

General Rules for Loop Invariant Proofs

We use loop invariants to help us understand why an algorithm is correct. We must show three things about a loop invariant:

Initialization: It is true prior to the first iteration of the loop.

Maintenance: If it is true before an iteration of the loop, it remains true before the next iteration.

Termination: When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.

When the first two properties hold, the loop invariant is true prior to every iteration of the loop. Note the similarity to mathematical induction, where to prove that a property holds, you prove a base case and an inductive step. Here, showing that the invariant holds before the first iteration is like the base case, and showing that the invariant holds from iteration to iteration is like the inductive step.

The third property is perhaps the most important one, since we are using the loop invariant to show correctness. It also differs from the usual use of mathematical induction, in which the inductive step is used infinitely; here, we stop the “induction” when the loop terminates.

HeapSort loop invariant

Build – Max – Heap(A)

- 1** $heap\text{-}size[A] \leftarrow length[A]$
- 2** **for** $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
- 3** **do** MAX-HEAPIFY(A, i)

To show why BUILD-MAX-HEAP works correctly, we use the following loop invariant:

At the start of each iteration of the for loop of lines 2– 3, each node $i + 1, i + 2, \dots, n$ is the root of a max-heap.

Loop Invariant Proof

At the start of each iteration of the for loop of lines 2– 3, each node $i + 1, i + 2, \dots, n$ is the root of a max-heap.

We need to show that this invariant is true prior to the first loop iteration, that each iteration of the loop maintains the invariant, and that the invariant provides a useful property to show correctness when the loop terminates.

Initialization: Prior to the first iteration of the loop, $i = \lfloor n/2 \rfloor$. Each node $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, n$ is a leaf and is thus the root of a trivial max-heap.

Maintenance: To see that each iteration maintains the loop invariant, observe that the children of node i are numbered higher than i . By the loop invariant, therefore, they are both roots of max-heaps. This is precisely the condition required for the call $\text{MAX-HEAPIFY}(A, i)$ to make node i a max-heap root. Moreover, the MAX-HEAPIFY call preserves the property that nodes $i + 1, i + 2, \dots, n$ are all roots of max-heaps. Decrementing i in the for loop update reestablishes the loop invariant for the next iteration.

Termination: At termination, $i = 0$. By the loop invariant, each node $1, 2, \dots, n$ is the root of a max-heap. In particular, node 1 is.