the seller as well as to other bidders.

- At date 1, the seller proposes an auction rule $M \in \mathcal{M}$.

- At date 2, the (uninformed) representative of each cartel $C \in \mathcal{C}$ simultaneously proposes a collusive side contract (to be described in detail later). Each member of $C \in \mathcal{C}$ then accepts or rejects the contract. If all bidders of C accept, then that cartel's side contract is enforced; or else, the members of C play the subsequent game non-cooperatively. Neither the side contract proposed for a cartel C nor its members' decision on accepting that contract is observed by the bidders outside C (and by the seller).

- At date 3, each bidder, $i \in N$, chooses $\tilde{\theta}_i \in \overline{\Theta}_i$; i.e., he accepts or rejects M, and reports from Θ_i if he accepts. (If the side contract of a cartel was accepted, then its members report according to the side contract.)

- At date 4, if collusion by a cartel is active, then the outcome of their side contract arises. If no collusion is active, then M results.

☐ COLLUSION TECHNOLOGY:

We assume that each cartel has at its disposal four instruments: (a) its members' participation decisions, (b) participating members' communication with the seller (e.g., bids), (c) reallocation of the good within the cartel, in case a member of that cartel receives the good, and (d) side payments that the cartel members can exchange in a budget-balanced fashion. These four instruments together encompass all possible ways in which a cartel can coordinate their members' behavior.

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[7]There are other papers considering rich cartel structures. Graham, Marshall, and Richard (1990) allow for subcartels within cartels, and Marshall, Meurer, Richard, and Stromquist (1994) provide a numerical analysis of the first-price auction with multiple competing cartels. Lopomo, Marshall and Marx (2005) allow for non-collusive bidders.

To formally describe possible manipulations utilizing all these instruments, fix a possible cartel $C \in \mathcal{C}$, and an auction rule $M = (q, t) \in \mathcal{M}$ the seller may propose. We then suppose that an uninformed representative of each cartel C, $|C| \geq 2$ proposes a side contract to its members, given that bidders outside C behave non-collusively (or equivalently their representatives offer null contracts). The latter presumption is made since later we shall focus on how non-collusive behavior can be supported as an equilibrium, which requires a *unilateral* deviation by each cartel to be prevented. Instead of considering a possible side contract by each cartel, it is convenient to think of a *manipulation*, the outcome that will emerge when that side contract is enforced *and* all others, including noncollusive bidders and members of different cartels, report truthfully.

Formally, an outcome, $\tilde{M} = (\tilde{q}, \tilde{t}) : \Theta \mapsto \mathcal{Q} \times \mathbb{R}^n$ is a *manipulation of M by cartel C*, if there exists a function, $\mu^C : \Theta_C \to \Delta\overline{\Theta}_C$, that maps from their types in Θ_C into a probability distribution over $\overline{\Theta}_C$ such that, $\forall \theta \in \Theta$,

$$\sum_{i \in C} \tilde{q}_i(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[\sum_{i \in C} q_i(\tilde{\theta}_C, \theta_{N \backslash C})], \qquad (RC_C^M)$$

$$\tilde{q}_i(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[q_i(\tilde{\theta}_C, \theta_{N \backslash C})], \forall i \in N \backslash C, \qquad (RC_{N \backslash C}^M)$$

$$\mathbb{E}\left[\sum_{i \in C} \tilde{t}_i(\theta)\right] = \mathbb{E}\left[\sum_{i \in C} \mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N \backslash C})]\right], \qquad (BB_C^M)$$

$$\tilde{t}_i(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N \backslash C})].\forall i \in N \backslash C. \qquad (BB_{N \backslash C}^M)$$

These conditions are explained as follows. First, condition (RC_C^M) requires the final assignment of the good to be "reallocationally consistent" in the sense that the good is allocated to any cartel member only if some member of that cartel obtains the good from the seller under some manipulation of reports/participation decision. Condition (BB_C^M) allows the cartel members to exchange side transfers in a budget-balanced fashion. Since budget balancing is required at the ex ante level, we are allowing for the cartel to finance (from a competitive capital market) across different realizations of its members' type profiles.[8] Conditions $(RC_{N \backslash C}^M)$ and $(BB_{N \backslash C}^M)$ simply assume that bidders outside C are not colluding: there is no reallocation and no exchange of side payments among all bidders outside C and between C and $N \backslash C$. This presumption would be without any loss if $N \backslash C$ were all noncollusive. Even if $N \backslash C$ may involve some bidding cartels, the above conditions are still sufficient for our purpose: We shall consider a unilateral manipulation by each cartel when no other cartels are active.

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[8]Our results do not change, if budget balancing is required at the *ex post* level. Clearly, our collusion-proof implementation result would be stronger with the ex ante version of budget balancing, explaining our choice.

□ INCENTIVE FEASIBILITY OF COLLUSION

For collusive manipulation to work, the members of the cartel must have the incentive to carry it out. We say that $\tilde{M}$ is *feasible* if it satisfies

$$U_i^{\tilde{M}}(\theta_i) \geq u_i^{\tilde{M}}(\tilde{\theta}_i, \theta_i), \quad \forall i \in C, \theta_i \in \Theta_i, \tilde{\theta}_i \in \overline{\Theta}_i, \qquad (IC_C^*)$$

and

$$U_i^{\tilde{M}}(\theta_i) \geq U_i^M(\theta_i), \quad \forall i \in C, \theta_i \in \Theta_i. \qquad (IR_C^M)$$

These conditions are explained as follows. First of all, (IC_C^*) requires the outcome resulting from collusion to be incentive compatible to all members of cartel. Since the cartel members face asymmetric information about one another, this condition must hold, regardless of the specifics of how the cartel is formed and how the members bargain over their collusive arrangement. Next, (IR_C^M) requires that each member of the cartel must do as well with the proposed manipulation as they would by vetoing that manipulation and acting non-cooperatively. Clearly, what each member will get in the latter event depends on the inferences made by the other members of the carte about him. Condition (IR_C^M) assumes that no new inferences about the members' types are made in such an event. This "passivity" of out-of-equilibrium beliefs is an important element of LM's weak collusion-proofness notion. Although (IR_C^M) assumes passive out-of-equilibrium beliefs, it in fact accommodates all non-pessimistic beliefs for our purpose. If a collusive proposal is made unattractive to a bidder with a passive belief about what will happen when he refuses the proposal, it will be unattractive to him if his beliefs were more optimistic about that event. In this sense, the real restriction arising from (IR_C^M) is for out-of-equilibrium beliefs to be non-pessimistic. This restriction serves as a reasonable discipline over belief formation.[9]

Lastly, note these conditions are imposed only for the members of the cartel, since the manipulation constitutes its deviation unobserved by outsiders of that cartel. We turn next to the weak notion of collusion-proof auctions.

Definition WCP. *An auction rule $M \in \mathcal{M}$ is weak collusion-proof (henceforth, WCP), if, for each cartel $C \in \mathcal{C}$ with $|C| \geq 2$, no feasible manipulation of M by C makes any member of C strictly better off.*

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[9]In fact, it is not too difficult to construct *a* non-collusive equilibrium, supported by an arbitrarily *optimistic* belief. The seller can simply make available an option which would pay an arbitrarily large amount to a bidder (say paid by a different bidder) if the bidders were to coordinate in the *right* way; the very optimistic belief that such a coordination would occur in the event of rejecting a collusive offer can sustain a non-collusive equilibrium. Clearly, such an equilibrium is not believable, and the "passivity" restriction can be seen to place a discipline against such an equilibrium by limiting the degree of optimism entertainable by the potential colluders when rejecting a collusive offer.

To explain this notion, suppose the seller offers an auction rule M. If M is WCP, then, for each cartel C, there exists no feasible manipulation that would make some members of C strictly better (without making the other members of C worse off), given that all other cartels are inactive. Our WCP notion is the same as the WCP of LM, except that we allow for randomization and reallocation possibilities in the collusive bidders and that we allow for proper subsets of bidders to be collusive. It is in fact a natural generalization of their notion to allow for these new features. Aside from the restrictions mentioned above, the WCP notion involves an implicit restriction that *the seller employs an auction rule in $\mathcal{M}$ which admits for each cartel no feasible manipulation that will make any member strictly better off.* Following LM's well-known weak-collusion-proof principle, one can easily see that the restriction entails no loss.[10]

WCP auctions are worth studying for several reasons. First, WCP auctions offer a reasonable protection against collusion since it is often unrealistic for cartel members to punish more severely than bidding non-cooperatively. Second, the weak notion provides a conservative test of when collusion imposes a real cost to the seller: If for instance there is no WCP auction that would allow a seller to earn the second-best revenue, then one can safely conclude that collusion matters, for the seller would not fare any better if the bidders can collude more effectively. Finally, the restrictions involved in the WCP notion are not crucial for the results obtained. We will later show that under some condition, the main result can be strengthened to the strong notion of collusion-proofness.

4   Properties of WCP Auctions

We first establish a couple of properties of WCP auctions (Lemmas 1 and 2), and then use these properties to obtain an important characteristic of an optimal WCP auction.

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[10]The argument is sketched as follows. Suppose the seller offers any indirect mechanism that maps from any message spaces from the bidders to allocations/payments. Suppose each cartel behaves as modeled above, namely it implements a feasible manipulation whenever it makes any member strictly better off. Now consider any equilibrium emanating from this mechanism, and the associated outcome $\bar{M} = (\bar{q}, \bar{t}) \in \mathcal{M}$. $\bar{M}$ cannot admit any manipulation for any cartel that makes any member strictly better off, or else $\bar{M}$ cannot arise in equilibrium. This proves that $\bar{M}$ is an WCP auction implementing the same outcome and revenue for the seller as the original indirect mechanism.

4.1 Properties of WCP Auctions

Fix a cartel $C \in \mathcal{C}$, and an auction rule $M = (q, t)$ that the seller proposes. It is useful to have a few definitions. Let $q_i^C(\theta_C) := \mathbb{E}_{\tilde{\theta}_{N \setminus C}}[q_i(\theta_C, \tilde{\theta}_{N \setminus C})]$. Let $q_C(\theta) := \sum_{i \in C} q_i(\theta)$ and $Q_C(\theta_C) := \mathbb{E}_{\tilde{\theta}_{N \setminus C}}[q_C(\theta_C, \tilde{\theta}_{N \setminus C})]$ denote the probability that the auction rule allocates to good to a member of the cartel given the value profile of all bidders and that of the cartel members, respectively. Let $\mathcal{Q}_C := [0, \sup_{\theta_C \in \overline{\Theta}_C} Q_C(\theta_C)]$ be the set of all probabilities with which the cartel can secure the good to its members under M. This set contains zero since all its members can boycott the auction i.e., $Q_C(\theta_\emptyset, \cdots, \theta_\emptyset) = 0$. This set is convex since the cartel members can randomize between boycotting and reporting some profile $\theta_C \in \Theta_C$. We then obtain our first property of WCP auction rules.

LEMMA 1. *If $M = (q, t) \in \mathcal{M}$ is WCP, then for each $C \in \mathcal{C}$ there exists a convex function, $r : \mathcal{Q}_C \to \mathbb{R}_+$ with $r(0) = 0$,[11] such that*

$$\mathbb{E}_{\tilde{\theta}_{N \setminus C}}\left[\sum_{i \in C} t_i(\theta_C, \tilde{\theta}_{N \setminus C})\right] = r(Q_C(\theta_C)), \forall \theta_C \in \overline{\Theta}_C.$$

To see how this property restricts the auction rules, suppose all bidders belong to one cartel, and suppose the seller wishes to implement a deterministic allocation (i.e., $q(\cdot) \in \{0, 1\}$). Lemma 1 implies that, for the auction to be weak collusion-proof, it must charge a single price if and only if the good is sold. More generally, the seller cannot collect any fee from a cartel whenever its members do not obtain the good. This feature arises from the abilities of the bidders to collude on their participation decisions; were they charged positive entry fees, they could all simply refuse to participate. Similarly, the collusive bidders can never be charged different prices for the same probability of obtaining the good; or else, they could manipulate their reports (or bids) to pick the lowest price for a given probability of obtaining the good. The surplus generated from such manipulation can be shared among all cartel members via appropriate side transfers and reallocations so that (IC^*) and (IR_C^M) conditions are satisfied. Finally, the sale price is (weakly) convex in the probability of the object being allocated to any cartel member, since the cartel members can at least randomize between non-participation and any probability of allocation attainable by some reports.

The next property is obtained for a class of allocation rules satisfying *monotonicity: for all $C \in \mathcal{C}$, $q_C(\cdot)$ is nondecreasing in θ_C and, for all $C \in \mathcal{C}$ and for all $i \in C$, $q_i(\cdot)$ is nondecreasing in θ_i and nonincreasing in θ_{-i}.* Let $\mathcal{M}_0 \subset \mathcal{M}$ denote the set of auction rules satisfying this

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[11]Of course, the function $r(\cdot)$ differs depending on the identity of cartel, though we suppress such dependence in our notation for simplicity.

monotonicity. The monotonic allocation rules are reasonable, and would naturally arise from the standard auctions such as first- and second-price auctions. Note that the monotonicity is satisfied by the second-best allocation also. Next, we define the average price charged to the cartel per unit probability:

$$
p(\theta_C) := \begin{cases} \dfrac{r(Q_C(\theta_C))}{Q_C(\theta_C)} & \text{if } Q_C(\theta_C) > 0, \\ 0 & \text{otherwise.} \end{cases}
$$

LEMMA 2. *If $M = (q, t) \in \mathcal{M}_0$ is WCP, then $\forall\, C \in \mathcal{C}$, $\forall\, i \in C$ and for almost every $\theta_C \in \Theta_C$,*

$$
\theta_i \in \arg \max_{\theta_i' \in \{\theta_\emptyset\} \cup [\underline{\theta}_i, \theta_i]} \left(K_i(\theta_i) - p(\theta_i', \theta_{C-i}) \right) q_i^C(\theta_i', \theta_{C-i}). \tag{2}
$$

This lemma characterizes the extent to which each cartel can "behave like a single agent." Specifically, condition (2) resembles an incentive compatibility constraint for a "single" agent who may consume one of $|C|$ alternative values. But this resemblance is not perfect. First, that agent realizes "pseudo" value $K_i(\theta_i)$ rather than true value θ_i. Second, the bidder's constraint is required only in one direction, i.e., not to under-report or withdraw from the auction. Third, the agent may not be able to shift his consumption among the alternative uses. All together, these features serve to limit the extent to which the cartel can coordinate their members' behavior. For instance, the fact that pseudo values, rather than true values, matter means that the cartel can be forced to sustain some ex post loss. Since $K_i(\theta_i) > \theta_i$, an average price of $p(\theta_C) > \theta_i$ need not violate the above constraint. The cartel's limited ability to coordinate their behavior arises from the fact that any collusive defection requires a consensus from all types of bidders. Different types of bidders may have conflicting interests, say about consumption of the good by any particular type θ_i. For instance, if $\theta_i < p(\theta_C)$, then the cartel wishes to cancel such consumption, but the highest type of bidder i would not agree as long as $K_i(\theta_i) > p(\theta_C)$.

4.2 Exclusion Principle for Optimal WCP Auctions

The properties of WCP auction imply an important property of the optimal WCP auction for the seller, which will be useful for interpreting subsequent results. We show that the optimal auction rule in the monotonic class must entail some exclusion of collusive bidders. To gain some idea behind this result, suppose to the contrary that optimal WCP auction sells the object to some members of any given cartel C. Lemma 1 implies that the seller can only charge a sale price to collusive bidders, regardless of their types. Meanwhile, Lemma 2 says that this price cannot be too high relative to the pseudo value, $K_i(\theta_i)$, of the collusive

bidder who consumes the good. Taken together, these lemmas imply that the seller must either charge a low sale price, or else she should exclude types with low pseudo values from consuming the good. It turns out the seller wishes to exclude cartel members always.

THEOREM 1. (EXCLUSION PRINCIPLE) *Assume that there are more than one bidder $i \in C$ with $\underline{\theta}_i = \underline{\theta} := \max_{j \in C} \underline{\theta}_j$. Then, the optimal WCP mechanism in $\mathcal{M}_0$ requires that the object not be sold to any member of C with a positive probability.*

PROOF: Let $M = (q, t)$ denote the optimal WCP auction rule. Suppose to the contrary that $\sum_{i \in C} q_i(\theta) = 1$ for all $\theta \in \Theta$, which implies by Lemma 1 that $\sum_{i \in C} t_i(\theta) = r$ for some r. Then, Lemma 2 requires that

$$r \le \max_{i \in C} K_i(\underline{\theta}_i) = \underline{\theta}.$$

Thus, the revenue cannot exceed $\underline{\theta}$. We now generate a contradiction by constructing a WCP auction rule which raises a higher revenue than $\underline{\theta}$: Sell the object at a fixed price $\tilde{r}$, which is slightly greater than $\underline{\theta}$, if and only if at least one member of C has a value higher than $\tilde{r}$. This take-it-or-leave offer is clearly WCP and generates a revenue equal to $R(\tilde{r}) := \tilde{r}(1 - \prod_{i \in C} F_i(\tilde{r}))$. And $R(\tilde{r}) > R(\underline{\theta}) = \underline{\theta}$ for $\tilde{r}$ slightly above $\underline{\theta}$ since

$$\left.\frac{d}{d\tilde{r}} R(\tilde{r})\right|_{\tilde{r} = \underline{\theta}} = (1 - \prod_{i \in C} F_i(\underline{\theta})) - \underline{\theta} \sum_{i \in C} f_i(\underline{\theta}) \prod_{j \neq i} F_j(\underline{\theta}) = 1 > 0,$$

where the last equality holds because for each $i \in C$, there exists at least one bidder $j \neq i$ for whom $F_j(\underline{\theta}) = 0$. ∎

Recall from Theorem 0 that an optimal auction excludes some low valuation bidders even in the absence of collusion. Yet, exclusion never arises if there is some buyer i whose valuation is always high so that $J_i(\underline{\theta}) \ge 0$. Collusion tilts the tradeoff toward more exclusion, since the seller can only charge a single sale price, whereas absent collusion bidding competition generates higher payment from high valuation types beyond the reserve price. Consequently, an optimal collusion-proof auction *always* excludes some types of collusive bidders.

5   WCP Implementation of the Second-Best Outcome

In this section, we ask the question of whether the second-best outcome is WCP implementable. Toward this end, we use the results of previous section to obtain a necessary condition for the second-best outcome to be WCP implementable (Theorem 2). We then show that, the necessary condition is also sufficient for the WCP implementability of the second-best outcome (Theorems 3 and 4).

To begin, fix any bidder $i \in C$ for some $C \in \mathcal{C}$ with $|C| \geq 2$. For each profile $\theta_{N \backslash C} \in \Theta_{N \backslash C}$, let

$$\phi_i(\theta_{N \backslash C}) := \inf\{\theta_i \in \Theta_i \mid J_i(\theta_i) \geq \max\{\max_{j \in N \backslash C} J_j(\theta_j), 0\}\}$$

denote the lowest type of bidder i that can obtain the good with positive probability in the second-best allocation, given the type profile of bidders outside the cartel C.

CONDITION (SB): (i) If $\mathcal{C} = \{N\}$, then

$$K_i(\hat{\theta}_i) \left(\mathbb{E} \left[\sum_{i \in N} q_i^*(\theta) \right] \right) \geq \mathbb{E} \left[\sum_{i \in N} J_i(\theta_i) q_i^*(\theta) \right], \forall i \in N.$$

(ii) If $\mathcal{C} \neq \{N\}$, then, for each $C \in \mathcal{C}$ with $|C| \geq 2$,

$$\mathbb{E} \left[\sum_{i \in C} K_i(\phi_i(\theta_{N \backslash C})) q_i^*(\theta) \right] \geq \mathbb{E} \left[\sum_{i \in C} J_i(\theta_i) q_i^*(\theta) \right].$$

This condition is explained as follows. The RHS of the inequalities represent the amounts of surplus that *should* be extracted from the cartel to implement the second-best payoff for the seller. As will be proven next, the LHS of the inequalities represent the highest payments that can be collected from the cartel in any WCP auction implementing the second-best allocation q^*. Thus, the inequalities are necessary for the second-best outcome to be WCP implementable.[12]

THEOREM 2. (NECESSITY) CONDITION (SB) *is necessary for the second-best outcome to be WCP implementable.*

5.1 WCP Implementation of the Second-Best Outcome: Symmetric Bidders

Here we show that CONDITION (SB) is also sufficient for the second-best outcome to be WCP implementable when, bidders are symmetric. We begin with the symmetry assumption:

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[12]In fact, the necessity of CONDITION (SB) can be extended to any, not necessarily second-best, auction outcome $M = (q, t) \in \mathcal{M}_0$ that satisfies the following properties: For each bidder $i \in N$, (i) there is a threshold type $\theta_i^m = \inf\{\theta_i \in \Theta_i | Q_i(\theta_i) > 0\}$, (ii) there are functions $\phi_{ij} : [\theta_i^m, \bar{\theta}_i] \to [\theta_j^m, \bar{\theta}_j], j \neq i$, each of which is continuous and strictly increasing where $\phi_{ij}(\theta_i) < \bar{\theta}_j$, and which satisfy $\theta_j < \phi_{ij}(\theta_i), \forall j \neq i$ if and only if $q_i(\theta) = 1$, and (iii) the lowest type $\underline{\theta}_i$ obtains zero payoff. Then, the proof of Theorem 2 can be slightly modified to establish that CONDITION (SB) is necessary for M to be WCP implementable if $q^*(\cdot)$ is replaced by $q(\cdot)$ and the function $\phi_i(\cdot)$ is redefined as $\phi_i(\theta_{N \backslash C}) = \inf\{\theta_i \in [\theta_i^m, \bar{\theta}_i] | \theta_j \leq \phi_{ij}(\theta_i), \forall j \in N \backslash C\}$.

$F_i(\cdot) =: F(\cdot)$ for all $i \in N$, for some common cdf $F(\cdot)$ which has a positive density f. The associated virtual valuations J and K are defined analogously, and their monotonicity properties are maintained. Likewise, we let $\hat{\theta} := \inf\{\theta | J(\theta) \geq 0\}$. CONDITION (SB) is now more succinctly described in this environment. Define first $\theta_C^{(1)} := \max_{i \in C} \theta_i$ and $\theta_{N \setminus C}^{(1)} := \max\{\max_{i \in N \setminus C} \theta_i, \hat{\theta}\}$. (We adopt a convention that $\theta_{N \setminus C}^{(1)} := \hat{\theta}$ when $\mathcal{C} = \{N\}$.) Then, CONDITION (SB) simplifies to:

CONDITION (SB'): For each C with $|C| \geq 2$, $\mathbb{E}\left[K(\theta_{N \setminus C}^{(1)}) \Big| \theta_C^{(1)} > \theta_{N \setminus C}^{(1)} \right] \geq \mathbb{E}\left[J(\theta_C^{(1)}) \Big| \theta_C^{(1)} > \theta_{N \setminus C}^{(1)} \right]$.

We shall provide an intuition behind this condition later. Here, we establish the sufficiency of this condition, by constructing an auction rule that WCP implements the second-best outcome. Suppose that a second-price auction is held with a reserve price $\hat{\theta}$, and consider the associated auction rule $M^* = (q^*, t^*)$ (defined over $\overline{\Theta}$).[13] We then construct a new auction rule $\hat{M} = (\hat{q}, \hat{t})$. The allocation rule $\hat{q}$ is constructed so that $\hat{q}(\cdot) = q^*(\cdot)$. To construct the payment rule $\hat{t}$, we first determine the sale price against each cartel $C \in \mathcal{C}$ (with $|C| \geq 2$). Let $\alpha_C \in [0, 1]$ satisfy

$$\mathbb{E}\left[\alpha_C K(\theta_{N \setminus C}^{(1)}) + (1 - \alpha_C) J(\theta_{N \setminus C}^{(1)}) \Big| \theta_C^{(1)} > \theta_{N \setminus C}^{(1)} \right] = \mathbb{E}\left[J(\theta_C^{(1)}) \Big| \theta_C^{(1)} > \theta_{N \setminus C}^{(1)} \right]. \tag{3}$$

CONDITION (SB') allows such an α_C to be well defined. The sale price against cartel C is then set at $H_C(\theta_{N \setminus C}^{(1)}) := \alpha_C K(\theta_{N \setminus C}^{(1)}) + (1 - \alpha_C) J(\theta_{N \setminus C}^{(1)})$. This sale price is defined in terms of the highest type of bidders outside C and is set above $J(\theta_{N \setminus C}^{(1)})$ just enough to extract $J(\theta_C^{(1)})$ on average from the highest valuation bidder in C. Let $\delta_C(\theta) := H_C(\theta_{N \setminus C}^{(1)}) \sum_{i \in C} q_i^*(\theta)$ denote the expected sale price charged against cartel C.

We now describe the payment rule $\hat{t}$. For each noncollusive bidder i (i.e., $\{i\} \in \mathcal{C}$), we set $\hat{t}_i(\theta) := t_i^*(\theta)$, $\forall \theta \in \overline{\Theta}$. For each cartel $C \in \mathcal{C}$, let $C(\theta_C) := \{i \in C | \theta_i \neq \theta_\emptyset\}$ be the set of its members who participate in the auction, given θ_C. For each $i \in C$, if $C(\theta_C) = C$, then we set

$$\begin{aligned}
\hat{t}_i(\theta) &:= \frac{1}{|C|} \delta_C(\theta) + \mathbb{E}_{\tilde{\theta}_{-i}} \left[t_i^*(\theta_i, \tilde{\theta}_{-i}) - \frac{1}{|C|} \delta_C(\theta_i, \tilde{\theta}_{-i})) \right] \\
&\quad - \frac{1}{|C| - 1} \sum_{k \in C \setminus \{i\}} \mathbb{E}_{\tilde{\theta}_{-k}} \left[t_k^*(\theta_k, \tilde{\theta}_{-k}) - \frac{1}{|C|} \delta_C(\theta_k, \tilde{\theta}_{-k}) \right]
\end{aligned} \tag{4}$$

and, if $C(\theta_C) \subsetneq C$, then we set

$$\hat{t}_i(\theta) := \begin{cases} \delta_C(\overline{\theta}) & \text{if } i \in C(\theta_C) \\ 0 & \text{if } i \in C \setminus C(\theta_C). \end{cases} \tag{5}$$

---

[13]Recall that the outcome is well defined even when some bidders do not participate, a situation described by $\theta_\emptyset$ being chosen by these bidders.

Two properties of the current construction are important. First, in case all cartel members participate, (4) implies $\mathbb{E}_{\tilde{\theta}_{-i}}[\hat{t}_i(\theta_i, \tilde{\theta}_{-i})] = \mathbb{E}_{\tilde{\theta}_{-i}}[t_i^*(\theta_i, \tilde{\theta}_{-i})]$, $\forall i \in N$, $\forall \theta_i \in \Theta_i$,[14] so that each bidder has the same interim incentives as with M. This property means that the new auction rule $\hat{M}$ inherits the incentive and participation properties of the original auction M^*. Hence, $\hat{M}$ satisfies (IC^*) and implements V^*. Second, it can be checked from (4) and (5) that $\sum_{i \in C} \hat{t}_i(\theta) = \delta_C(\theta) = H_C(\theta_{N \setminus C}^{(1)}) \sum_{i \in C} q_i^*(\theta)$ if all cartel members participate (or else, each participating cartel member pays a high "punishment price" equal to $\delta_C(\bar{\theta})$). That is, upon participating, each cartel $C \in \mathcal{C}$ is charged a single sale price, $H(\theta_{N \setminus C}^{(1)})$, which depends only on non-cartel members' types and is payable only when the object is allocated to its member. This property satisfies Lemma 1 and also ensures that the cartel cannot manipulate the sale price charged against its members. These two properties deliver weak collusion-proof implementation of the second-best outcome:

THEOREM 3. (SUFFICIENCY) *Given* CONDITION (SB′), *the auction rule $\hat{M}$ is WCP and achieves the second-best revenue.*

We first provide some intuition behind the idea of WCP implementation of the second-best; we do so with an example.

EXAMPLE 1. *Suppose there are two bidders each with valuation drawn uniformly from $[0,1]$. According to Theorem 0, the second-best outcome allocates the object efficiently for valuation exceeding $\hat{\theta} = \frac{1}{2}$, and yields revenue of $\frac{5}{12}$. This also satisfies* CONDITION (SB′), *so the second-best is WCP implementable. The WCP auction rule charges a sale price of $r^* := \mathbb{E}[J(\theta_N^{(1)})|\theta_N^{(1)} > \hat{\theta}] = 5/9$ to the bidders, regardless of who win and what their bids are. Without collusion, each bidder receives the interim payoff of*

$$U^{\hat{M}}(\theta) = \begin{cases} 0 & \text{if } \theta \in [0, \frac{1}{2}) \\ \frac{1}{2}\theta^2 - \frac{1}{8} & \text{if } \theta \in [\frac{1}{2}, 1]. \end{cases}$$

Since the cartel is charged a sale price of $5/9$, it suffers an ex post loss whenever the highest valuation is in the interval $[\frac{1}{2}, \frac{5}{9}]$. Why can they not simply boycott the auction in this situa-

---

[14]To see this, note first, by (4) and symmetry,

$$\mathbb{E}[\delta_C(\theta)] = \mathbb{E}\left[J(\theta_C^{(1)})1_{\{\theta_C^{(1)} > \theta_{N \setminus C}^{(1)}\}}\right] = \mathbb{E}\left[\sum_{i \in C} t_i^*(\theta)\right] = |C|\mathbb{E}\left[t_i^*(\theta)\right], \forall i \in C, \tag{6}$$

from which it follows that

$$\mathbb{E}_{\tilde{\theta}_{-i}}[\hat{t}_i(\theta_i, \tilde{\theta}_{-i})] = \mathbb{E}_{\tilde{\theta}_{-i}}[t_i^*(\theta_i, \tilde{\theta}_{-i})] - \frac{1}{|C| - 1}\sum_{k \in C \setminus \{i\}} \mathbb{E}_{\tilde{\theta}}\left[t_k^*(\tilde{\theta}) - \frac{1}{|C|}\delta_C(\tilde{\theta})\right] = \mathbb{E}_{\tilde{\theta}_{-i}}[t_i^*(\theta_i, \tilde{\theta}_{-i})],$$

where the last equality follows from (6).

tion? Indeed, their joint surplus will increase by doing so. The problem, however, is that the increased surplus cannot be allocated to benefit all types; some types will be strictly worse off and thus object to that move. To illustrate, suppose indeed that the bidders boycott auction whenever no bidder has valuation exceeding 5/9, and, otherwise, the high-valuation bidder consumes the object. Under this collusive arrangement, labeled $\tilde{M}$, each bidder's interim payoff is[15]

$$U^{\tilde{M}}(\theta) = \begin{cases} \frac{32}{2187} & \text{if } \theta \in [0, \frac{5}{9}) \\ \frac{1}{2}\theta^2 - \frac{611}{4374} & \text{if } \theta \in [\frac{5}{9}, 1]. \end{cases}$$
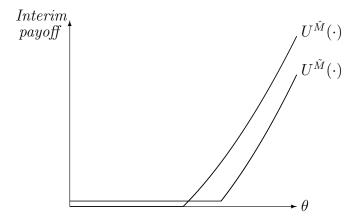

Figure 2: Comparison of interim payoffs from $\hat{M}$ and $\tilde{M}$

As can be seen from Figure 2, a bidder benefits from this collusion when his valuation is less than 0.528 but is strictly worse off if his valuation is higher. Hence, a collusive arrangement $\tilde{M}$ is not feasible. (The same is true for any other feasible manipulations.) Even though the net expected surplus may rise with some collusive manipulation, incentive compatibility facing the collusive bidders limits the way surplus can be allocated across types to make them uniformly better off. In this sense, our WCP auction exploits the informational asymmetry facing the collusive bidders.

We next provide some intuition about CONDITION (SB′). Theorem 1 identifies exclusion of the collusive bidders as a necessary property of optimal WCP auctions. Indeed, exclusion is a crucial weapon with which the seller can control the collusive power of cartel. The seller can exclude cartel members in two ways. If a cartel is not all-inclusive, the seller can do so by selling the good to the bidders outside that cartel. Leveraging such a non-cartel sale is an

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[15]This payoff can be obtained by applying the transfer rule in (29) with $n = 2$, $r = 5/9$, $\rho_i = 0$, and $q_i(\theta_i, \theta_{-i})$ being equal to 1 if $\theta_i > \max\{\theta_{-i}, 5/9\}$ and 0 otherwise.

effective way to extract rents from the cartel. If a cartel is all inclusive, then no such leverage exists, but the seller can still exclude the cartel by the threat of no sale, which again limits the cartel's power.

This exclusion insight also helps to understand CONDITION (SB′). We will develop the sense in which the condition specifies the precise degree of exclusion necessary for the WCP implementation of the second best. Suppose first there is no all-inclusive cartel (i.e., $\mathcal{C} \neq \{N\}$). This means either that there is a noncollusive bidder (i.e., $\mathcal{C}$ contains a singleton element) or that there are multiple proper cartels (i.e., $\mathcal{C}$ contains multiple non-singleton elements). Either way, the second-best naturally involves some degree of exclusion, since for each cartel $C \in \mathcal{C}$, the second-best assigns the good to an agent outside C with positive probability. Remarkably, with symmetric bidders, the resulting exclusion is sufficient to satisfy CONDITION (SB′), *regardless of the specific cartel structure:*

PROPOSITION 1. *If $\mathcal{C} \neq \{N\}$, CONDITION (SB′) holds, so the second-best is WCP implementable.*

Let us now consider the all-inclusive cartel (i.e., $\mathcal{C} = \{N\}$). In this case, CONDITION (SB′) may sometimes bind. To see this, suppose $\hat{\theta} = \underline{\theta}$. Then, the collusive bidders are never excluded in the second-best outcome, which would contradict Theorem 1.

Our exclusion insight suggests that CONDITION (SB′) may depend on the extent to which the second best prescribes exclusion. Since exclusion arises through the threshold $\hat{\theta}$ in this case, and since the threshold is determined by the virtual value J, one would expect CONDITION (SB′) to depend on the shape of the virtual value function. This intuition can be made precise.

Let $\mathcal{F}$ be an arbitrary family of cdf's such that, if $F \in \mathcal{F}$, then F has a strictly positive density f, and $J_F(\theta) := \theta - \frac{1-F(\theta)}{f(\theta)}$ is strictly increasing in θ, in $(\underline{\theta}_F, \overline{\theta}_F) \subset \mathbb{R}_+$. For each $F \in \mathcal{F}$, extend J_F to $\mathbb{R}_+$ by letting $J_F(\theta) := \lim_{\theta \downarrow \underline{\theta}} J_F(\theta)$ for $\theta \in [0, \underline{\theta}_F]$ and $J_F(\theta) := \overline{\theta}_F$ for $\theta \geq \overline{\theta}_F$. We then say that $F \in \mathcal{F}$ *virtual-value dominates* $G \in \mathcal{F}$, or $F \trianglerighteq G$, if $J_F(\theta) \geq J_G(\theta)$ for all $\theta \in \mathbb{R}_+$.

PROPOSITION 2. *Assume an all-inclusive cartel. Suppose F virtual-value dominates G, for $F, G \in \mathcal{F}$. Then, G satisfies CONDITION (SB′) whenever F does; that is, the second-best is WCP implementable with valuation distribution G whenever the same is true with valuation distribution F. Hence, if $\mathcal{F}_I = \{F_m\}_{m \in I}$ is a subset of $\mathcal{F}$ indexed by a parameter in some interval I such that $F_m \trianglerighteq F_{m'}$ for $m > m'$, then there exists $\hat{m} \in cl(I)$ such that the second-best is WCP implementable with F_m if $m < \hat{m}$ but not if $m > \hat{m}$.*

The second characterization is illustrated in the next example. The example suggests that,

at least for a well known family of distributions, one does not require much exclusion to WCP implement the second-best outcome.

EXAMPLE 2. *Suppose that there are n bidders each with a value drawn from the distribution $F(\theta) = \theta^m$ on the unit interval. This family constitutes a subset of $\mathcal{F}$, indexed by $m \in I = [1, \infty)$. One can check that the exclusion threshold is given by $\hat{\theta}(m) = \left(\frac{1}{m+1}\right)^{1/m}$. Let $\hat{m}(n)$ denote the threshold value of m below which* CONDITION (SB') *holds when there are n bidders. At this threshold value, the probability of exclusion necessary and sufficient for WCP implementation of the second best is given by $F(\hat{\theta}(\hat{m}(n)))^n$, which can be calculated as*

$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
0.0127	0.0071	0.0036	0.0017	0.0009
.

For instance, with four bidders, it takes exclusion of only 0.36% or more to WCP implement the second-best!

5.2 WCP Implementation of the Second-Best Outcome: Asymmetric Bidders

We now turn to the case of asymmetric bidders. In this case, the optimal noncollusive auction, as characterized in Theorem 0, requires bidders to be treated differently based on their ex ante distribution of types. This presents an extra challenge for the WCP implementation since, as shown in Lemma 1, the same price is charged no matter which member of the cartel receives the good. This does not mean, however, that the collusive bidders cannot be treated differently, for different interests of the heterogeneous types can be exploited to make (IR_N^M) difficult to satisfy. Indeed, we will show that the second-best outcome is WCP implementable at least with respect to the all-inclusive cartel (i.e., $\mathcal{C} = \{N\}$), under a condition that is not much stronger than CONDITION (SB).

Consider now a strict inequality version of (SB)-(i):

CONDITION (SB*): $K_i(\hat{\theta}_i) > r^* := \dfrac{\mathbb{E}\left[\sum_{i \in N} J_i(\theta_i) q_i^*(\theta)\right]}{\mathbb{E}\left[\sum_{i \in N} q_i^*(\theta)\right]}, \forall i \in N.$

This condition follows from applying Lemma 1 and 2 to the case of asymmetric bidders. Note that given the deterministic allocation at the second-best, the sales price has to be constant at the level, r^*, that achieves the second-best revenue if paid whenever the item is sold. Given this price, the *pseudo threshold type*, $K_i(\hat{\theta}_i)$, of each bidder i should not find it profitable to cancel its consumption so that $K_i(\hat{\theta}_i) > r^*$ is required. Otherwise, all types will agree on the cancelation to prevent the second-best from being WCP implementable.

THEOREM 4. *Assume $\mathcal{C} = \{N\}$ and* CONDITION (SB^*) *holds. Suppose also that $(J_i(\cdot) - r^*)f_i(\cdot)$ is increasing in the interval $[\underline{\theta}_i, J_i^{-1}(r^*)]$ for all $i \in N$.*[16] *Then, there exists an auction rule which is WCP and achieves the second-best outcome.*

6 Strong Collusion-Proof Implementation

Thus far, we have focused on weak collusion-proof implementation. Weak collusion-proof auctions protect a seller from a wide range of manipulations collusive bidders may employ. At the same time, it involves some restrictions. First, it rules out collusion supported by pessimistic beliefs on the part of members of a cartel about what may happen when they refuse to collude. Although such pessimistic beliefs may not be very plausible, the restriction on beliefs is nonetheless unsatisfactory. Second, the concept of weak collusion-proofness presumes that a cartel is organized by a third party who is uninformed of the members' types. Clearly, such an assumption simplifies the modeling of the collusive behavior, but has no clear empirical justification. We show, however, that these restrictions are not crucial for our main result: Given an additional condition, the second-best outcome can be implemented in a way robust to the specific beliefs entertained off the equilibrium path and to the particular way in which a cartel is formed and its proposal is made.

To begin, we define a strong collusion-proof auction. To that end, we consider any arbitrary indirect auction rule, $A = (B, \xi, \tau)$, where $B = (B_1, ..., B_n)$ is the profile of the message spaces, B_i for bidder i, and $(\xi, \tau) : B \to \mathcal{Q} \times \mathbb{R}^n$ is the outcome function mapping from messages to an allocation and payments. As before, we assume that B_i includes the option of non-participation by i and that the outcome function assigns the null outcome to the bidder invoking that option.

We next consider an extensive form game Γ_A that auction A induces. In that game, each member of any cartel $C \in \mathcal{C}$ (which may include a third party maximizing joint payoffs of C) may propose a side contract which maps from $\prod_{i \in C}(\Theta_i \cup \{\theta_\emptyset^C\})$ to a probability distribution over $\prod_{i \in C} B_i$, where $\theta_\emptyset^C$ denotes *an agent's refusal to participate in the side contract*. As before, we require the side contract to be reallocationally consistent (in the sense the side contract allocates the good to a member of a cartel only when some member of the cartel obtains the good from the seller) and budget balanced among those members of C who do not choose $\theta_\emptyset^C$. That the side contract can depend on the identities of cartel members refusing to participate means that, like Dequiedt (2007), we allow the remaining members to commit themselves to punish those refusing to join. If a side contract has been proposed, the members of the

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[16]This condition can be shown to hold if $F_i(\cdot)$ is concave, for instance.

cartel then simultaneously decide whether to reject all side contracts or accept one of them, and the messages in B are chosen according to the agreed-upon side contract,[17] the seller determines the outcome based on the messages according to the auction rule, and the good is reallocated and side transfers are exchanged according to the agreed-upon side contracts. Let $\mathcal{E}_A$ be the set of all Bayesian Nash equilibria that Γ_A admit in undominated strategies. The strong collusion-proof auctions are then defined as follows.

Definition SCP. *Expected revenue V is strong collusion-proof (henceforth, SCP) implementable if there exists an indirect auction rule, $A = (B, \xi, \tau)$, such that $\mathcal{E}_A$ is nonempty and that the seller receives expected revenue of V in every equilibrium of $\mathcal{E}_A$.*

It is worth emphasizing that SCP implementation restricts neither cartel members' out-of-equilibrium beliefs nor, as noted above, their ability to punish a defector. In fact, an SCP auction implementing V would guarantee the seller the revenue in every Bayesian Nash equilibrium, let alone every Perfect Bayesian equilibrium. More important, an SCP auction — or more precisely the revenue it implements — is robust to who proposes the collusive proposal. In this respect, the current SCP notion is stronger than any existing notions proposed by the existing authors.

We restrict attention to the case in which bidders are ex ante symmetric and there exists only one cartel $C \in \mathcal{C}$ with $|C| > 1$. The cartel may or may not be all-inclusive. We show that the second-best revenue V^* is SCP implementable, given CONDITION (SB$'$) and an additional condition. We construct the auction rule that accomplishes this. That auction rule, labeled $A = (B, \xi, \tau)$, builds on our WCP auction, $\hat{M}$. Recall that $\hat{M}$ provides a dominant strategy incentive for each noncollusive bidder to participate and report truthfully, a feature we retain in A.

We augment the WCP auction $\hat{M}$ by adding a message, r_z, for each cartel member, which is interpreted as a statement: "I reject $\hat{M}$ but would like to buy the item with an eagerness of z," where z is a positive integer of his choosing. Let $B_i := \overline{\Theta}_i \cup \{r_z, z = 1, 2, \cdots\}$ if $i \in C$ and $B_i := \overline{\Theta}_i$ otherwise. We then define the auction rule (ξ, τ) such that it coincides with $\hat{M}$ if no bidder in C announces r_z, but if some bidder in C announces r_z, then the auction rule is constructed as follows. Consider the following condition:

CONDITION (R): *For each $C \in \mathcal{C}$, there exists some $\theta' \in [\hat{\theta}, \bar{\theta}]$ satisfying*

$$U^{\hat{M}}(\bar{\theta}) \leq F^{n-|C|}(\theta') \left(\bar{\theta} - \mathbb{E}\left[H_C(\theta_{N \setminus C}^{(1)}) \, \middle| \, \theta_{N \setminus C}^{(1)} \leq \theta' \right] \right). \tag{7}$$

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[17]That is, a bidder's subsequent bidding is not bound by any side contract that he rejected.

If CONDITION (R) is satisfied, there are two possible situations: (a) the inequality in (7) holds strictly for all $\theta' \in [\hat{\theta}, \bar{\theta}]$ or (b) there exists some $\theta^r \in [\hat{\theta}, \bar{\theta}]$ satisfying

$$U^{\hat{M}}(\bar{\theta}) = F^{n-|C|}(\theta^r) \left(\bar{\theta} - \mathbb{E} \left[H_C(\theta^{(1)}_{N \setminus C}) \middle| \theta^{(1)}_{N \setminus C} \leq \theta^r \right] \right). \tag{8}$$

We design the auction rule (ξ, τ) based on which of the two cases hold. In either case, if there is a collusive bidder who announces r_z, then we select the bidder who announces r_z with the highest z (with a tie broken randomly among bidders with the same z). In case (a), the selected bidder receives the object with probability 1 at a fixed price $T_i^*(\bar{\theta})$. In case (b), the bidder receives the object at price $H_C(\theta^{(1)}_{N \setminus C})$ if $\theta^{(1)}_{N \setminus C} \leq \theta^r$, or else the bidder does not receive the object and he pays nothing. Last, no other bidders receive the object or pay any amount to the seller.

In words, the auction rule A gives an option for a collusive bidder with the highest valuation to secure his non-collusive payoff, $U^{\hat{M}}(\bar{\theta})$, by announcing r_z with the highest integer z. This extra option serves to limit the cartel's ability to punish a defector, which in turn constrains the set of side contracts sustainable in equilibrium. At the same time, the option itself should not be too profitable for the cartel since it would then become an object of collusion. We solve this problem by ensuring that the extra option does not introduce new allocation/transfers beyond those that are already available to the cartel C in $\hat{M}$. CONDITION (R) ensures that, given such a set of allocation/tranfers, the highest type should be able to achieve at least his non-collusive payoff.

THEOREM 5. *Suppose that there is a single cartel $C \subset N$. Then, given CONDITION (SB$'$) and (R), the second-best revenue V^* is SCP implementable.*

How restrictive is CONDITION (R)? The condition appears to hold in many circumstances. Specifically, the next Proposition reports that the condition holds if either the cartel is all-inclusive or it has sufficiently many members compared with the non-cartel members.

PROPOSITION 3. CONDITION (R) *is satisfied if $|C| = n$. If $|C| < n$, then for a fixed number $k\ (= n - |C|)$ of non-collusive bidders, there exists $\hat{c} \geq 2$ such that CONDITION (R) is satisfied if $|C| \geq \hat{c}$.*

As shown next, the threshold $\hat{c}$ is not very large at least for the uniform family.

EXAMPLE 3. *Consider again the uniform family with a bidder's valuation distributed uniformly over $[m, m+1]$ for $m \geq 1$. Then, $\hat{\theta}(m) = m$, so the object is sold with probability 1 at the second-best. Using (3) and letting $k = n - |C|$, one can calculate $\alpha_C = \frac{2(k+|C|)}{(k+|C|+1)(k+1)}$. Then,*

CONDITION (R) *is satisfied if and only if*

$$\frac{2}{k+1}\left(\frac{2-\alpha_C}{2}\right)^{k+1} \geq \frac{1}{k+|C|}. \tag{9}$$

Table 2 below depicts the threshold $\hat{c}$ for each k. As is clear, CONDITION (R) *requires the cartel to be bigger when there are more non-collusive bidders. This is because, with more non-collusive bidders, the message r_z becomes a less profitable option for the highest type so that the inequality (7) becomes more difficult to satisfy.*

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
$\hat{c}$	2	2	3	3	3	4	4	4	5	5	5	6	6	7	7	7	$\cdots$

Table 1: Threshold values for Condition (R)

Of course, CONDITION (R) is only sufficient. This, along with the fact that the condition is often not very restrictive, suggests that the particular notion of collusion-proofness does not play a significant role. In particular, the second-best outcome is SCP implementable whenever it is WCP implementable if the cartel is all-inclusive or if an additional reasonable condition holds.

7 Related Literature

Our model builds on the seminal framework of LM which embeds the collusive side contracting as a constraint facing the principal's grand contracting problem. Their main insight was that the agents' asymmetric information entails transactions costs on their collusion and that the principal can exploit these transaction costs. A further generalization by CK reveals that, in a broad set of circumstances, the principal can exploit the transaction cost to such an extent that she enjoys any payoff she would in a collusion-free environment. These papers assume that agents make their participation decisions non-cooperatively, however, assuming away any collusion on participation.

As mentioned, two recent papers study collusion on participation. Dequiedt (2007) considers a model in which an object is sold to one of two bidders each with either high or low valuation, and the cartel can commit to punish a bidder who refuses to join the collusive agreement to his reservation utility. He finds that the seller can do no better than text-book monopoly pricing; i.e., no real bidding competition can be induced from the bidders.[18] Hence,

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[18]This particular result holds when the reallocation is allowed, as in our model. Dequiedt also studies the case where the reallocation is not allowed, and obtains a somewhat different result.

the second-best is achieved only when it excludes the low valuation type, and the seller may exclude the low type even when the second-best would prescribe selling to both types.

This latter implication anticipates our exclusion principle, although the discrete case means exclusion may not always occur in the Dequiedt's model.[19] At the same time, it appears that optimal WCP auction can do no better than the Dequiedt's optimal collusion-proof auction, in the binary type case. Further, our strong notion of collusion, which does allow for the kinds of punishment possibility he assumes, often does not limit the seller's ability to implement the second best — and to induce nontrivial bidding competition (recall Theorem 5). These two facts suggest that the discrete types play an important role in his model. Of course, this does not mean the extent of collusive punishment does not matter in settings beyond simple auctions, and his innovation suggests an important direction for future research.

Pavlov (2008) has independently obtained some results of this paper. He focuses on all-inclusive cartel with symmetric bidders, and thus obtains analogues of Theorems 2 and 3 for that particular case. The current paper differs primarily in its treatment of the general cartel structure which allows for both non-collusive bidders and/or multiple cartels. Handling these cases required us to use different arguments than have been used by Pavlov. For instance, Lemma 2, which is critical for Theorem 2, has no analogue in his analysis.[20] In fact, we believe that Lemma 2 has a stand alone value, for it may lead to a more general analysis of optimal collusion-proof mechanism; the shadow cost of collusion characterized in the lemma may offer a useful clue on the way in which allocation should be distorted to deal with collusion in the general settings beyond simple auctions. Furthermore, our result on asymmetric bidders (Theorem 4), our general exclusion principle (Theorem 1) and our characterization of WCP implementation of second best based on the exclusion insight (Proposition 2), and our strong collusion-proof implementation (Theorem 5 and Proposition 3) are novel and have no analogues in his analysis. At the same time, Pavlov analyzes the set of feasible collusive agreements and considers the case in which the mechanism designer can prohibit the agents from reallocating the good. Overall, we believe that this paper, Dequiedt (2007) and Pavlov (2008) can be best viewed as complementary.

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[19]Since our exclusion principle relies on an atomless distribution, it does not rule out the possibility of non-exclusion with discrete types, which is indeed what Dequiedt finds optimal in some case.

[20]Instead, Pavlov requires a WCP auction to maximize a weighted sum of cartel members, which leads to his condition for the case of all-inclusive cartel. It is unclear whether this method can yield the general condition we obtain.

8 Conclusion

We have studied optimal collusion-proof auctions when a group of bidders can collude not only on their messages (e.g., "bids"), but also on their participation decisions. Despite this strong collusive power, we have shown that the asymmetric information facing the collusive bidders can be exploited to significantly weaken their collusive power, by eliminating the scope of collusive arrangements that could make all cartel members uniformly better off regardless of their types. We show that the second-best outcome is achievable if a cartel is not all-inclusive (which will be the case either if there is a noncollusive bidder or if there are multiple bidding cartels), or if the outcome involves a nontrivial probability of the object not being allocated to any bidder. More generally, we have shown that the optimal collusion-proof auction rule involves a positive probability of the object not being allocated to a collusive bidder.

Our results have two broad implications. First, unlike the prevailing impression based on the existing literature, the presence of bidder collusion need not mean that the seller can do no better than textbook monopoly pricing. Our seller can do as well as if there is no collusion in a broad set of circumstances. Second, an auction rule different from standard one may be more desirable when bidder collusion is a serious issue. We have identified group-based pricing and exclusion as features crucial for dealing with collusion.

It is legitimate to ask the practical relevance of our collusion-proof auctions, but the answer is not immediately clear. Our collusion-proof auction may be implemented in various ways, some of which may not even resemble an auction. For instance, our auction may be implemented by a seller who negotiates with a group of organized bidders for a single sale price. Indeed, it is quite common for a procurer of a service or a good to negotiate with a prime contractor acting as a representative of a group of contractors. Whether such a collective negotiation approach serves as a response to possible collusion among contractors is an interesting, yet unresolved, question.

It is of course quite possible that our collusion-proof auction has no real world correspondence. To the extent that this is true, there could be several reasons. First, collusion may not be a serious enough problem in many situations to warrant non-standard auctions. More important, it may be because of the two restrictive features of our auction design. First, our collusion-proof auction involves Bayesian implementation, which does rely on bidders' common knowledge of priors, which are clearly unrealistic in many situations. Relaxing the common knowledge of priors by strengthening the solution concept, say to dominant strategies, seems to be an important next step in the research of collusion-proof auctions. Such an extension need not imply, though, that collusion would become more difficult to deal with has been suggested here, since it affects the contracting problem at both ends, i.e., for both the

auction designer and for the colluding bidders. Collusion may indeed become easier to prevent if dominant-strategy incentives are very difficult to provide in a budget-balanced fashion among the colluders. Second, we have assumed that the seller has accurate information about the cartel structure; i.e., which bidders belong to what cartel. While this assumption may make sense for some scenarios, it may not for many others. In a sense, an important lesson from the current paper may be the highlighting of these features as challenges to overcome in collusion-proof auction design.

Appendix: Proofs

Proof of Lemma 0. It suffices to show that (IC^*) implies the following: for all $i \in N$ and all $\tilde{\theta}_i \in \Theta_i$,

$$U_i^M(\theta_i) = \mathbb{E}\left[K_i(\tilde{\theta}_i)Q_i(\tilde{\theta}_i)1_{\{\tilde{\theta}_i \leq \theta_i\}} + J_i(\tilde{\theta}_i)Q_i(\tilde{\theta}_i)1_{\{\tilde{\theta}_i \geq \theta_i\}} - T_i(\tilde{\theta}_i)\right]. \tag{10}$$

Note first that a well-known necessary condition for (IC^*) is: for all $i \in N$ and all $\tilde{\theta}_i, \theta_i \in \Theta_i$,

$$U_i^M(\theta_i) - U_i^M(\tilde{\theta}_i) = \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)da. \tag{11}$$

We show that (11) implies (10). Since $U_i^M(\tilde{\theta}_i) = \tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i)$, (11) becomes

$$U_i^M(\theta_i) = \tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i) + \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)da.$$

Taking expectation on both sides regarding $\tilde{\theta}_i$ yields

$$
\begin{aligned}
U_i^M(\theta_i) &= \mathbb{E}[\tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i)] + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)da\,dF_i(\tilde{\theta}_i) \\
&= \mathbb{E}[\tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i)] + \int_{\underline{\theta}_i}^{\theta_i} \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)da\,dF_i(\tilde{\theta}_i) + \int_{\theta_i}^{\overline{\theta}} \int_{\tilde{\theta}_i}^{\theta_i} Q_i(a)da\,dF_i(\tilde{\theta}_i) \\
&= \mathbb{E}[\tilde{\theta}_i Q_i(\tilde{\theta}_i) - T_i(\tilde{\theta}_i)] + \int_{\underline{\theta}_i}^{\theta_i} Q_i(\tilde{\theta}_i)F_i(\tilde{\theta}_i)d\tilde{\theta}_i - \int_{\theta_i}^{\overline{\theta}} Q_i(\tilde{\theta}_i)(1 - F_i(\tilde{\theta}_i))d\tilde{\theta}_i \\
&= \mathbb{E}\left[K_i(\tilde{\theta}_i)Q_i(\tilde{\theta}_i)1_{\{\tilde{\theta}_i \leq \theta_i\}} + J_i(\tilde{\theta}_i)Q_i(\tilde{\theta}_i)1_{\{\tilde{\theta}_i \geq \theta_i\}} - T_i(\tilde{\theta}_i)\right],
\end{aligned}
$$

where the third equality follows from the integration by parts. ∎

Proof of Lemma 1. To begin with, define $T_C(\theta_C) := \mathbb{E}_{\tilde{\theta}_{N\setminus C}}[\sum_{i \in C} t_i(\theta_C, \tilde{\theta}_{N\setminus C})]$ and $\mathcal{Q}_C^s := \{Q \in \mathcal{Q}_C | Q = Q_C(\theta_C)$ for some $\theta_C \in \overline{\Theta}_C\}$. Then, let us define $r : \mathcal{Q}_C \to \mathbb{R}_+$ as the greatest convex function such that for all $Q \in \mathcal{Q}_C^s$,

$$r(Q) \leq \inf\{T_C(\theta_C)|Q_C(\theta_C) = Q\}.$$

We show that $r(Q_C(\theta_C)) = T_C(\theta_C)$ for almost every θ_C. Suppose not. Then, it must be that for some $\epsilon > 0$,

$$\mathbb{E}\left[r(Q_C(\theta_C))\right] + \epsilon < \mathbb{E}\left[T_C(\theta_C)\right]. \tag{12}$$

Also, by the definition of $r(\cdot)$, it is possible to find $\mu^C : \Theta_C \to \Delta\overline{\Theta}_C$ satisfying that for all θ_C,

$$\mathbf{E}_{\mu^C(\theta_C)}\left[T_C(\tilde{\theta}_C)\right] \leq r(Q(\theta_C)) + \epsilon \text{ and } \mathbf{E}_{\mu^C(\theta_C)}[Q_C(\tilde{\theta}_C)] = Q_C(\theta_C). \tag{13}$$

We now show that M cannot be WCP with respect to C by constructing a weakly feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ of M by cartel C with which some bidder is better off than with M.

Let the cartel manipulate its type reports using $\mu^C(\cdot)$, whereafter, the object is reallocated to bidder i with probability $w_i(\theta_C) := \frac{q_i^C(\theta_C)}{Q_C(\theta_C)}$ so that $\sum_{i \in C} w_i(\theta_C) = 1$, satisfying (RC_C^M). Note that the interim allocation for each collusive bidder $i \in C$ is preserved since

$$
\begin{aligned}
\tilde{q}_i^C(\theta_C) &= \omega_i(\theta_C)\mathbb{E}_{\tilde{\theta}_{N\setminus C}}\left[\mathbf{E}_{\mu^C(\theta_C)}[q_C(\tilde{\theta}_C, \tilde{\theta}_{N\setminus C})]\right] \\
&= \omega_i(\theta_C)\mathbf{E}_{\mu^C(\theta_C)}[Q_C(\tilde{\theta}_C)] = \omega_i(\theta_C)Q_C(\theta_C) = q_i^C(\theta_C), \tag{14}
\end{aligned}
$$

where the second equality follows from changing the order of expectations, the third from (13), and the last from the definition of $\omega_i(\cdot)$.

Next, the cartel manipulates the transfer rule as follows: Letting $t_i^\mu(\theta) := \mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N\setminus C})]$, set $\tilde{t}_j(\theta) = t_j^\mu(\theta)$ for each $j \in N\setminus C$, and for each $i \in C$,

$$\tilde{t}_i(\theta) = t_i^\mu(\theta) + \mathbb{E}_{\tilde{\theta}_{-i}}\left[t_i(\theta_i, \tilde{\theta}_{-i}) - t_i^\mu(\theta_i, \tilde{\theta}_{-i})\right] - \frac{1}{|C| - 1}\sum_{j \in C\setminus\{i\}}\mathbb{E}_{\tilde{\theta}_{-j}}\left[t_j(\theta_j, \tilde{\theta}_{-j}) - t_j^\mu(\theta_j, \tilde{\theta}_{-j})\right] + \sigma_i,$$

where $\sum_{i \in C} \sigma_i = 0$. Note that $\sum_{i \in C} \tilde{t}_i(\theta) = \sum_{i \in C} t_i^\mu(\theta)$, which satisfies (BB_C^M) while $(BB_{N\setminus C}^M)$ is obviously satisfied. Also,

$$\mathbb{E}_{\tilde{\theta}_{-i}}\left[\tilde{t}_i(\theta_i, \tilde{\theta}_{-i})\right] = \mathbb{E}_{\tilde{\theta}_i}\left[t(\theta_i, \tilde{\theta}_{-i})\right] - \kappa_i, \tag{15}$$

where

$$\kappa_i := \frac{1}{|C| - 1}\sum_{j \in C\setminus\{i\}}\mathbb{E}\left[t_j(\theta) - t_j^\mu(\theta)\right] - \sigma_i.$$

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Then, one can choose $\sigma_i's$ so that $\kappa_i \geq 0, \forall i \in C$, since

$$\sum_{i \in C} \kappa_i = \mathbb{E}\left[\sum_{i \in C}(t_i(\theta) - t_i^\mu(\mu))\right] = \mathbb{E}\left[T_C(\theta_C) - \mathbf{E}_{\mu^C(\theta_C)}[T_C(\tilde{\theta}_C)]\right] > 0, \tag{16}$$

where the inequality follows from (12) and (13). So, (IC^*) and (IR_N^C) are satisfied for collusive bidders, due to (14), (15), and $\kappa_i \geq 0, \forall i \in C$, which means that $\tilde{M}$ is a weakly feasible manipulation of M. Also, some collusive bidder is better off than in M since $\kappa_j > 0$ for some $j \in C$. ∎

Proof of Lemma 2. To begin, we adopt the convention that $\theta_\emptyset < \underline{\theta}_i$ for all $i \in N$. Observe that $Q_C(\cdot)$ and $q_i^C(\cdot)$ inherit the monotonicity of $q_C(\cdot)$ and $q_i(\cdot)$, respectively, and hence are a.e. continuous. Also, since $r(\cdot)$ is convex with $r(0) = 0$, $p(\cdot)$ is nondecreasing and hence a.e. continuous also. Suppose to the contrary that (2) does not hold for almost every type profile. Then, we can find some bidder $k \in C$ and a positive measure set $\hat{\Theta}_{C-k} \subset \Theta_{C-k}$ such that for each $\theta_{C-k} \in \hat{\Theta}_{C-k}$, there exist $\theta_k \in \Theta_k$ and $\theta_k' \in \overline{\Theta}$ satisfying

$$(K_k(\theta_k) - p(\theta_k, \theta_{C-k}))q_i^C(\theta_k, \theta_{C-k}) < (K_k(\theta_k) - p(\theta_k', \theta_{C-k}))q_i^C(\theta_k', \theta_{C-k}).$$

Then, the a.e. continuity of $q_i^C(\cdot)$ and $p(\cdot)$ guarantees that for each $\theta_{C-k} \in \hat{\Theta}_{C-k}$, we can find two types $\hat{\theta}_k(\theta_{C-k}) \in \overline{\Theta}$ and $\hat{\hat{\theta}}_k(\theta_{C-k}) > \hat{\theta}_k(\theta_{C-k})$ such that for all $\theta_k \in (\hat{\theta}_k(\theta_{C-k}), \hat{\hat{\theta}}_k(\theta_{C-k})]$,

$$(K_k(\theta_k) - p(\theta_k, \theta_{C-k}))q_i^C(\theta_k, \theta_{C-k}) < (K_k(\theta_k) - p(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k}))q_i^C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k}). \tag{17}$$

We now define

$$\hat{\Theta}_C := \{(\theta_k, \theta_{C-k}) \in \Theta | \theta_{C-k} \in \hat{\Theta}_{C-k} \text{ and } \theta_k \in (\hat{\theta}_k(\theta_{C-k}), \hat{\hat{\theta}}_k(\theta_{C-k})]\},$$

$\hat{q}_k^C(\theta_{C-k}) := q_k^C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})$, and $\hat{p}(\theta_{-k}) := p(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})$. Note that (17) holds for all $\theta_C \in \hat{\Theta}_C$.

In order to draw a contradiction, we construct a weakly feasible manipulation of M, $\tilde{M} = (\tilde{q}, \tilde{t})$, which makes bidder k better off.

Consider the following report manipulation, denoted $\mu^C : \Theta_C \to \Delta\overline{\Theta}_C$, and reallocation scheme by the cartel: if $\theta_C \notin \hat{\Theta}_C$, then report truthfully and do not perform any reallocation while if $\theta_C \in \hat{\Theta}_C$, then (i) report truthfully with probability $\frac{\sum_{i \in C \setminus \{k\}} q_i^C(\theta_C)}{Q_C(\theta_C)}$ and, once the object is assigned, reallocate it to bidder $i \in C \setminus \{k\}$ with probability $\frac{q_i^C(\theta)}{\sum_{i \in C \setminus \{k\}} q_i^C(\theta_C)}$, (ii) report $(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})$ (or $(\theta_\emptyset, \cdots, \theta_\emptyset)$ in case $\hat{\theta}_k(\theta_{C-k}) = \theta_\emptyset$) with probability $\frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k})), \theta_{C-k}}$ and, once the object is assigned, reallocate it to bidder k with probability 1, and (iii) choose

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$(\theta_\emptyset, \cdots, \theta_\emptyset)$ (or nonparticipation) with the remaining probability.[21] This manipulation will result in the following allocation probabilities: for bidder $i \in C \backslash \{k\}$,

$$\tilde{q}_i^C(\theta_C) = Q_C(\theta_C) \frac{\sum_{i \in C \backslash \{k\}} q_i^C(\theta_C)}{Q_C(\theta_C)} \frac{q_i^C(\theta_C)}{\sum_{i \in C-k} q_i^C(\theta_C)} = q_i^C(\theta_C) \text{ if } \theta_C \in \hat{\Theta}_C,$$

and simply $\tilde{q}_i^C(\theta_C) = q_i^C(\theta_C)$ if $\theta_C \notin \hat{\Theta}_C$. Likewise, for bidder k, if $\theta_C \notin \hat{\Theta}_C$, then $\tilde{q}_k^C(\theta_C) = q_k^C(\theta_C)$, and if $\theta_C \in \hat{\Theta}_C$, then

$$\tilde{q}_k^C(\theta_C) = Q_C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k}) \frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})} = \hat{q}_k^C(\theta_{C-k}). \tag{18}$$

It can be easily verified that $\tilde{q}_k^C(\cdot, \theta_{C-k})$ is nondecreasing for every θ_{C-k}.[22] Thus, the interim allocation $\tilde{Q}_i(\theta_i) = \mathbb{E}_{\tilde{\theta}_{C-i}}[\tilde{q}_i^C(\theta_i, \tilde{\theta}_{C-i})]$ is also nondecreasing for each $i \in C$.

After the manipulation, the cartel's aggregate payment becomes

$$\mathbb{E}_{\tilde{\theta}_{N \backslash C}} \left[\mathbf{E}_{\mu^C(\theta_C)} [\sum_{i \in C} t_i(\tilde{\theta}_C, \tilde{\theta}_{N \backslash C})] \right]$$

$$= \begin{cases} r(Q_C(\theta_C)) \dfrac{\sum_{i \in C \backslash \{k\}} q_i^C(\theta_C)}{Q_C(\theta_C)} + r(Q_C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})) \dfrac{\hat{q}_k^C(\theta_{C-k})}{Q(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})} & \text{if } \theta_C \in \hat{\Theta}_C \\ r(Q_C(\theta_C)) & \text{otherwise,} \end{cases}$$

which yields

$$\mathbb{E} \left[\mathbf{E}_{\mu^C(\theta_C)} [\sum_{i \in C} t_i(\tilde{\theta}_C, \theta_{N \backslash C})] \right]$$

$$= \mathbb{E} \left[r(Q_C(\theta_C)) \right] + \mathbb{E}_{\theta_C \in \hat{\Theta}_C} \left[r(Q(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})) \frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})} - r(Q_C(\theta_C)) \frac{q_k^C(\theta_C)}{Q_C(\theta_C)} \right]$$

---

[21]It is important to make sure that the probability of reporting truthfully or $(\hat{\theta}(\theta_{C-k}), \theta_{C-k})$ does not exceed 1, for which it suffices to verify that $\frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})} \leq \frac{q_k^C(\theta_C)}{Q_C(\theta_C)}$. This holds trivially if $\hat{\theta}_k(\theta_{C-k}) = \theta_\emptyset$. If $\hat{\theta}_k(\theta_{C-k}) \neq \theta_\emptyset$, it holds since

$$\frac{\hat{q}_k^C(\theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})} = 1 - \frac{\sum_{i \in C \backslash \{k\}} q_i^C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})}{Q_C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k})} \leq 1 - \frac{\sum_{i \in C \backslash \{k\}} q_i^C(\theta_C)}{Q_C(\theta_C)} = \frac{q_k^C(\theta_C)}{Q_C(\theta_C)},$$

where the inequality holds since $Q_C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k}) \leq Q_C(\theta_k, \theta_{C-k})$ and $q_i^C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k}) \geq q_i^C(\theta_k, \theta_{C-k})$ for all $i \neq k$, by the monotonicity of $Q_C(\cdot)$ and $q_i^C(\cdot)$.

[22]To see this, consider arbitrary θ_k and θ_k' with $\theta_k' > \theta_k$, and θ_{C-k}: (i) if $(\theta_k, \theta_{C-k}), (\theta_k', \theta_{C-k}) \in \hat{\Theta}_C$, then $\tilde{q}_k^C(\theta_k, \theta_{C-k}) = \tilde{q}_k^C(\theta_k', \theta_{C-k}) = \hat{q}_k^C(\theta_{C-k})$, (ii) if $(\theta_k, \theta_{C-k}) \in \hat{\Theta}_C$ and $(\theta_k', \theta_{C-k}) \notin \hat{\Theta}_C$, then $\hat{\theta}_k(\theta_{C-k}) < \theta_k \leq \hat{\theta}_k(\theta_{C-k}) < \theta_k'$ and thus $\tilde{q}_k^C(\theta_k, \theta_{C-k}) = q_k^C(\hat{\theta}_k(\theta_{C-k}), \theta_{C-k}) \leq q_k^C(\theta_k', \theta_{C-k}) = \tilde{q}_k^C(\theta_k', \theta_{C-k})$. And other cases can be dealt with similarly.

$$= \mathbb{E}\left[\sum_{i\in C} t_i(\theta)\right] + \mathbb{E}_{\theta_C \in \hat{\Theta}_C}\left[\hat{p}(\theta_{C-k})\hat{q}_k^C(\theta_{C-k}) - p(\theta_C)q_k^C(\theta_C)\right] \tag{19}$$

Next, $\tilde{t}(\cdot)$ is constructed as follows. For each $j \in N\backslash C$, set $\tilde{t}_j(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[t_j(\tilde{\theta}_C, \theta_{N\backslash C})]$. For each $i \in C$, we set

$$\tilde{t}_i(\theta) = \mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N\backslash C})] + Y_i(\theta_i) - \frac{1}{|C|-1}\sum_{j\in C\backslash\{i\}} Y_j(\theta_j) + \rho_i,$$

where

$$Y_i(\theta_i) := \theta_i \tilde{Q}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \tilde{Q}_i(a)da - \mathbb{E}_{\theta_{-i}}[\mathbf{E}_{\mu^C(\theta_C)}[t_i(\tilde{\theta}_C, \theta_{N\backslash C})]],$$

and

$$\rho_i := \frac{1}{|C|-1}\mathbb{E}_{\theta_{-i}}\left[\sum_{j\in C\backslash\{i\}} Y_j(\theta_j)\right] - U_i^M(\underline{\theta}_i) \text{ for } i \in C\backslash\{k\}, \text{ and } \rho_k = -\sum_{i\in C\backslash\{k\}}\rho_i.$$

By construction, then $\tilde{t}$ satisfies (BB_C^M) and $(BB_{N\backslash C}^M)$.

We now complete the proof by showing that $\tilde{M}$ is a weakly feasible manipulation and makes bidder k better off. To this end, observe that for an arbitrary $\theta_k \in \Theta_k$,

$$U_k^{\tilde{M}}(\theta_k) + \sum_{i\in C\backslash\{k\}} U_i^{\tilde{M}}(\underline{\theta}_i)$$

$$= \mathbb{E}\left[\left(K_k(\tilde{\theta}_k)\tilde{q}_k(\tilde{\theta})\right)1_{\{\tilde{\theta}_k<\theta_k\}} + \left(J_k(\tilde{\theta}_k)\tilde{q}_k(\tilde{\theta})\right)1_{\{\tilde{\theta}_k>\theta_k\}} + \sum_{i\in N\backslash\{k\}} J_i(\tilde{\theta}_i)q_i(\tilde{\theta}) - \sum_{i\in C}\tilde{t}_i(\tilde{\theta})\right]$$

$$= \mathbb{E}\left[\left(K_k(\tilde{\theta}_k)q_k(\tilde{\theta})\right)1_{\{\tilde{\theta}_k<\theta_k\}} + \left(J_k(\tilde{\theta}_k)q_k(\tilde{\theta})\right)1_{\{\tilde{\theta}_k>\theta_k\}} + \sum_{i\in N\backslash\{k\}} J_i(\tilde{\theta}_i)q_i(\tilde{\theta}) - \sum_{i\in C}t_i(\tilde{\theta})\right]$$

$$+ \mathbb{E}_{\tilde{\theta}_C \in \hat{\Theta}_C}\left[K_k(\tilde{\theta}_k)(\tilde{q}_k^C(\tilde{\theta}_C) - q_k^C(\tilde{\theta}_C))1_{\{\tilde{\theta}_k<\theta_k\}} + J_k(\tilde{\theta}_k)(\tilde{q}_k^C(\tilde{\theta}_C) - q_k^C(\tilde{\theta}_C))1_{\{\tilde{\theta}_k>\theta_k\}}\right.$$
$$\left. - \left(\hat{p}(\tilde{\theta}_{C-k})\hat{q}_k^C(\tilde{\theta}_{C-k}) - p(\tilde{\theta}_C)q_k^C(\tilde{\theta}_C)\right)\right]$$

$$= U_k^M(\theta_k) + \sum_{i\in C\backslash\{k\}} U_i^M(\underline{\theta}_i)$$

$$+ \mathbb{E}_{\tilde{\theta}_C \in \hat{\Theta}_C}\left[\left((K_k(\tilde{\theta}_k) - \hat{p}(\tilde{\theta}_{C-k}))\hat{q}_k^C(\tilde{\theta}_{C-k}) - (K_k(\tilde{\theta}_k) - p_k(\tilde{\theta}_C))q_k^C(\tilde{\theta}_C)\right)1_{\{\tilde{\theta}_k<\theta_k\}}\right.$$
$$\left. + \left((J_k(\tilde{\theta}_k) - \hat{p}(\tilde{\theta}_{C-k}))\hat{q}_k^C(\tilde{\theta}_{C-k}) - (J_k(\tilde{\theta}_k) - p_k(\tilde{\theta}_C))q_k^C(\tilde{\theta}_C)\right)1_{\{\tilde{\theta}_k>\theta_k\}}\right]$$

$$> U_k^M(\theta_k) + \sum_{i\in C\backslash\{k\}} U_i^M(\underline{\theta}_i). \tag{20}$$

The first equality follows from Lemma 0, the second from (19), the third from the rearrangement and (18), and the inequality from (17) and the fact that for all $\tilde{\theta}_C \in \hat{\Theta}_C$,

$$J_k(\tilde{\theta}_k)(\hat{q}_k^C(\tilde{\theta}_{C-k}) - q_k^C(\tilde{\theta}_C)) \geq K_k(\tilde{\theta}_k)(\hat{q}_k^C(\tilde{\theta}_{C-k}) - q_k^C(\tilde{\theta}_C)),$$

since $\hat{q}_k^C(\tilde{\theta}_{C-k}) \leq q_k^C(\tilde{\theta}_C)$ and $J_k(\tilde{\theta}_k) < K_k(\tilde{\theta}_k)$.

From the construction of $\tilde{t}(\cdot)$, one can easily verify that $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^M(\underline{\theta}_i)$, $\forall i \in C\backslash\{k\}$. Then, (20) implies $U_k^{\tilde{M}}(\theta_k) > U_k^M(\theta_k)$ for all $\theta_k \in \Theta_k$ or bidder k is better off. The construction of $\tilde{t}(\cdot)$ and monotonicity of $\tilde{Q}_i(\cdot)$, $\forall i \in C$ guarantee that $\tilde{M}$ satisfies (IC^*) for all collusive bidders. The proof will be complete if (IR_C^M) holds for all $i \in C\backslash\{k\}$:

$$U_i^{\tilde{M}}(\theta_i) = U_i^{\tilde{M}}(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \tilde{Q}_i(\tilde{\theta})d\tilde{\theta} = U_i^M(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} Q_i(\tilde{\theta})d\tilde{\theta} = U_i^M(\theta_i), \forall\theta_i \in \Theta_i$$

since $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^M(\underline{\theta}_i)$ and $\tilde{Q}_i(\cdot) = Q_i(\cdot)$, $\forall i \in C\backslash\{k\}$. $\blacksquare$

Proof of Theorem 2. Suppose that an auction rule $M = (q, t) \in \mathcal{M}$ WCP implements the second-best outcome. Then, $q(\cdot) = q^*(\cdot)$ and $U_i(\underline{\theta}_i) = 0$ for all $i \in N$, which implies by Lemma 0 that for any $C \subset N$,

$$\mathbb{E}\left[\sum_{i\in C} t_i(\theta)\right] = \mathbb{E}\left[\sum_{i\in C} J_i(\theta_i)q_i^*(\theta_i)\right]. \tag{21}$$

By Lemma 1, there exists a convex function $r(\cdot)$ that represents the total payment for the cartel.

We first consider the case $\mathcal{C} = \{N\}$. Since $q_N^*(\theta) = 0$ or 1 for all $\theta \in \Theta$, Lemma 1 implies that $p(\theta) = r^*$ whenever $q_N^*(\theta) = 1$. We first prove $\hat{\theta}_i > \underline{\theta}_i$ for all $i \in N$. Suppose not. Then, there exists k such that $J_k(\underline{\theta}_k) \geq \max\{\max_{i \in N\backslash\{k\}} J_i(\underline{\theta}_i), 0\}$. It follows that $q_k^*(\underline{\theta}_1, \cdots, \underline{\theta}_n) > 0$, so $p(\underline{\theta}_1, \cdots, \underline{\theta}_n) = r^*$. Since $r^* \geq V^* > \underline{\theta}_i = K_i(\underline{\theta}_i)$, we have a contradiction to (2).

We next consider the case $\mathcal{C} \neq \{N\}$. Fix any C with $|C| \geq 2$. If no such C exists, there is no collusion, so we are done. For each bidder $i \in C$ and his type $\theta_i \in \Theta_i$, let $X_i(\theta_i) := \Pr\{\theta_{C-i} \in \Theta_{C-i} \mid J_i(\theta_i) > \max_{k \in C\backslash\{i\}} J_k(\theta_k)\}$ be the probability that i has the highest virtual value among the collusive bidders, and let $Y_i(\theta_i) := \Pr\{\theta_{N\backslash C} \in \Theta_{N\backslash C} \mid J_i(\theta_i) > \max\{\max_{k \in N\backslash C} J_k(\theta_k), 0\}\}$. Letting $p_i(\theta_i) := \frac{r(Y_i(\theta_i))}{Y_i(\theta_i)}$ for each $i \in C$, Lemma 2 implies that, $\forall\theta_i \geq \hat{\theta}_i$

$$(K_i(\theta_i) - p_i(\theta_i)) Y_i(\theta_i) \geq \max\{0, \max_{\theta_i' \in [\hat{\theta}_i, \theta_i]} (K_i(\theta_i) - p_i(\theta_i')) Y_i(\theta_i')\}.$$

By the envelope theorem argument, $\forall\theta_i \geq \hat{\theta}_i$,

$$(K_i(\theta_i) - p_i(\theta_i))Y_i(\theta_i) \geq (K_i(\hat{\theta}_i) - p_i(\hat{\theta}_i))Y_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da \geq \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da$$

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or

$$p_i(\theta_i)Y_i(\theta_i) \le K_i(\theta_i)Y_i(\theta_i) - \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da.$$

Thus, we have

$$
\begin{aligned}
\mathbb{E}\left[\sum_{i \in C} t_i(\theta)\right] &= \mathbb{E}\left[\sum_{i \in C} r(Y_i(\theta_i))X_i(\theta_i)\right] \\
&= \mathbb{E}\left[\sum_{i \in C} p_i(\theta_i)Y_i(\theta_i)X_i(\theta_i)\right] \\
&\le \mathbb{E}\left[\sum_{i \in C}\left(K_i(\theta_i)Y_i(\theta_i) - \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da\right)X_i(\theta_i)\right]. \qquad (22)
\end{aligned}
$$

Letting $Z_i(\theta_i) = \int_{\theta_i}^{\bar{\theta}_i} X_i(s)dF_i(s)$,

$$
\begin{aligned}
&\mathbb{E}\left[\left(K_i(\theta_i)Y_i(\theta_i) - \int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da\right)X_i(\theta_i)\right] \\
&= \int_{\hat{\theta}_i}^{\bar{\theta}_i} K_i(\theta_i)Y_i(\theta_i)X_i(\theta_i)dF_i(\theta_i) - \int_{\hat{\theta}_i}^{\bar{\theta}_i}\int_{\hat{\theta}_i}^{\theta_i} K_i'(a)Y_i(a)da X_i(\theta_i)dF_i(\theta_i) \\
&= \int_{\hat{\theta}_i}^{\bar{\theta}_i} K_i(\theta_i)X_i(\theta_i)Y_i(\theta_i)dF_i(\theta_i) - \int_{\hat{\theta}_i}^{\bar{\theta}_i} K_i'(\theta_i)Y_i(\theta_i)Z_i(\theta_i)d\theta_i \\
&= \int_{\hat{\theta}_i}^{\bar{\theta}_i} K_i(\theta_i)X_i(\theta_i)Y_i(\theta_i)dF_i(\theta_i) - K_i(\theta_i)Y_i(\theta_i)Z_i(\theta_i)\Big|_{\hat{\theta}_i}^{\bar{\theta}_i} \\
&\quad + \int_{\hat{\theta}_i}^{\bar{\theta}_i} K_i(\theta_i)\left(Y_i'(\theta_i)Z_i(\theta_i) + Y_i(\theta_i)Z_i'(\theta_i)\right)d\theta_i \\
&= K_i(\hat{\theta}_i)Y_i(\hat{\theta}_i)Z_i(\hat{\theta}_i) + \int_{\hat{\theta}_i}^{\bar{\theta}_i} K_i(\theta_i)Y_i'(\theta_i)Z_i(\theta_i)d\theta_i \\
&= \mathbb{E}\left[K_i(\phi_i(\theta_{N\setminus C}))q_i^*(\theta)\right].
\end{aligned}
$$

The second and fourth equalities follow from integration by parts. To verify the fifth equality, note that $Y_i(\hat{\theta}_i) = \Pr\{\phi_i(\theta_{N\setminus C}) = \hat{\theta}_i\}$, $Y_i(s) = \Pr\{\phi_i(\theta_{N\setminus C}) \le s\}$ for each $s > \hat{\theta}_i$, and $Z_i(s) = \mathbb{E}[q_i^*(\theta)|\phi_i(\theta_{N\setminus C}) = s]$. Combine this derivation with (21) and (22) to obtain (ii) of CONDITION (SB). ∎

Proof of Theorem 3. Since $\hat{M}$ implements V^*, it suffices to prove that $\hat{M}$ is WCP. To this end, consider any $C \in \mathcal{C}$ with $|C| \ge 2$. Suppose all bidders outside C report truthfully, but cartel C contemplates a manipulation of $\hat{M}$, $\tilde{M} = (\tilde{q}, \tilde{t})$, that satisfies (IC_C^*) and $(IR_C^{\hat{M}})$. Then, there exists a function $\mu^C : \Theta_C \mapsto \Delta\overline{\Theta}_C$ such that $(R_C^{\hat{M}}), (R_{N\setminus C}^{\hat{M}}), (BB_C^{\hat{M}})$ and $(BB_{N\setminus C}^{\hat{M}})$

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hold. Since the same sale price is charged against a cartel no matter how many of its members participate, it cannot gain from non-participation of its members. Hence, without loss, we assume that μ^C places no weight on $\overline{\Theta} \backslash \Theta$.

We first prove that $\tilde{q}(\theta) = q^*(\theta)$ for almost every $\theta \in \Theta$. To this end, suppose this is not the case. Then,

$$\alpha_C(\sum_{i \in C} U_i^{\tilde{M}}(\bar{\theta})) + (1 - \alpha_C)(\sum_{i \in C} U_i^{\tilde{M}}(\underline{\theta}))$$

$$= \mathbb{E}\left[\sum_{i \in C} H_C(\theta_i)\tilde{q}_i(\theta) - \sum_{i \in C} \tilde{t}_i(\theta)\right]$$

$$= \mathbb{E}\left[\sum_{i \in C} H_C(\theta_i)\tilde{q}_i(\theta) - \sum_{i \in C} \mathbb{E}_{\mu^C(\theta_C)}[\hat{t}_i(\tilde{\theta}_C, \theta_{N \backslash C})]\right]$$

$$\leq \mathbb{E}\left[\sum_{i \in C} H_C(\theta_i)\tilde{q}_i(\theta) - \mathbb{E}_{\mu^C(\theta_C)}[\delta_C(\tilde{\theta}_C, \theta_{N \backslash C})]\right]$$

$$= \mathbb{E}\left[\sum_{i \in C} H_C(\theta_i)\tilde{q}_i(\theta) - H_C(\theta_{N \backslash C}^{(1)})\mathbb{E}_{\mu^C(\theta_C)}[\sum_{i \in C} q_i^*(\tilde{\theta}_C, \theta_{N \backslash C})]\right]$$

$$= \mathbb{E}\left[\sum_{i \in C}[H_C(\theta_i) - H_C(\theta_{N \backslash C}^{(1)})]\tilde{q}_i(\theta)\right]$$

$$< \mathbb{E}\left[\sum_{i \in C}[H_C(\theta_i) - H_C(\theta_{N \backslash C}^{(1)})]q_i^*(\theta)\right] \tag{23}$$

$$= \alpha_C(\sum_{i \in C} U_i^{\hat{M}}(\bar{\theta})) + (1 - \alpha_C)(\sum_{i \in C} U_i^{\hat{M}}(\underline{\theta})).$$

The first equality follows from Lemma 0, the second from equation $(BB_C^{\hat{M}})$, the third from the definition of $\delta_C(\cdot)$, the fourth from $(R_C^{\hat{M}})$, and the last equality from the above string of equalities repeated in the reverse order. The weak inequality follows from the construction of $\hat{t}_i(\cdot)$ for $i \in C$ as in (4) and (5). Lastly, the strict inequality follows from the definition of α_C and the strict monotonicity of $H_C(\cdot)$. To see this, we compare the LHS and RHS of the inequality (23) at the ex-post level: (i) if $\theta_k > \max\{\max_{i \in N \backslash \{k\}} \theta_i, \hat{\theta}\}$ for some $k \in C$, then $q_k^*(\theta) = 1 \neq \tilde{q}_k(\theta)$ implies that

$$LHS = \sum_{i \in C}(H_C(\theta_i) - H_C(\theta_{N \backslash C}^{(1)}))\tilde{q}_i(\theta_i) < H_C(\theta_k) - H_C(\theta_{N \backslash C}^{(1)}) = RHS,$$

(ii) if $\theta_k > \max\{\max_{i \in N \backslash \{k\}} \theta_i, \hat{\theta}\}$ for some $k \in N \backslash C$, then any manipulated allocation different

from $q^*(\cdot)$ implies $\tilde{q}_k(\theta) < 1$ and $\tilde{q}_{k'}(\theta) > 0$ for some $k' \in C$,[23] and thus

$$LHS = \sum_{i \in C}(H_C(\theta_i) - H_C(\theta_{N \backslash C}^{(1)}))\tilde{q}_i(\theta_i) = \sum_{i \in C}(H_C(\theta_i) - H_C(\theta_k))\tilde{q}_i(\theta_i) < 0 = RHS,$$

(iii) if $\max_{i \in N} \theta_i < \hat{\theta}$, then $\tilde{q}(\theta) \neq q^*(\theta) = \mathbf{0}$ implies that the LHS is negative while the RHS is zero. In sum, the LHS of (23) is always less than the RHS, which means that $\tilde{M}$ worsens the (interim) payoff of either the highest type or the lowest type of at least one collusive bidder. This contradicts that $\tilde{M}$ satisfies $(IR_C^{\tilde{M}})$. We have thus proven that $\tilde{q}(\theta) = q^*(\theta)$ for almost every θ.

It follows from this result that the gross surplus realized within C from $\tilde{M}$ is the same as from $\hat{M}$, and, combined with (3), that the cartel pays the same expected payments with $\tilde{M}$ as with $\hat{M}$. Hence, the net total expected payoff accruing to C from $\tilde{M}$ is the same as from $\hat{M}$. Together with $(IR_C^{\hat{M}})$, this implies that no bidder of C is strictly better off from manipulation $\tilde{M}$. Since this is true for all feasible manipulation of $\hat{M}$, we conclude that $\hat{M}$ is WCP. ∎

Proof of Proposition 1. First, we prove that CONDITION (SB′) holds for any C with $2 \leq |C| < n$. To this end, observe that

$$\mathbb{E}\left[K(\theta_{N \backslash C}^{(1)})1_{\{\theta_C^{(1)} > \theta_{N \backslash C}^{(1)}\}}\right]$$

$$= K(\hat{\theta})(1 - F^{|C|}(\hat{\theta}))F^{n-|C|}(\hat{\theta}) + \int_{\hat{\theta}}^{\bar{\theta}}\left(\theta + \frac{F(\theta)}{f(\theta)}\right)(1 - F^{|C|}(\theta))dF^{n-|C|}(\theta)$$

$$= K(\hat{\theta})(1 - F^{|C|}(\hat{\theta}))F^{n-|C|}(\hat{\theta}) + \int_{\hat{\theta}}^{\bar{\theta}}\theta(1 - F^{|C|}(\theta))dF^{n-|C|}(\theta)$$

$$+ \int_{\hat{\theta}}^{\bar{\theta}}(n - |C|)(1 - F^{|C|}(\theta))F^{n-|C|}(\theta)d\theta. \tag{24}$$

Observe also that

$$\mathbb{E}\left[J(\theta_C^{(1)})1_{\{\theta_C^{(1)} > \theta_{N \backslash C}^{(1)}\}}\right] = \int_{\hat{\theta}}^{\bar{\theta}}\left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right)F^{n-|C|}(\theta)dF^{|C|}(\theta)$$

$$= -(1 - F^{|C|}(\theta))\theta F^{n-|C|}(\theta)\Big|_{\hat{\theta}}^{\bar{\theta}} + \int_{\hat{\theta}}^{\bar{\theta}}(1 - F^{|C|}(\theta))d\left(\theta F^{n-|C|}(\theta)\right) - \int_{\hat{\theta}}^{\bar{\theta}}|C|(1 - F(\theta))F^{n-1}(\theta)d\theta$$

$$= \hat{\theta}(1 - F^{|C|}(\hat{\theta}))F^{n-|C|}(\hat{\theta}) + \int_{\hat{\theta}}^{\bar{\theta}}(1 - F^{|C|}(\theta))F^{n-|C|}(\theta)d\theta$$

---

[23]This follows from the fact that noncollusive bidders always report truthfully so collusive bidders can change the allocation only by announcing that one of them has at least $\theta_k > \hat{\theta}$, and getting themselves allocated the object.

$$+ \int_{\hat{\theta}}^{\bar{\theta}} \theta(1 - F^{|C|}(\theta))dF^{n-|C|}(\theta) - \int_{\hat{\theta}}^{\bar{\theta}} |C|(1 - F(\theta))F^{n-1}(\theta)d\theta,$$

where the second equality follows from integration by parts. Subtracting this expression from (24) yields

$$\mathbb{E}\left[\sum_{i \in C} \left(K_i(\phi_i(\theta_{N\backslash C})) - J_i(\theta_i)\right)q_i^*(\theta)\right]$$

$$= \mathbb{E}\left[\left(K(\theta_{N\backslash C}^{(1)}) - J(\theta_C^{(1)})\right)1_{\{\theta_C^{(1)} > \theta_{N\backslash C}^{(1)}\}}\right]$$

$$= (K(\hat{\theta}) - \hat{\theta})(1 - F^{|C|}(\hat{\theta}))F^{n-|C|}(\hat{\theta})$$

$$+ \int_{\hat{\theta}}^{\bar{\theta}} \left[(n - |C| - 1)(1 - F^{|C|}(\theta))F^{n-|C|}(\theta) + |C|(1 - F(\theta))F^{n-1}(\theta)\right] d\theta > 0,$$

satisfying CONDITION (SB′). ∎

Proof of Proposition 2. It suffices to prove the first statement, from which the second follows. Assume that CONDITION (SB′) for F. Then, by the text following Proposition 1, we must have $\hat{\theta}_F > \underline{\theta}_F$. Since $F \unrhd G$ implies $\hat{\theta}_G \geq \hat{\theta}_F$ and $\underline{\theta}_G \leq \underline{\theta}_F$, we must have $\hat{\theta}_G > \underline{\theta}_G$. It also follows from $F \unrhd G$ that F first-order stochastically dominates (FOSD) G (see Maskin and Riley (2000), for instance.)

Given $C = N$, CONDITION (SB′) simplifies to

$$K(\hat{\theta}) \geq \mathbb{E}\left[J(\theta_N^{(1)})|\theta_N^{(1)} > \hat{\theta}\right], \tag{25}$$

which we can rewrite, using the fact that $K_F(\hat{\theta}_F) = J_F(\hat{\theta}_F) + \frac{1}{f(\hat{\theta}_F)} = \frac{1}{f(\hat{\theta}_F)}$, as:

$$\frac{1}{f(\hat{\theta}_F)} \geq \frac{\int_{\underline{\theta}_F}^{\bar{\theta}_F} \max\{J_F(\theta), 0\}dF^n(\theta)}{1 - F^n(\hat{\theta}_F)}$$

or

$$\left(\frac{1 - F(\hat{\theta}_F)}{f(\hat{\theta}_F)}\right)\left(\sum_{k=1}^{n} F^{k-1}(\hat{\theta}_F)\right) \geq \int_{\underline{\theta}_F}^{\bar{\theta}_F} \max\{J_F(\theta), 0\}dF^n(\theta), \tag{26}$$

and similarly for G. We show below that if F is replaced by G, then (a) the LHS of (26) (weakly) increases while (b) the RHS of (26) (weakly) decreases, which implies that if the inequality (26), and thus (25), is satisfied with F, then it is also satisfied with G, as desired.

First, the fact that F FOSD G and $\hat{\theta}_G \geq \hat{\theta}_F$ yields

$$\sum_{k=1}^{n} G^{k-1}(\hat{\theta}_G) \geq \sum_{k=1}^{n} F^{k-1}(\hat{\theta}_F). \tag{27}$$

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Next, we must have

$$\frac{1 - G(\hat{\theta}_G)}{g(\hat{\theta}_G)} \geq \frac{1 - F(\hat{\theta}_F)}{f(\hat{\theta}_F)}, \tag{28}$$

or else,

$$\hat{\theta}_F - \frac{1 - G(\hat{\theta}_G)}{g(\hat{\theta}_G)} > \hat{\theta}_F - \frac{1 - F(\hat{\theta}_F)}{f(\hat{\theta}_F)} = 0 = \hat{\theta}_G - \frac{1 - G(\hat{\theta}_G)}{g(\hat{\theta}_G)},$$

which yields $\hat{\theta}_F > \hat{\theta}_G$, a contradiction. Now, (27) and (28) together imply that the LHS of (26) becomes (weakly) higher if F is replaced by G, as stated in (a).

Next, (b) follows since

$$\int \max\{J_F(\theta), 0\} dF^n(\theta) \geq \int \max\{J_G(\theta), 0\} dF^n(\theta) \geq \int \max\{J_G(\theta), 0\} dG^n(\theta),$$

where the first inequality follows from the fact that $F \unrhd G$ and the second follows since F FOSD G, so F^n also FOSD G^n, and since $\max\{J_G(\cdot), 0\}$ is nondecreasing. ∎

For the remainder of proofs, we will often use the following transfer rule: Given an allocation rule $q_i(\cdot)$ and a sale price r, if all bidders participate, then for each $i \in N$

$$t_i(\theta) := \frac{1}{n} r \sum_{j \in N} q_j(\theta) + \left(T_i(\theta_i) - \frac{1}{n} r \mathbb{E}_{\tilde{\theta}_{-i}} \left[\sum_{j \in N} q_j(\theta_i, \tilde{\theta}_{-i}) \right] \right)$$

$$- \frac{1}{n-1} \sum_{k \in N \setminus \{i\}} \left(T_k(\theta_k) - \frac{1}{n} r \mathbb{E}_{\tilde{\theta}_{-k}} \left[\sum_{j \in N} q_j(\theta_k, \tilde{\theta}_{-k}) \right] \right) + \rho_i, \tag{29}$$

where

$$T_i(\theta_i) := \theta_i \mathbb{E}_{\tilde{\theta}_{-i}} \left[q_i(\theta_i, \tilde{\theta}_{-i}) \right] - \int_{\underline{\theta}_i}^{\theta_i} \mathbb{E}_{\tilde{\theta}_{-i}} \left[q_i(a, \tilde{\theta}_{-i}) \right] da$$

and $\rho_i \in \mathbb{R}$ with $\sum_{i \in N} \rho_i = 0$. If some bidder does not participate, then the payment of r is equally shared among those who participate while others make no payments. Note from this and (29) that $\sum_{i \in N} t_i(\theta) = r \sum_{i \in N} q_i(\theta)$, $\forall \theta \in \overline{\Theta}$, implying that bidders pay a sale price r as long as at least one bidder participates. Note also that if all bidders participate, then

$$\mathbb{E}_{\tilde{\theta}_{-i}} \left[t_i(\theta_i, \tilde{\theta}_{-i}) \right] = T_i(\theta_i) + c_i, \forall i, \forall \theta_i,$$

for some constant c_i, implying that the incentive compatibility is satisfied as long as the interim allocation probability is nondecreasing. The (IR) constraint will be checked later wherever required.

Proof of Theorem 4. Construct a transfer rule $\hat{t}(\cdot)$ by substituting $q^*(\cdot)$ and r^* into (29). It is straightforward that one can choose ρ_i's to make $\hat{t}(\cdot)$ satisfy (IR) condition. Thus, $\hat{M} = (q^*, \hat{t})$ satisfies (IC^*) and implements the second-best outcome absent collusion.

To prove that $\hat{M}$ is WCP consists of several steps.

STEP 1. *Suppose that a feasible manipulation, $M = (q, t)$, of $\hat{M}$ (by N) satisfies $U_i^M(\underline{\theta}_i) > U_i^{\hat{M}}(\underline{\theta}_i)$ for some $i \in N$. Then, there exists another feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ that satisfies*

$$U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i), \forall i \in N, \quad and \quad \mathbb{E}\left[\sum_{i \in N} U_i^{\tilde{M}}(\theta_i)\right] > \mathbb{E}\left[\sum_{i \in N} U_i^{\hat{M}}(\theta_i)\right]. \tag{30}$$

Proof. Consider a bidder k for whom $U_k^M(\underline{\theta}_k) > U_k^{\hat{M}}(\underline{\theta}_k)$. We construct a mechanism $\tilde{M}$ which satisfies (IC^*), $(IR_N^{\hat{M}})$, and $U_k^{\tilde{M}}(\underline{\theta}_k) = U_k^{\hat{M}}(\underline{\theta}_k)$.

We first construct an 'auxiliary' mechanism M' which will be used to construct $\tilde{M}$. Let us begin by defining $\Theta' \subset \Theta$ as

$$\Theta' := \{\theta \in \Theta | \theta_k \in [K_k^{-1}(r^*), \hat{\theta}_k) \text{ and } \theta_i < \hat{\theta}_i, \forall i \neq k\},$$

which must have a positive measure due to CONDITION (SB^*). The allocation rule is constructed as

$$q'(\theta) = (q_k'(\theta), q_{-k}'(\theta)) := \begin{cases} (1, \mathbf{0}) & \text{if } \theta \in \Theta', \\ (q_k^*(\theta), q_{-k}^*(\theta)) & \text{otherwise.} \end{cases}$$

Clearly, $q'(\cdot)$ results in a nondecreasing interim allocation probability for each bidder. Construct a transfer rule $t_i'(\cdot)$ by substituting $q'(\cdot)$ and r^* into (29) with ρ_i's to be determined later. Then, $M' = (q', t')$ satisfies (IC) since q' satisfies the required monotonicity.[24] Note that M' can be obtained by manipulating $\hat{M}$ in the following way: if $\theta \in \Theta'$, then bidder i report some $\theta_i' > \hat{\theta}_i$ and others report truthfully, and if $\theta \notin \Theta'$, then all bidders report truthfully. Also, we have

$$U_k^{M'}(\hat{\theta}_k) + \sum_{i \in N \setminus \{k\}} U_i^{M'}(\underline{\theta}_i)$$

$$= \mathbb{E}\left[K_k(\theta_k)q_k'(\theta)1_{\{\theta_k \leq \hat{\theta}_k\}}\right] + \mathbb{E}\left[J_k(\theta_k)q_k'(\theta)1_{\{\theta_k \geq \hat{\theta}_k\}}\right] + \mathbb{E}\left[\sum_{i \in N \setminus \{k\}} J_i(\theta_i)q_i'(\theta)\right] - \mathbb{E}\left[\sum_{i \in N} t_i'(\theta)\right]$$

$$= \mathbb{E}\left[K_k(\theta_k)q_k'(\theta)1_{\{\theta_k \leq \hat{\theta}_k\}}\right] + \mathbb{E}\left[\sum_{i \in N} J_i(\theta_i)q_i'(\theta)1_{\{\theta_i \geq \hat{\theta}_i\}}\right] - \mathbb{E}\left[r^* \sum_{i \in N} q_i'(\theta)\right]$$

---

[24]Note that M' need not satisfy (IR) since it is just an auxiliary mechanism used to construct $\tilde{M}$.

$$
\begin{aligned}
&= && \mathbb{E}\left[(K_k(\theta_k) - r^*)q'_k(\theta)1_{\{\theta_k \le \hat{\theta}_k\}}\right] + \mathbb{E}\left[\sum_{i \in N}(J_i(\theta_i) - r^*)q'_i(\theta)1_{\{\theta_i \ge \hat{t}_i\}}\right] \\
&= && \mathbb{E}\left[(K_k(\theta_k) - r^*)1_{\{\theta \in \Theta'\}}\right] + \mathbb{E}\left[\sum_{i \in N}(J_i(\theta_i) - r^*)q^*_i(\theta)\right] \\
&> && \mathbb{E}\left[\sum_{i \in N}(J_i(\theta_i) - r^*)q^*_i(\theta)\right] = \sum_{i \in N} U_i^{\hat{M}}(\underline{\theta}_i) = U_k^{\hat{M}}(\hat{\theta}_k) + \sum_{i \in N\setminus\{k\}} U_i^{\hat{M}}(\underline{\theta}_i),
\end{aligned}
$$

where the inequality follows since $K_k(\theta_k) > r^*$ for $\theta \in \Theta'$. Thus, we can pick $\rho = (\rho_1, \ldots, \rho_n)$ with $\sum_{i \in N} \rho_i = 0$ such that $U_k^{M'}(\hat{\theta}_k) > U_k^{\hat{M}}(\hat{\theta}_k)$, and $U_i^{M'}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i)$ for each $i \ne k$.

For such ρ, we have

$$
\begin{aligned}
U_k^{M'}(\underline{\theta}_k) &= && -\sum_{i \in N\setminus\{k\}} U_i^{M'}(\underline{\theta}_i) + \mathbb{E}\left[\sum_{i \in N}(J(\theta_i) - r^*)q'_i(\theta)\right] \\
&= && -\sum_{i \in N\setminus\{k\}} U_i^{\hat{M}}(\underline{\theta}_i) + \mathbb{E}\left[\sum_{i \in N}(J_i(\theta_i) - r^*)q^*_i(\theta)\right] + \mathbb{E}\left[(J_k(\theta_k) - r^*)1_{\{\theta \in \Theta'\}}\right] \\
&< && -\sum_{i \in N\setminus\{k\}} U_i^{\hat{M}}(\underline{\theta}_i) + \mathbb{E}\left[\sum_{i \in N}(J_i(\theta_i) - r^*)q^*_i(\theta)\right] = U_k^{\hat{M}}(\underline{\theta}_k),
\end{aligned}
$$

where the inequality follows since $J_k(\theta_k) \le 0 < r^*$ for $\theta \in \Theta'$. In sum, under M', the bidders' payoffs satisfy

$$
\begin{aligned}
&U_i^{M'}(\theta_i) = U_i^{\hat{M}}(\theta_i), \ \forall \theta_i, \forall i \ne k \\
&U_k^{M'}(\underline{\theta}_k) < U_k^{\hat{M}}(\underline{\theta}_k) \text{ and } U_k^{M'}(\theta_k) > U_k^{\hat{M}}(\theta_k) \text{ if } \theta_k \ge \hat{\theta}_k.
\end{aligned}
\tag{31}
$$

Finally, we construct $\tilde{M}$ satisfying the desired properties. For dosing so, consider a linear combination of M and M', denoted $M^\lambda := \lambda M + (1-\lambda)M' = (\lambda q + (1-\lambda)q', \lambda t + (1-\lambda)t')$ for $\lambda \in [0.1]$. Note that for any λ, M^λ satisfies (IC) since both M and M' satisfy (IC). Note also that M^λ is a manipulation of $\hat{M}$ since both M and M' are manipulations of $\hat{M}$. Letting $U_i^\lambda(\cdot) := U_i^{M^\lambda}(\cdot)$, (31) implies

$$
U_i^\lambda(\theta_i) = \lambda U_i^M(\theta_i) + (1-\lambda)U_i^{\hat{M}}(\theta_i) \ge U_i^{\hat{M}}(\theta_i), \ \forall \lambda, \forall i \ne k,
\tag{32}
$$

$$
U_k^\lambda(\theta_k) = \lambda U_k^M(\theta_k) + (1-\lambda)U_k^{M'}(\theta_k) > U_k^{\hat{M}}(\theta_k), \ \forall \lambda < 1, \forall \theta_k \ge \hat{\theta}_k,
\tag{33}
$$

$$
U_k^0(\underline{\theta}_k) = U_k^{M'}(\underline{\theta}_k) < U_k^{\hat{M}}(\underline{\theta}_k) \text{ and } U_k^1(\underline{\theta}_k) = U_k^M(\underline{\theta}_k) > U_k^{\hat{M}}(\underline{\theta}_k).
\tag{34}
$$

From (34) and the linearity of $U_k^\lambda(\cdot)$ regarding λ, there exists $\tilde{\lambda} \in (0,1)$ satisfying $U_k^{\tilde{\lambda}}(\underline{\theta}_k) = U_k^{\hat{M}}(\underline{\theta}_k)$, which implies

$$
U_k^{\tilde{\lambda}}(\theta_k) \ge U_k^{\tilde{\lambda}}(\underline{\theta}_k) = U_k^{\hat{M}}(\underline{\theta}_k) = U_k^{\hat{M}}(\theta_k) \text{ for } \theta_k < \hat{\theta}_k.
\tag{35}
$$

Letting $\tilde{M} \equiv M^{\tilde{\lambda}}$, $\tilde{M}$ satisfies $(IR_N^{\hat{M}})$ due to (32), (33), and (35).

If there is any other bidder i for whom $U_i^{\tilde{M}}(\underline{\theta}_i) > U_i^{\hat{M}}(\underline{\theta}_i)$, then we can start from $\tilde{M}$ constructed above and repeat the same procedure as above to construct another $\tilde{M}$ under which $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i)$. To repeat in this fashion will yield $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i)$ for all $i \in N$, establishing the first equation of (30). The second equation follows immediately from $(IR_N^{\hat{M}})$ and (33). ∥

STEP 2. *For any feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ of $\hat{M}$ that satisfies $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i), \forall i \in N$, we have*

$$\int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \leq 0, \forall i \in N. \tag{36}$$

The inequality holds strictly unless $\tilde{Q}_i(\theta_i) = Q_i^*(\theta_i), \forall \theta_i \leq J_i^{-1}(r^*)$.

Proof. It follows from the assumption on $\tilde{M}$ that for all $i \in N$ and all $\theta_i \in \Theta_i$,

$$
\begin{aligned}
X_i(\theta_i) := \int_{\underline{\theta}_i}^{\theta_i} [\tilde{Q}_i(a) - Q_i^*(a)] da &= U_i^{\tilde{M}}(\theta_i) - U_i^{\tilde{M}}(\underline{\theta}_i) - [U_i^{\hat{M}}(\theta_i) - U_i^{\hat{M}}(\underline{\theta}_i)] \\
&= U_i^{\tilde{M}}(\theta_i) - U_i^{\hat{M}}(\theta_i) \geq 0.
\end{aligned}
$$

Then, the integration by parts yields

$$
\begin{aligned}
&\int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \\
&= \left. (J_i(\theta_i) - r^*) f_i(\theta_i) X_i(\theta_i) \right|_{\underline{\theta}_i}^{J_i^{-1}(r^*)} - \int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} X_i(\theta_i) d\left[(J_i(\theta_i) - r^*) f_i(\theta_i) \right] \\
&= -\int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} X_i(\theta_i) d\left[(J_i(\theta_i) - r^*) f_i(\theta_i) \right] \leq 0,
\end{aligned}
$$

since $(J_i(\cdot) - r^*) f_i(\cdot)$ is increasing. The inequality is strict unless $X_i(\theta_i) = 0$ for all $\theta_i \leq J_i^{-1}(r^*)$, that is $\tilde{Q}_i(\theta_i) = Q_i^*(\theta_i)$ for all $\theta_i \leq J_i^{-1}(r^*)$. ∥

To state the next step, we define $\Theta^* := \{\theta \in \Theta | \max_{i \in N} J_i(\theta_i) \geq r^*\}$.

STEP 3. *For any feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ of $\hat{M}$ by N that satisfies $U_i^{\tilde{M}}(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i), \forall i \in N$, we have $\tilde{Q}_i(\theta_i) = \hat{Q}_i^*(\theta_i), \forall i \in N, \forall \theta_i \in \Theta_i$.*

Proof. Consider another allocation rule, $\bar{q}(\cdot)$, with $\bar{q}_i(\theta) = \tilde{q}_i(\theta)$ if $\theta_i \geq J_i^{-1}(r^*)$ and $\bar{q}_i(\theta) = q_i^*(\theta)$ otherwise, and let $\bar{Q}_i(\theta_i) := \mathbb{E}_{\theta_{-i}}[\bar{q}(\theta_i, \theta_{-i})]$, for each $i \in N$. (Whether $\bar{Q}_i(\cdot)$ is

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monotonic or whether $\bar{q}_i(\cdot)$ is implementable is irrelevant for the subsequent argument.) Then, it holds that

$$
\begin{aligned}
& \sum_{i \in N} \int_{J_i^{-1}(r^*)}^{\bar{\theta}_i} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \\
=\ & \sum_{i \in N} \int_{\underline{\theta}_i}^{\bar{\theta}_i} (J_i(\theta_i) - r^*)(\bar{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \\
=\ & \mathbb{E}\left[\sum_{i \in N} (J_i(\theta_i) - r^*)(\bar{q}_i(\theta) - q_i^*(\theta)) \right] \\
=\ & \mathbb{E}_{\theta \in \Theta^*}\left[\sum_{i \in N} (J_i(\theta_i) - r^*)(\bar{q}_i(\theta) - q_i^*(\theta)) \right] \\
=\ & \mathbb{E}_{\theta \in \Theta^*}\left[\sum_{i \in N} (J_i(\theta_i) - r^*)\bar{q}_i(\theta) \right] - \mathbb{E}_{\theta \in \Theta^*}\left[\max_{i \in N} J_i(\theta_i) - r^* \right] \le 0, \qquad (37)
\end{aligned}
$$

where the inequality follows from the definition of $q^*(\cdot)$ and becomes strict unless $\tilde{q}(\theta) = q^*(\theta)$ for almost all $\theta \in \Theta^*$. Thus, we have

$$
\begin{aligned}
0 \ \le\ & \sum_{i \in N}[\tilde{U}_i(\underline{\theta}_i) - \hat{U}_i(\underline{\theta}_i)] - \sum_{i \in N} \int_{J_i^{-1}(r^*)}^{\bar{\theta}_i} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i \\
=\ & \sum_{i \in N} \int_{\underline{\theta}_i}^{J_i^{-1}(r^*)} (J_i(\theta_i) - r^*)(\tilde{Q}_i(\theta_i) - Q_i^*(\theta_i)) f_i(\theta_i) d\theta_i.
\end{aligned}
$$

In order not to contradict Step 2, this inequality and the inequality (36) both must hold as equality, which in turn implies that the inequality (37) also must hold as equality. Then, (36) and (37) can hold as equality only if $\tilde{Q}_i(\theta_i) = Q_i^*(\theta_i), \forall \theta_i \le J_i^{-1}(r^*)$, and $\tilde{q}(\theta) = q^*(\theta)$, for almost all $\theta \in \Theta^*$, which yields the desired result. ‖

STEP 4. $\hat{M}$ is WCP.

Proof. Consider any feasible manipulation $M = (q, t)$. We claim that $U_i^M(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i), \forall i \in N$. Suppose not. By Step 1, we can find another feasible manipulation $\tilde{M} = (\tilde{q}, \tilde{t})$ satisfying (30). Then, by Step 3,

$$
U_i^{\tilde{M}}(\theta_i) = U_i^{\tilde{M}}(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \tilde{Q}_i(a) da = U_i^{\hat{M}}(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} Q_i^*(a) da = U_i^{\hat{M}}(\theta_i),
$$

which contradicts the inequality in (30). Thus, it must be that $U_i^M(\underline{\theta}_i) = U_i^{\hat{M}}(\underline{\theta}_i), \forall i \in N$. Applying Step 3 again, we have $\tilde{Q}_i(\cdot) = Q_i^*(\cdot)$ for all $i \in N$, implying that M yields the same interim payoffs as $\hat{M}$ to the bidders, which means that $\hat{M}$ is WCP. ‖ ∎

Proof of Theorem 5. Recall first that the original auction M^* is the second-price auction. Since the auction A has the same allocation/payment rule as M^* for bidders outside C, it is weakly dominant for them to participate in A and report their true types. Fix any Bayesian Nash equilibrium in $\mathcal{E}_A$. Letting $\tilde{M} = (\tilde{q}, \tilde{t})$ denote the mechanism resulting from the equilibrium play of bidders, we show that $\tilde{M}$ must yield V^* to the seller. First of all, $\tilde{M}$ must satisfy (IC^*) since in the equilibrium, each bidder is forming a correct belief about what types propose or accept/reject a collusive side contract, and thereafter playing sequentially rational strategy.[25] We now establish that both $\hat{M}$ and $\tilde{M}$ must yield the same interim payoffs for all collusive bidders. Let us first consider case (a). Then, since (7) holds strictly for all θ',

$$T_i^*(\bar{\theta}) > \mathbb{E}\left[H_C(\theta_{N\setminus C}^{(1)})\right]. \tag{38}$$

Define

$$\breve{\Theta}_C := \{\theta_C \in \Theta_C \mid \exists i \in C \text{ with type } \theta_i \text{ that announces } r_z\}.[26]$$

Then, for all $\theta_C \in \Theta_C$,

$$
\begin{aligned}
\mathbb{E}\left[\sum_{i\in C} \tilde{t}_i(\theta)\right] &= \mathbb{E}\left[H_C(\theta_{N\setminus C}^{(1)})\left(\sum_{i\in C}\tilde{q}_i(\theta)\right)1_{\{\theta_C\notin\breve{\Theta}_C\}} + T_i^*(\bar{\theta})1_{\{\theta_C\in\breve{\Theta}_C\}}\right] \\
&\geq \mathbb{E}\left[H_C(\theta_{N\setminus C}^{(1)})\left(\sum_{i\in C}\tilde{q}_i(\theta)\right)1_{\{\theta_C\notin\breve{\Theta}_C\}} + H_C(\theta_{N\setminus C}^{(1)})1_{\{\theta_C\in\breve{\Theta}_C\}}\right] \\
&\geq \mathbb{E}\left[H_C(\theta_{N\setminus C}^{(1)})\sum_{i\in C}\tilde{q}_i(\theta)\right], \tag{39}
\end{aligned}
$$

where the first inequality follows from (38). Thus,

$$
\begin{aligned}
\alpha_C(\sum_{i\in C} U_i^{\tilde{M}}(\bar{\theta})) + (1-\alpha_C)(\sum_{i\in C} U_i^{\tilde{M}}(\underline{\theta})) &= \mathbb{E}\left[\sum_{i\in C} H_C(\theta_i)\tilde{q}_i(\theta) - \sum_{i\in C}\tilde{t}_i(\theta)\right] \\
&\leq \mathbb{E}\left[\sum_{i\in C}[H_C(\theta_i) - H_C(\theta_{N\setminus C}^{(1)})]\tilde{q}_i(\theta)\right] \\
&\leq \mathbb{E}\left[\sum_{i\in C}[H_C(\theta_i) - H_C(\theta_{N\setminus C}^{(1)})]q_i^*(\theta)\right] \\
&= \alpha_C(\sum_{i\in C} U_i^{\hat{M}}(\bar{\theta})) + (1-\alpha_C)(\sum_{i\in C} U_i^{\hat{M}}(\underline{\theta})),
\end{aligned}
$$

---

[25]Since $\theta_\emptyset \in B_i$ for each $i \in C$, every type of bidder i can secure at least zero (or individual rational) payoff whenever participating in A so that $\tilde{M}$ must satisfy (IR) condition.

[26]A mixed strategy which randomizes between r_z and some other messages can be accommodated without changing the subsequent result.

where the first and last equalities follow from Lemma 0, the first inequality follows from (39), and the second follows from the definition of $q^*(\cdot)$. Indeed, both inequalities must hold as equality since (i) $U_i^{\tilde{M}}(\underline{\theta}) \geq 0 = U_i^{\hat{M}}(\underline{\theta})$ and (ii) $U_i^{\tilde{M}}(\bar{\theta}) \geq U_i^{\hat{M}}(\bar{\theta})$ for all $i \in C$. First, (i) is immediate from the fact that $\tilde{M}$ satisfies (IC^*). To show (ii), suppose to the contrary that $U_i^{\tilde{M}}(\bar{\theta}) < U_i^{\hat{M}}(\bar{\theta})$ for some $i \in C$. Then, bidder i has a profitable deviation to announce r_z with a sufficiently high z, since it will yield him an (interim) payoff arbitrarily close to $\bar{\theta} - T_i^*(\bar{\theta}) = U_i^{\hat{M}}(\bar{\theta})$, a contradiction. Now, both inequalities hold with equalities only if $\breve{\Theta}_C$ is a measure zero set and $\tilde{q}_i(\cdot) = q_i^*(\cdot)$, which implies that the interim payoffs in $\hat{M}$ and $\tilde{M}$ can only differ by constants. That (i) and (ii) hold with equalities in turn implies that those constants have to be zero. Consequently, $\hat{M}$ and $\tilde{M}$ must yield the same interim payoffs for all parties, which implies that $\tilde{M}$ must yields the seller her second-best payoff V^*.

The proof is similar for the case (b), upon two observations. First, adding the message r_z does not give the cartel any new opportunity to manipulate $\hat{M}$ since announcing r_z results in the same outcome as each collusive bidder announcing θ^r. Second, since (8) holds, the highest type of any bidder $i \in C$ can announce r_z (with sufficiently large z) to obtain at least its noncollusive payoff $U_i^{\hat{M}}(\bar{\theta})$.

Last, we prove for the case (a) that $\mathcal{E}_A$ is non-empty. (A similar proof follows for the case (b).) To this end, we show that there exists a weak perfect Bayesian, and thus Bayesian Nash, equilibrium in which each cartel member proposes no side contract. If no one proposes a side contract, then each collusive bidder i with type θ_i plays $\hat{M}$ and obtains his equilibrium payoff $U_i^{\hat{M}}(\theta_i)$. If a side contract is proposed, then each collusive bidder i responds as follows: "Report $\theta_\emptyset$ if $\theta_i < T_i^*(\bar{\theta})$ or else report r_{z_i} for some integer $z_i > 1$." This response is supported by the out-of-equilibrium belief of bidder i that each bidder $j \neq i$ in C reports $r_{z'}$ for some $z' < z_i$ if $\theta_j > T_j^*(\bar{\theta})$, and $\theta_\emptyset$ otherwise.

We now show that this strategy profile constitutes a weak perfect Bayesian equilibrium. First of all, a deviation by some collusive bidder or third party to a side contract will trigger the response as above and yield each collusive bidder i with θ_i at most $\max\{\theta_i - T_i^*(\bar{\theta}), 0\}$, which is no greater than $U_i^{\hat{M}}(\theta_i)$, his equilibrium payoff. Second, once a side contract has been proposed (out of equilibrium), it is optimal for a collusive bidder i with type $\theta_i > T_i^*(\bar{\theta})$ to report r_{z_i} and obtain $\theta_i - T_i^*(\bar{\theta}) > 0$, given his belief that every other collusive bidder will report either $r_{z'}$ or $\theta_\emptyset$. Also, bidder i with $\theta_i < T_i^*(\bar{\theta})$ optimally reports $\theta_\emptyset$ to obtain zero payoff since (i) reporting some r_z instead is clearly suboptimal and (ii) reporting some type from Θ_i yields either zero payoff (in case some other collusive bidder reports r_z) or at most $\theta_i - \delta_C(\bar{\theta}) < 0$ (in case every other collusive bidder reports $\theta_\emptyset$), and thus is suboptimal too. ∎

Proof of Proposition 3. Suppose that $|C| = n$ or cartel is all-inclusive. Then, by setting

$\theta' = \bar{\theta}$, (7) becomes

$$T_i^*(\bar{\theta}) \geq H_N(\hat{\theta}). \tag{40}$$

To see that (40) always holds, observe

$$T_i^*(\bar{\theta}) > \mathbb{E}\left[\frac{T_i^*(\theta_i)}{Q_i^*(\theta_i)}\frac{Q_i^*(\theta_i)}{\mathbb{E}\left[Q_i^*(\theta_i)\right]}\right] = \frac{\mathbb{E}\left[T_i^*(\theta_i)\right]}{\mathbb{E}\left[Q_i^*(\theta_i)\right]} = \frac{\mathbb{E}\left[J(\theta_i)Q_i^*(\theta_i)\right]}{\mathbb{E}\left[Q_i^*(\theta_i)\right]} = \frac{\mathbb{E}\left[J(\theta_i)q_i^*(\theta)\right]}{\mathbb{E}\left[q_i^*(\theta)\right]},$$

where the inequality holds since $\frac{T_i^*(\cdot)}{Q_i^*(\cdot)}$ is increasing,[27] Since the bidders are symmetric, the last expression is the same for all i, so

$$\frac{\mathbb{E}\left[J(\theta_i)q_i^*(\theta)\right]}{\mathbb{E}\left[q_i^*(\theta)\right]} = \frac{\mathbb{E}\left[\sum_{j\in N}J(\theta_j)q_j^*(\theta)\right]}{\mathbb{E}\left[\sum_{j\in N}q_j^*(\theta)\right]} = H_N(\hat{\theta}),$$

where the last equality follows from (3). We thus conclude that (40) holds always, proving the first statement of the proposition.

Now consider $|C| < n$ and fix $k = n - |C|$. Note first that rewriting (3) yields

$$\int_{\hat{\theta}}^{\bar{\theta}}\left(\int_{\underline{\theta}}^{\theta} H_C(\max\{\hat{\theta}, s\})dF^k(s)\right)dF^{|C|}(\theta)$$

$$= \mathbb{E}\left[H_C(\theta_{N\backslash C}^{(1)})\mathbb{1}_{\{\theta_C^{(1)}>\theta_{N\backslash C}^{(1)}\}}\right] = \mathbb{E}\left[J(\theta_C^{(1)})\mathbb{1}_{\{\theta_C^{(1)}>\theta_{N\backslash C}^{(1)}\}}\right] \leq \int_{\hat{\theta}}^{\bar{\theta}}\bar{\theta}F^k(\theta)dF^{|C|}(\theta), \tag{41}$$

where the inequality holds since $J(\theta) \leq \bar{\theta}, \forall\theta$. Let $\bar{\alpha} := \sup\{\alpha_C \in [0,1]|\,|C| = 2, 3\cdots\}$ and $\bar{H}(\cdot) := \bar{\alpha}K(\cdot) + (1-\bar{\alpha})J(\cdot)$. Then, it is possible to find some $\theta' \in [\hat{\theta}, \bar{\theta}]$ and $\epsilon > 0$ such that

$$\int_{\underline{\theta}}^{\theta'}\bar{H}(\max\{\hat{\theta}, s\})dF^k(s) < \bar{\theta}F^k(\theta') - \epsilon. \tag{42}$$

Or else, we must have

$$\bar{H}(\hat{\theta})F^k(\hat{\theta}) = \int_{\underline{\theta}}^{\hat{\theta}}\bar{H}(\max\{\hat{\theta}, s\})dF^k(s) \geq \bar{\theta}F^k(\hat{\theta}),$$

so $\bar{H}(\hat{\theta}) \geq \bar{\theta}$. Then, since $\bar{H}(\cdot)$ is strictly increasing,

$$\int_{\underline{\theta}}^{\theta}\bar{H}(\max\{\hat{\theta}, s\})dF^k(s) > \bar{\theta}F^k(\theta), \forall\theta > \hat{\theta}$$

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[27]Note that $\frac{T_i^*(\cdot)}{Q_i^*(\cdot)}$ corresponds to the symmetric equilibrium bidding function in the first-price auction and thus is increasing.

and thus

$$\int_{\hat{\theta}}^{\bar{\theta}} \left(\int_{\underline{\theta}}^{\theta} \bar{H}(\max\{\hat{\theta}, s\}) dF^k(s) \right) dF^{|C|}(\theta) > \int_{\hat{\theta}}^{\bar{\theta}} \bar{\theta} F^k(\theta) dF^{|C|}(\theta),$$

which implies that (41) is violated for some C with α_C sufficiently close to $\bar{\alpha}$, a contradiction.

Given (42) and $\bar{H}(\cdot) \geq H_C(\cdot)$ for all C,

$$\int_{\underline{\theta}}^{\theta'} H_C(\max\{\hat{\theta}, s\}) dF^k(s) < \bar{\theta} F^k(\theta') - \epsilon,$$

or

$$\int_{\underline{\theta}}^{\theta'} \left[\bar{\theta} - H_C(\max\{\hat{\theta}, s\}) \right] dF^k(s) > \epsilon. \tag{43}$$

Observe that the LHS of (43) coincides with the RHS of (7). Meanwhile, the LHS of (7) can be written as

$$\int_{\hat{\theta}}^{\bar{\theta}} F^{n-1}(s) ds = \int_{\hat{\theta}}^{\bar{\theta}} F^{k+|C|-1}(s) ds,$$

which converges to zero as $|C| \to \infty$. The second statement of the proposition then follows. ∎

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