Worst Case Analysis

e.g. Heap

n INSERTS $\lg n$

n DELETE-MINS $\lg n$

$O(n \lg n)$

seq. of

n INSERTS took $O(n)$

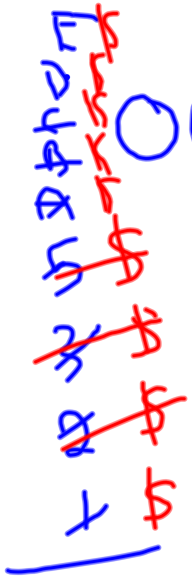
Stack

Push

Pop

Multipop

O(1) time



O(k)

Push(1)
Push(2)
Push(3)
Pop( )
Push(5)
Multipop(S, 2)
5 times { Push
Multipop (S, 6)
...

n Push, Pop, MP n ops each $O(n)$ time take $O(n^2)$ time.

Can have 1 MP take $\Omega(n)$ time

Can you have $\Omega(n)$ MP's each taking $\Omega(n)$ time.

Push MP MP MP Push MP MP Push..

Any sequence of n Push Pop MPs

take $O(n)$ time in total.

($O(1)$ time / op <u>amortized</u>)

Amortized Analysis

3 methods of AA.

- Aggregate Analysis
- Banker's Method
- Potential function Method.

Let $m(i)$ = # of pops done in i^{th} multipop

Let p = # of pushes done by the alg.

(#pushes $\geq$ #pops)

$$\sum_i m(i) \leq p$$

$$\text{Time}^{(n \text{ ops})} = \#\text{pushes} + \#\text{pops} + \sum m(i)$$

$$\leq p + p + p \leq 3p$$

$$\leq 3n.$$

Banker's

OPS	Real cost c_i	Amort. cost $\hat{c}_i$
Push	1	2 (1)
Pop	1	0
MP	k	0

- Assign $\hat{c}_i$
- Show $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$ for any seq. of ops.
- Bound on n ops is $\sum_{i=1}^{n} \hat{c}_i$. $\left(\sum_{i=1}^{n} \hat{c}_i \leq 2n \Rightarrow \sum_{i=1}^{n} c_i \leq 2n \right)$

Proof that $\sum \hat{c}_i \geq \sum r_i$

	$\hat{c}_i$
Push	2
Pop	0
Mp	0

Whenever you push() use \$1 to pay for the push, store the other \$1 in the "bank" to pay for the item being popped.

$\Rightarrow$ Balance always ≥ 0.

Potential

Def

$\Phi(D_i)$ = potential after i^{th} op.

Real costs c_i

Am costs $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} \left(\Phi(D_i) - \Phi(D_{i-1}) \right)$$

$$\begin{aligned}
&\Phi(\cancel{D_1}) - \Phi(D_0) \\
&\Phi(\cancel{D_2}) - \Phi(\cancel{D_1}) \\
&\Phi(\cancel{D_3}) - \Phi(\cancel{D_2}) \\
&\quad \vdots \\
&\Phi(D_n) - \Phi(\cancel{D_{n-1}})
\end{aligned}$$

$$\Rightarrow \sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

Want $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i \Rightarrow \Phi(D_n) - \Phi(D_0) \geq 0$

$$\Phi(D_n) \geq \Phi(D_0)$$

If $\Phi(D_n) \geq \Phi(D_0)$ then $\sum_{i=1}^{n} \hat{c}_i$ is an upper bound on $\sum_{i=1}^{n} c_i$.

$\Phi(D_i) = \#$ items in stack

$\Phi(D_0) = 0$

$\Phi(D_i) \geq 0 \quad \forall i \qquad \Phi$ is valid

Compute $\hat{c}_i = c_i + \Delta\Phi$

Push $\hat{c}_i = 1 + 1 = 2$

Pop $\hat{c}_i = 1 + (-1) = 0$

MP(k) $\hat{c}_i = k + (-k) = 0$

Cost of n ops $\leq \sum_{i=1}^{n} \hat{c}_i \leq 2n$

3 bit counter

charge \$2/inc.
use \$1 to pay for new 1
store \$ to pay for flip back to 0.

\$\$

	cost	$\hat{\text{cost}}$
000	~~3~~	~~2~~ 0
001	1	2
010	2	2
011	1	2
100	3	2
101	1	2
110	2	2
111	1	2

total $\leq 2n$

cost(increment) = # bits flipped

k bit counter, each inc. costs $\leq k$

n be a power of 2

n INC

LSB flipped n times

next bit " $n/2$ time

" " $n/4$ time

⋮

MSB flipped $n/2^{k-1}$ times

total flips

$$n + \frac{n}{2} + \frac{n}{4} + \dots \leq 2n$$

n INC in $\leq 2n$ total time

$O(1)$ / op amortized

$\Phi(D_i) = \#$ 1's in the counter

$$\Phi(D_0) = 0$$

$$\Phi(D_i) \geq 0 \quad \forall i$$

$$\hat{c}_i = (c_i) + (\Phi(D_i) - \Phi(D_{i-1}))$$

let $f_{01} = \#$ bits flipped $0 \to 1$

$f_{10} =$ " " " $1 \to 0$

$$= (f_{01} + \cancel{f_{10}}) + (f_{01} - \cancel{f_{10}})$$

$$= 2f_{01}$$

$$\leq 2$$

Total cost for n ops is $\leq 2n$.