Hiring Problem

best = 0
for i = 1 to n
    interview candidate i
    if i is better than best
        hire i
        best = i

count # hirings.

$O(n)$ hirings

- $1 \leq$ hirings $\leq n$.

avg. is not $\frac{n}{2}$.

Assume all orderings of candidates are equally likely.

$$\text{expected value}(X) = \sum_{\substack{\text{outcomes}\\\text{of } X}} \Pr(X \text{ occurs}) \cdot \text{value}(X).$$

h = # hirings

$h(\pi_i)$ = # hirings when the order of the candidates is permutation π_i

$$E[h] = \sum_{\substack{\text{perm.}\\ \pi_i}} \frac{1}{n!} h(\pi_i).$$

$$= \sum_{j=1}^{n} \Pr(\text{hire exactly } j \text{ times}) \cdot j$$

Indicator random variables

A event

$$I\{A\} = \begin{cases} 1 & \text{if event occurs} \\ 0 & \text{o.w.} \end{cases}$$

Q. Expected number of heads in one coin flip.

Let Y be a r.v. denoting heads or tails

$$X_H = I\{Y = \text{"Heads"}\} = \begin{cases} 1 & \text{if } Y = H \\ 0 & \text{if } Y = T \end{cases}$$

$$E[X_H] = \Pr(\overset{(Y=H)}{X_H = 1}) \cdot 1 + \Pr(\overset{(Y=T)}{X_H = 0}) \cdot 0$$

$$= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}.$$

n Coin Flips

E[# of heads] ?

$$= \sum_{i=0}^{n} \Pr(\text{exactly } i \text{ heads}) \cdot i$$

⋮

X_j = # heads on flip i.

X = # heads on all n flips

linearity of expectation

$$E[Y+Z] = E[Y] + E[Z]$$

$$X = \sum_{j=1}^{n} X_j$$

$$E[X] = E\left[\sum_{j=1}^{n} X_j\right]$$

$$= \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2}.$$

Hiring problem,

X_i = # of candidates hired on the day when the i^{th} is interviewed

X = total # of hirings

$$X = \sum_{i=1}^{n} X_i$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i].$$

$$E[X_i] = Pr(X_i=1)\cdot 1 + Pr(X_i=0)\cdot 0$$
$$= Pr(i^{th} \text{ candidate is hired}).$$

Candidate1 $E[X_1] = 1$

$E[X_2] = \frac{1}{2}$

$E[X_3] = \frac{1}{3}$

$E[X_i] =$

Pr(candidate i is better than all candidates $1 \ldots i-1$)

$= \frac{1}{i}$

$\rightarrow \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n$

Assumption: candidates come in a random order.

Remove assumption: Have the algorithm randomly order the candidates.

Bad input: increasing order

Bad case: my random permuting algorithm accidentally sorts.

Function

Random(a, b) - returns an integer between a & b, uniformly at random.

Birthday Paradox

n people

$E[\#\text{ of pairs w/ same birthday}]$

$X_{ij} = $ IRV i & j having same b.d.

$X = $ # pairs w/ same b.d.

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{365}$$

$$= \binom{n}{2} \frac{1}{365} = \frac{n(n-1)}{2} \cdot \frac{1}{365}$$

$n=23$ .69

$n=28$ 1.03

$n=64$ 5.5

Streaks

n coins, whats the longest streak of heads?

X_{ik} = IRV heads on flips i to $i+k-1$

$$X_k = \sum_{i=1}^{n-k+1} X_{ik} = \text{\# of streaks of length } k$$

$$E[X_k] = \sum_{i=1}^{n-k+1} E[X_{ik}]$$

$$= \sum_{i=1}^{n-k+1} \frac{1}{2^k} = \frac{n-k+1}{2^k}.$$

$$\frac{n-k+1}{2^k}$$ # streaks of length k

$k=3 \quad \approx \frac{n}{8}$

$k=n \quad \approx \frac{1}{2^n}$

$k = c\lg n$

$$\frac{n - c\cancel{\lg n} \cancel{+1}}{2^{c\lg n}} \approx \frac{n}{n^c} = \frac{1}{n^{c-1}}.$$

if $c=\frac{1}{2}$ $\qquad \approx \frac{1}{n^{-1/2}} = \sqrt{n}$

$c=4$ $\qquad \approx \frac{1}{n^3}$

Length of longest streak $= \Theta(\lg n)$