Let $T(n)$ = running time of MS on n items

$$T(n) = 1 + T\left(\tfrac{n}{2}\right) + T\left(\tfrac{n}{2}\right) + O(n)$$

$$\boxed{T(n) = 2T\left(\tfrac{n}{2}\right) + O(n)}$$

want a "O" solution.

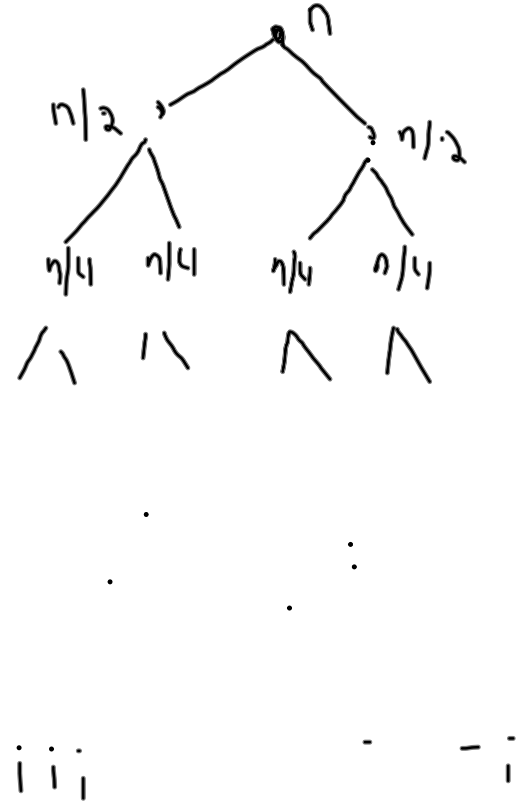
$$T(n) = \begin{cases} 2T\left(\tfrac{n}{2}\right) + O(n) & n > 1 \\ O(1) & n = 1 \end{cases}$$

slappiness - base case, floors, ceilings

Divide + Conquer Recurrences

$$T(n) = \underline{a}\,T(n/\underline{b}) + \underline{f(n)}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



Level	Work
0	n
1	$2\left(\frac{n}{2}\right) = n$
2	
3	
$\vdots$	
i	$2^i\left(\frac{n}{2^i}\right) = n$
$\lg n$	$2^{\lg n}\left(\frac{n}{2^{\lg n}}\right) = n$

Sum work over levels

$$\sum_{i=0}^{\lg n} n = n(\lg n + 1) = O(n \lg n)$$

3

$$T(n) = T\left(\frac{n}{2}\right) + n$$

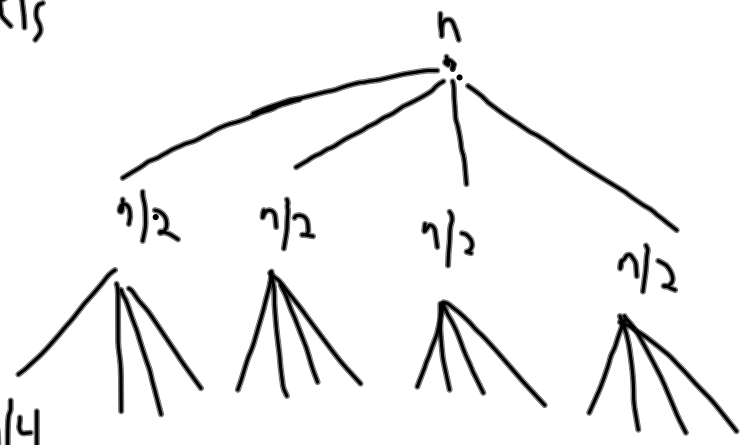
levels		Work
0	n	n
1	$n/2$	$n/2$
2	$n/4$	$n/4$
i	$n/2^i$	$n/2^i$
$\lg n$	1	

$$\sum_{i=0}^{\lg n} n/2^i = n \sum_{i=0}^{\lg n} 1/2^i \le 2n = O(n)$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

levels — Work

0 — n



1 — $4\left(\frac{n}{2}\right) = 2n$

2 — $n/4$ — $16\left(\frac{n}{4}\right) = 4n$

i — $4^i\left(\frac{n}{2^i}\right) = 2^i n$

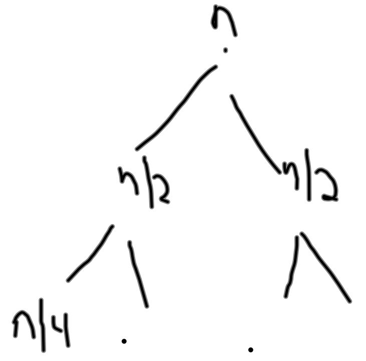
$\lg n$ — n^2 leaves, 1 unit of work each — $2^{\lg n} n = n^2$

$$\sum_{i=0}^{\lg n} 2^i n = n^2 + n^2/2 + n^2/4 + \cdots + n^2/n$$
$$= n^2\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{n}\right) \le 2n^2 = O(n^2)$$

ZS

$$T(n) = 2T\left(\frac{n}{2}\right) + n\lg n$$

Lev



$n\lg n$

$2\left(\frac{n}{2}\lg\left(\frac{n}{2}\right)\right)$

$4\left(\frac{n}{4}\lg\left(\frac{n}{4}\right)\right)$

$2^i\left(\frac{n}{2^i}\lg\left(\frac{n}{2^i}\right)\right)$

$= n\lg(n/2^i)$

$= n\left(\lg n - \lg 2^i\right)$

$= n\lg n - in$

$\lg n$

$$\sum_{i=0}^{\lg n}(n\lg n - in) = n\lg n(\lg n+1) - n\frac{\lg n(\lg n+1)}{2}$$

$$= n\lg^2 n + n\lg n - \frac{n}{2}\lg^2 n - \frac{n\lg n}{2}$$

$$= \frac{n}{2}\lg^2 n + \frac{n}{2}\lg n = O(n\lg^2 n).$$

$$T(n) = T(n-1) + n$$

n	n
$n-1$	$n-1$
$n-2$	$n-2$
$\vdots$	
1	1

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = O(n^2)$$

Proof by induction

$$T(n) = 3T\left(\frac{n}{3}\right) + n$$

Prove $T(n) = O(n \lg n)$

$T(n) \le c n \lg n$ for some c.

Assume true for smaller values of n

Prove for n

$$T(n) = 3T\left(\frac{n}{3}\right) + n \qquad \lg_b n = O(\lg_a n)$$

$$\le \cancel{3}\left(c \frac{n}{\cancel{3}} \lg\left(\frac{n}{3}\right)\right) + n$$

$$= cn\lg n - cn\lg 3 + n$$

$$= cn\lg n - n(c\lg 3 - 1) < cn\lg n$$

if $c\lg 3 - 1 > 0$

choose $c = 4$. $\left(c > \frac{1}{\lg 3}\right)$

$$T(n) = 2T\left(\tfrac{n}{2}\right) + \overset{kn}{\cancel{O(n)}}$$

$\le kn$ for some k. (exists, but we don't pick)

we pick

Prove $T(n) \le c\,n\lg n$

Induction:

$$T(n) = 2T\left(\tfrac{n}{2}\right) + kn$$

$$\le \cancel{2}\left(c\frac{n}{\cancel{2}}\lg\left(\tfrac{n}{2}\right)\right) + kn$$

$$= cn\lg n - cn + kn$$

$$= cn\lg n \quad (c-k)n \le cn\lg n$$

$cn\lg n + (c+k)n$

if $c+k < 0$

$c < -k$

Proof fails

if $c-k \ge 0$

$c \ge k$.

Choose $c = k$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

Prove $T(n) \leq cn^2$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$\leq 4\left(c\left(\frac{n}{2}\right)^2\right) + n$$

$$= cn^2 + n \overset{?}{\leq} cn^2$$

if $n < 0$

Proof fails

Inductive hypothesis is too weak.

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

Prove $T(n) \le cn^2 - dn$ $\quad c, d > 0$

$$T(n) \le 4\left(c\left(\frac{n}{2}\right)^2 - d\,\frac{n}{2}\right) + n$$

$$= cn^2 - 2dn + n$$

$$= cn^2 - n(2d-1)$$

$$= cn^2 - dn - n(d-1) \le cn^2 - dn$$

choose $c = 1, d = 1$ $\quad$ if $d - 1 \geq 0$

$\therefore T(n) = O(n^2)$ $\quad$ $d \geq 1$