Online Appendix to "Large Sample Mean-Field Stochastic Optimization"

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1 Technical Results for a Multiplicative Noise Scale Function

We provide the proofs for the case of a multiplicative noise scale function $\sigma(t,\theta,x):[0,T]\times\Theta\times\mathbb{R}^d\to\mathbb{R}_+$. For $i=1,\ldots,N$, the controlled state process $X^{\theta,i}=(X_t^{\theta,i})_{t\in[0,T]}$ with multiplicative noise scale function $\sigma(t,\theta,x)$ satisfies the differential equation

$$dX^{\theta,i}(t)=f\left(t,\theta(t),X^{\theta,i}(t),\frac{1}{N}\sum_{j=1}^N\rho(X^{\theta,j}(t))\right)dt+\varepsilon^i\sigma\left(t,\theta(t),X^{\theta,i}(t)\right)dW^i(t). \tag{1}$$

Obviously, if the multiplicative noise scale function $\sigma(t,\theta,x)\equiv 1$, the above dynamics reduces to the forward dynamics with additive noise given in Equation (1) of the main manuscript.

To study the case with multiplicative noise injection, we need to impose mild assumptions on the multiplicative noise scale function $\sigma(t,\theta,x)$:

($\mathbf{A}_\sigma$) (i) $\sigma(t,\theta,x)$ is continuous in (t,θ,x) and is twice continuously differentiable in x;

(ii) σ satisfies the following Lipschitz condition: for all $\theta_1,\theta_2\in\Theta$ and $x,y\in\mathbb{R}^d$,

$$|\sigma(t,\theta_1,x)-\sigma(t,\theta_2,y)|\leq L(|\theta_1-\theta_2|+|x-y|),$$

where $L>0$ is independent of t and (θ_1,θ_2,x,y).

Then, the condition **($\mathbf{A}_\sigma$)** together with the Lipschitz conditions imposed in (iii)-(iv) of Assumption **($\mathbf{A}_{\varepsilon,f,\rho}$)** in the main manuscript guarantees the existence of a unique strong solution for the stochastic system (1) with multiplicative noise scale function.

The first task is to study the existence of an optimal solution to the relaxed control problem with finite sample size, i.e.,

$$\alpha_N:=\inf_{Q\in\mathcal{Q}(\nu)}J_N(Q), \tag{2}$$

where we recall that, for any $Q\in\mathcal{Q}(\nu)$ (the set of all relaxed controls, formally specified in **Def. 3.1** of the main manuscript), the objective functional with finite sample size is

$$J_N(Q):=\mathbb{E}^Q\left[L_N(\tilde{X}^\theta(T),\tilde{Y}(0))+\int_0^T R_N(\theta(t),\theta'(t);\tilde{X}^\theta(t),\tilde{Y}(0))dt\right]. \tag{3}$$

However, the controlled state process $(X^{\theta,i}(t))_{t\in[0,T]}$ is the strong solution to Eq. (1) driven by (ζ,W,θ). Then, we still have the following result (which is the equivalent of Proposition 3.1 in the main manuscript where the noise function is additive):

Proposition 1.1. *Let* **($\mathbf{A}_{\varepsilon,f,\rho}$)**, **($\mathbf{A}_\Theta$)** *and* **($\mathbf{A}_\sigma$)** *hold. Then, there exists an optimal solution to the relaxed control problem* (2) *in the finite sample case.*

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Proof. The proof argument is very similar to that in the proof of Proposition 3.1 where noise is additive. The main difference in the multiplicative injection noise case is that one needs to prove the following convergence result: for all $t \in [0, T]$,

$$\lim_{k\to+\infty} \mathbb{E}\left[\sup_{s\in[0,t]} \left|\tilde{X}_k^*(s) - \tilde{X}^*(s)\right|_N^2\right] = 0, \tag{4}$$

where, for $k \geq 1$ and $i \in \mathbb{N}$, $X_k^{*,i} = (X_k^{*,i}(t))_{t\in[0,T]}$ is the strong solution of SDE (1) driven by $(\zeta_k^*, W_k^*, \theta_k^*)$. In other words, under $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, we have $(X_k^{*,i}(0), Y_k^{*,i}(0)) = \zeta_k^{*,i}$, and for $t \in (0, T]$,

$$dX_k^{*,i}(t) = f\left(t, \theta_k^*(t), X_k^{*,i}(t), \frac{1}{N}\sum_{j=1}^N \rho(X_k^{*,j}(t))\right) dt + \varepsilon^i \sigma(t, \theta_k^*(t), X_k^{*,i}(t)) dW_k^{*,i}(t). \tag{5}$$

For $i \in \mathbb{N}$, $X^{*,i} = (X^{*,i}(t))_{t\in[0,T]}$ is the strong solution of SDE (1) driven by (ζ^*, W^*, θ^*). That is, under $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, we have $(X^{*,i}(0), Y^{*,i}(0)) = \zeta^{*,i}$, and for $t \in (0, T]$,

$$dX^{*,i}(t) = f\left(t, \theta^*(t), X^{*,i}(t), \frac{1}{N}\sum_{j=1}^N \rho(X^{*,j}(t))\right) dt + \varepsilon^i \sigma(t, \theta^*(t), X^{*,i}(t)) dW^{*,i}(t). \tag{6}$$

Then, it follows from (5) and (6) that

$$\begin{aligned}\sup_{s\in[0,t]} \left|X_k^{*,i}(s) - X^{*,i}(s)\right|^2 \leq C_T \Bigg\{ &\left|X_k^{*,i}(0) - X^{*,i}(0)\right|^2 + \int_0^t \left|\theta_k^*(s) - \theta^*(s)\right|^2 ds \\ &+ \int_0^t \left|X_k^{*,i}(s) - X^{*,i}(s)\right|^2 ds + \int_0^t \left|\frac{1}{N}\sum_{j=1}^N (\rho(X_k^{*,j}(s)) - \rho(X^{*,j}(s)))\right|^2 ds \\ &+ \sup_{s\in[0,t]} \left|\int_0^s \Delta\sigma_{ik}(s) dW^{*,i}(s)\right|^2 + \sup_{s\in[0,t]} \left|\int_0^s \sigma_{ik}(s) d\tilde{W}_k^{*,i}(s)\right|^2 \Bigg\},\end{aligned}$$

where, for $t \in [0, T]$, we define

$$\begin{aligned}\sigma_{ik}(t) &:= \varepsilon^i \sigma(t, \theta_k^{*,i}(t), X_k^{*,i}(t)), \quad \tilde{W}_k^{*,i}(t) := W_k^{*,i}(t) - W^{*,i}(t), \\ \Delta\sigma_{ik}(t) &:= \varepsilon^i \sigma(t, \theta_k^{*,i}(t), X_k^{*,i}(t)) - \varepsilon^i \sigma(t, \theta^{*,i}(t), X^{*,i}(t)).\end{aligned} \tag{7}$$

In light of assumptions $(\mathbf{A}_\sigma)$, $(\mathbf{A}_{\varepsilon,f,\rho})$ and $(\mathbf{A}_\Theta)$, we obtain that, for some constant $C_{T,N} > 0$,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} \left|\tilde{X}_k^*(s) - \tilde{X}^*(s)\right|_N^2\right] \leq e^{C_{T,N}} \mathbb{E}\Bigg[&\left|\tilde{X}_k^*(0) - \tilde{X}^*(0)\right|_N^2 + \int_0^t \left|\theta_k^*(s) - \theta^*(s)\right|^2 ds \\ &+ \frac{1}{N}\sum_{i=1}^N \sup_{s\in[0,t]} \left|\int_0^s \sigma_{ik}(s) d\tilde{W}_k^{*,i}(s)\right|^2\Bigg].\end{aligned} \tag{8}$$

We next show that

$$\lim_{k\to+\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\sup_{s\in[0,t]} \left|\int_0^s \sigma_{ik}(s) d\tilde{W}_k^{*,i}(s)\right|^2\right] = 0. \tag{9}$$

In view of Burkholder-Gundy-Davis inequality, we deduce that, for all $t \in [0, T]$,

$$\mathbb{E}\left[\sup_{s\in[0,t]} \left|\int_0^s \sigma_{ik}(s) d\tilde{W}_k^{*,i}(s)\right|^2\right] \leq C_T \mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 d\langle \tilde{W}_k^{*,i}\rangle_s\right],$$

for some $C_T > 0$. In the sequel, for an arbitrary constant $K > 0$, we decompose the expectation on the RHS of the above inequality into two parts:

$$\mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 d\langle \tilde{W}_k^{*,i}\rangle_s\right] = \mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|<K\}} d\langle \tilde{W}_k^{*,i}\rangle_s\right] + \mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|\geq K\}} d\langle \tilde{W}_k^{*,i}\rangle_s\right].$$

First of all, in view of (7), we have that

$$\langle \tilde{W}_k^{*,i}\rangle_s = \langle W_k^{*,i} - W^{*,i}\rangle_s \leq 2\langle W_k^{*,i}\rangle_s + 2\langle W^{*,i}\rangle_s = 4s, \quad \forall s \in [0,T].$$

Then, it holds that, for all $t \in [0,T]$,

$$\mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|\geq K\}} d\langle \tilde{W}_k^{*,i}\rangle_s\right] \leq 4\mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|\geq K\}} ds\right].$$

Note that the assumption $(\mathbf{A}_\sigma)$ implies that $\sigma(t,\cdot)$ has linear growth uniformly in $t \in [0,T]$. Then, we applying the standard moment estimates for SDEs, together with assumptions $(\mathbf{A}_\sigma)$, $(\mathbf{A}_{\varepsilon,f,\rho})$ and $(\mathbf{A}_\Theta)$, we can easily show that, for all $i = 1,\dots,N$,

$$\sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T |\sigma_{ik}(s)|^4 ds\right] < +\infty.$$

This shows that

$$\lim_{K\to+\infty} \sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|\geq K\}} ds\right] = 0.$$

Therefore, for any $\delta > 0$, there exists a constant $K > 0$ such that

$$\sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|\geq K\}} ds\right] < \delta.$$

Thus, for any $\delta > 0$, there exists a constant $K > 0$ independent of $k \in \mathbb{N}$ such that

$$\mathbb{E}\left[\int_0^T |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|\geq K\}} ds\right] < \delta, \quad \forall k \in \mathbb{N}.$$

For the above K, it holds that, for all $t \in [0,T]$,

$$\mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 d\langle \tilde{W}_k^{*,i}\rangle_s\right] = 4\mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|<K\}} d\langle \tilde{W}_k^{*,i}\rangle_s\right] + 4\mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|\geq K\}} d\langle \tilde{W}_k^{*,i}\rangle_s\right].$$

Note that, it holds that, for all $t \in [0,T]$,

$$\mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|<K\}} d\langle \tilde{W}_k^{*,i}\rangle_s\right] \leq K^2 \mathbb{E}\left[\langle \tilde{W}_k^{*,i}\rangle_t\right],$$

$$\sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T |\sigma_{ik}(s)|^2 \mathbf{1}_{\{|\sigma_{ik}(s)|\geq K\}} ds\right] < \delta.$$

This yields that, for all $t \in [0,T]$,

$$\mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 d\langle \tilde{W}_k^{*,i}\rangle_s\right] \leq 4K^2 \mathbb{E}\left[\langle \tilde{W}_k^{*,i}\rangle_t\right] + 4\delta.$$

Letting k go to infinity, we arrive at

$$\lim_{k\to+\infty} \mathbb{E}\left[\int_0^t |\sigma_{ik}(s)|^2 d\langle \tilde{W}_k^{*,i}\rangle_s\right] \leq 4K^2 \lim_{k\to+\infty} \mathbb{E}\left[\langle \tilde{W}_k^{*,i}\rangle_t\right] + 4\delta = 4\delta, \quad \forall t \in [0,T].$$

In the above inequality, we have used the following fact:

$$\lim_{k\to+\infty}\mathbb{E}\left[\langle\tilde{W}_k^{*,i}\rangle_t\right]=\lim_{k\to+\infty}\mathbb{E}\left[\left|W_k^{*,i}(t)-W^{*,i}(t)\right|^2\right]=0.$$

because $W_k^*\to W^*$ in $\mathcal{C}_p^{\mathbb{N}}$ as $k\to\infty$. This verifies the validity of (9). Then, the convergence (4) follows from using the identity (9) into (8). □

Next, we need to prove the equivalent of Theorem 4.1 in the main manuscript, but with multiplicative noise scale function. We next state the theorem:

Theorem 1.2. *Let* $(\mathbf{A}_{\varepsilon,f,\rho})$*,* $(\mathbf{A}_{\Theta})$ *and* $(\mathbf{A}_{\sigma})$ *hold. Suppose further that, for some* $\vartheta_0\in\mathcal{P}(\mathcal{P}_2(E)\times\mathcal{C}_m)$*,*

$$Q_N\circ(\mu^N(0),\theta_N)^{-1}\Rightarrow\vartheta_0,\quad N\to\infty. \tag{10}$$

Then $(\mathbb{Q}^N)_{N=1}^{\infty}$ *defined by Equation (15) in the main manuscript converges in* $\mathcal{P}_2(\mathcal{P}_2(E)\times\mathcal{C}_m\times\hat{S})$*. Moreover, if the law of a* $\mathcal{P}_2(E)\times\mathcal{C}_m\times\hat{S}$*-valued r.v.* $(\hat{\mu}_0,\hat{\theta},\hat{\mu})$ *defined on some probability space* $(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}})$ *is a limit point of* $(\mathbb{Q}^N)_{N=1}^{\infty}$*, then,* $\hat{\mathbb{P}}$*-a.s.,* $\hat{\mu}$ *is the unique solution to the following FPK equation in a random environment:* $\hat{\mu}(0)=\hat{\mu}_0$*, and for* $t\in(0,T]$*,*

$$\langle\hat{\mu}(t),\varphi(t)\rangle-\langle\hat{\mu}(0),\varphi(0)\rangle-\int_0^t\left\langle\hat{\mu}(s),\mathcal{A}^{\hat{\theta},\langle\hat{\mu}(s),\rho\rangle}\varphi(s)\right\rangle ds=0,\quad\forall\ \varphi\in\mathcal{D}. \tag{11}$$

Above, the operator $\mathcal{A}^{\theta,\eta}$ *in the multiplicative noise case becomes*

$$\mathcal{A}^{\theta,\eta}\varphi(s,e):=\nabla_s\varphi(s,e)+f(s,\theta(s),x,\eta)^{\top}\nabla_x\varphi(s,e)+\frac{1}{2}\mathrm{tr}\left[\varepsilon\varepsilon^{\top}\sigma^2(t,\theta(s),x)\nabla_{xx}^2\varphi(s,e)\right],\quad\varphi\in\mathcal{D}. \tag{12}$$

Even if the proof of the above Theorem 1.2 is very similar to that of Theorem 4.1 in the main manuscript (dealing with additive noise scale function), the main difference in the multiplicative case is to prove the well-posedness of FPK equation (11).

The well-posedness of FPK equation (11) *in the multiplicative noise case.* Compared with the proof of the additive noise case, we only need to verify the uniqueness of solutions to our FPK equation (11). We observe that Theorem 4.4 of [1] can also be applied to show uniqueness. This implies that we need to verify the conditions (DH1)-(DH4) imposed in Theorem 4.4 of [1] for the multiplicative noise case. The validity of (DH1) follows immediately from Assumption $(\mathbf{A}_{\sigma})$. Note that this is only related to the drift term in the FPK equation, and hence this condition can be verified in the same fashion as in the additive noise case. We next define

$$A(t,e):=\frac{1}{2}\left(\begin{array}{c}0\\ \tilde{\sigma}(t,e)\end{array}\right)(0,\tilde{\sigma}(t,e)^{\top}), \tag{13}$$

with $\tilde{\sigma}(t,e):=\varepsilon\sigma(t,\theta(t),x)$ for $t\in[0,T]$ and $e=(\varepsilon,y,x)\in E$. Then, it follows from Assumption $(\mathbf{A}_{\sigma})$ that, for any $\mu\in\mathcal{P}_2(E)$, there exists a constant $C_\mu>0$ which is independent of $(t,e)\in[0,T]\times E$ such that

$$\frac{\left|b_\omega(t,e,\mu)\right|^2}{1+|e|^2}+\frac{\left|A(t,e)\right|^2}{1+|e|^2}+\left|\varepsilon\nabla_x\sigma(t,\theta,x)\right|\leq C_\mu.$$

This yields the validity of conditions (DH2) and (DH4) in [1]. □

Because the multiplicative noise scale function satisfies Assumption $(\mathbf{A}_{\sigma})$, the remaining results in Section 4 of the main manuscript can be shown to hold in the case of a multiplicative noise function using arguments similar to those in the additive noise case. Moreover, using results from Section 4 of the main manuscript, one can establish the key result in Section 5 of the main manuscript–Theorem 5.1 for such multiplicative noise function.

References

[1] Manita, O.A., M.S. Romanov, and S.V. Shaposhnikov (2015): On uniqueness of solutions to nonlinear Fokker-Planck-Kolmogorov equations. *Nonlinear Anal.* 128, 199-226.