YIANNIS N. MOSCHOVAKIS, *Notes on Set Theory*, New York: Springer-Verlag, 1994. xiv + 272 pp. $39.00, ISBN 0-387-94180-0.

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This is an excellent introduction to axiomatic set theory, viewed both as a foundation of mathematics and as a branch of mathematics with its own subject matter, basic results, open problems. Written by one of the leading contributors to the field, it covers all the basic material, but with some novelties that make it interesting also to the advanced reader.

The book begins with a couple of chapters devoted to the first, basic results of Cantorian set theory, including detailed accounts of Cantor's diagonal arguments and of the Schroöder-Bernstein Theorem. While the style of presentation might strike some readers as rather demanding for a start (in spite of the useful figures accompanying the proofs), by the end of chapter 2 the effort is rewarded and the stage is set for appreciating the need for an axiomatic framework. This is introduced in Chapter 3, which follows rather faithfully Zermelo's 1908 pragmatic approach *vis-a-vis* Russell's paradox. The author also follows Zermelo in allowing for *Urelemente*, which is motivated mostly by a desire to remain neutral with regard to the opposition between "the small view" (that the set theoretic universe consists exclusively of those objects whose existence is guaranteed by the axioms) and "the large view" (that the universe is closed under the axioms, but may contain all sorts of animals to begin with). This neutrality represents a philosophical constant in the book. Thus, while the author takes very seriously the idea that every mathematical object can be construed as (i.e., faithfully represented by) a set, he is also careful in emphasizing the modal dimension of this idea. For instance, relations, functions, and the natural numbers can be represented by sets, but need not be.

The next two chapters introduce the basic material about the natural numbers (including induction and recursion) and cardinal arithmetic. By the end of Chapter 5, one can see how Zermelo's system would allow one to go a long way in the reconstruction of elementary discrete mathematics, the rational numbers, the reals. The emphasis, however, is not on the details of this development (which is left for the most part to the Problems or, as in the case of the real numbers, postponed to a rich Appendix.) The emphasis is on the fact that all of this is attainable within the framework of Zermelo's basic axioms. Indeed this is one of the main novelties of the book. Since Fraenkel's Axiom of Replacement is not introduced until the end, neither is von Neumann's theory of ordinals and cardinals. So $\mathbb{N}$ is not introduced as the set of the first ω ordinals, but as a structured set (unique up to isomorphism) satisfying Peano's postulates. This approach has some cost, for certain developments would be much simpler (and in some places also more natural) if the ordinals were brought in earlier. Yet the exercise is worthwile, for it shows how far one can actually go without Replacement. (It is a common claim that one can go as far as to develop 99% of modern mathematics, modulo the availability of some Choice principle, but one hardly gets a clear picture of what that really means unless one tries. Moreover, one learns that all the basic identities of cardinal arithmetic can be established as equinumerosities, which is something a beginner might easily overlook. The only danger is that if the book is used as a text, one may not be able to get far enough to see the ordinals at all.)

Beginning with Chapter 6, the material gets harder and the presentation more compact. Chapter 6 is itself a novelty for an introductory book, as it is centered around ideas and results in the context of the theory of partial orderings and partial functions. This leads to an elegant generalization of the recursion theorem which takes the form of a continuous fixed point theorem. Apart from its intrinsic interest, this also represents an interesting point of contact between classical set theory and contemporary theoretical computer science, as the author rightly emphasizes. (One could have taken this opportunity to mention other applications as well, e.g. partial semantics and revision theories.) The fixed point theorem in question is of course a special case of Zermelo's theorem, which is intimately related to the theory of wellorderings. This forms the topic of Chapter 7, which also deals with Transfinite Induction and Recursion. Coupled with Hartogs' Theorem, this provides the necessary machinery to apply recursive definitions and proofs by induction to situations far removed from the natural numbers.

Chapters 8 and 9 focus on the Axiom of Choice. There is a good blend of technical results and philosophical remarks in these chapters, although the starting point is somewhat misleading. The author argues that, naively understood, AC is obvious in view of the Powerset Axiom: "When we grant sethood to the class $\{X \mid X \quad A\}$ of all subsets of A, we truly mean *all* subsets of A, including those for which the membership criterion is not determined by some explicit law but by free choice, by chance if you will" [120]. But of course the Powerset Axiom has nothing to do with this. (A) consists of all subsets of A, but whether these include *inter alia* the range of a choice function for A is a totally independent question. (One should rather put it thus: if there were no choice set for A, it would seem that the notion of a *subset* includes some implicit restrictions which are not in the spirit of Zermelo's "limitation of size" approach. For, after all, a choice set must be a *sub*set.) Apart from this, these two chapters contain a lot of interesting data. Particualry valuable is the inclusion of material concerning weaker versions of AC, most notably the Axiom of Dependent Choices, which is enough to justify all classical mathematics. Moreover, on the technical side, there are some elegant applications of the material on fixed points introduced earlier. (See for instance the nice proofs of the various equivalents to AC.) This means that Chapter 6 is not skippable even if you are not interested in its relevance to computer science; however the effort is quite worthwhile.

Chapter 10 on Baire space, on the other hand, is in principle skippable without loss of continuity. It covers some results in the theory of pointsets which are of special interest to analysts. Their relevance to the Continuum Problem is also emphasized, though, so the non-analyst may use that as a reading key for this chapter.

Finally, Chapters 11 and 12 introduce the Axiom of Replacement and eventually the von Neumann ordinals and cardinals. This is all pretty standard, as is the brief illustration of von Neumann's picture of the set-theoretic universe (the cumulative hierachy of pure, well-founded sets). Still, it is nice to see at this point the further extension of the recursion principle allowed by Replacement. Indeed this gradual strengthening of the power of recursive definitions constitutes a valuable *fil rouge* through the book. By contrast, it would have been nice to see here more indications of the deductive strength of Replacement with respect to the overall axiomatic system. (The derivability of the Separation Axiom is only brought up in an exercise, and that of Pairing is not even mentioned.)

Throughout the book, the technical presentation is accompanied by useful historical remarks, including some indications of the main consistency and independence results about AC and CH (Gödel, Cohen, Solovay). Moreover, there is a rich Appendix devoted to the study of various natural set-theoretic models, including Rieger universes (which violate the Principle of Purity) and in particular Aczel's antifounded universe (which features a rich variety of non-well-founded sets). This is certainly admirable and confirms the recent change of attitude among set-theorists with regard to the axiom of Foundation. (This is the second book on elementary set theory that includes a section on non-well-founded sets, after Keith Devlin's *The Joy of Sets*, 1993.) Perhaps this is a place where the author could have been more generous with informal remarks and examples. Some decades ago, Fraenkel and Bar-Hillel argued that "one can accept [the Axiom of Foundation] not as an article of faith but as a convention for giving a more restricted meaning to the word 'set', to be discarded once it turns out that it impedes significant mathematical research" [*Foundations of Set Theory*, Second Revised Edition, p. 89]. Arguably, today there are indeed many interesting applications of non-well-founded sets apart from independence proofs—from the semantics of natural language and programming languages to knowledge representation and game theory. (Barwise and Moss' 1996 book, *Vicious Circles*, gives an excellent overview of such a variety of applications.) There are therefore good reasons for "discarding" Foundation from the axiomatic foundations of set theory. This is the author's own attitude. But he does little to help the reader appreciate it.

It is also a pity that the author decided not to back up his presentation with adequate bibliographic references: a generic reference to the Dover edition of Cantor's *Beiträge* and to van Heijenoort's collection of classics is hardly enough. Likewise, a more extensive index and a complete glossary of symbols would make the book more useful as a reference. As for the editing, there are unfortunately numerous errata which may be disturbing if the book is used as a text (although spotting them is certainly good exercise for the reader). Some nasty ones: on p. 19, the quantifier in GCH should have ‘ (A)’ in place of ‘A’; on p. 21, first line, ‘G’ should be ‘C’; on p. 59, the range of the functions h_y and f_y should be E, not $E \times N$, and the condition for $f_y(n+1)$ should refer to $f_y(n)$, not $f(n)$; on p. 69, first line, the second ‘N’ should be ‘$N \times N$’; on p. 95, just before (7.8), ‘x’ should read ‘y’ (twice); on p. 99, the formula following (7.13) should define σ_t, not σ_u (likewise, two lines above, ‘$u = Sv$’ should read ‘$t = Sv$’); on p. 107, line 3, ‘$\mathbf{seg}_V$’ should read ‘$\mathbf{seg}_U$’, and on line 10, ‘$<_o$’ should read ‘$<_{\chi(A)}$’; on p. 121, line 11 from bottom, ‘$A \not\leq_c h(A)$’ should read ‘$h(A) \not\leq_c A$’; on p. 123, line 11 from bottom should read ‘$f(0) = a$’; on p. 139, last line of Lemma **9.20**, ‘inf_c’ should read ‘sup_c’; and on p. 200, second last line, “every von Neumann cardinal” should read “every transfinite von Neumann cardinal” (similarly in exercise **12.21**, p. 201).

Even with these limitations, which can easily be overcome in a second edition, the book remains excellent and quite suitable as a textbook, ideal for advanced undergraduate or beginning graduate students in mathematics, philosophy, and computer science.