

# Huffman Codes

**Coding** is the problem of representing data in another representation. Typically, we want that representation to be concise. We will encode in binary in this lecture. We call the encoding of a character a **codeword**.

## Different types of codes

- **fixed length code.** Each codeword uses the same number of bits.
- **variable length code.** Codewords can use differing numbers of bits.

## Example

character	frequency	fixed length code	variable length code
a	.45		
b	.13		
c	.12		
d	.16		
e	.9		
f	.5		

# Huffman Codes

## Different types of codes

- **fixed length code.** Each codeword uses the same number of bits.
- **variable length code.** Codewords can use differing numbers of bits.

character	frequency	fixed length code	variable length code
a	.45	000	0
b	.13	001	101
c	.12	010	100
d	.16	011	111
e	.09	100	1101
f	.05	101	1100

**Evaluation of code:** Expected number of bits per codeword.

**Fixed length code:** 3

**Variable length code:**

$$.45(1) + .13(3) + .12(3) + .16(3) + .09(4) + .05(4) = 2.24$$

**Prefix free codes:** No codeword is a prefix of any other codeword.

**Decoding:** A variable length code must be prefix free.

# Trees

Codes can be represented as trees.

- Let  $T$  be a tree corresponding to a prefix code,
- Let  $f(c)$  denote the frequency of character  $c$ .
- Let  $d_T(c)$  denote the depth of  $c$ 's leaf in  $T$  ( $d_T(c)$  is also the length of the codeword for character  $c$ .)

The number of bits required to encode a file is thus

$$B(T) = \sum_{c \in C} f(c)d_T(c) ,$$

which we define as the **cost** of the tree  $T$ .

# Huffman Coding Algorithm

**Huffman(C)**

```
1   $n \leftarrow |C|$ 
2   $Q \leftarrow C$ 
3  for  $i \leftarrow 1$  to  $n - 1$ 
4      do allocate a new node  $z$ 
5           $left[z] \leftarrow x \leftarrow \text{EXTRACT-MIN}(Q)$ 
6           $right[z] \leftarrow y \leftarrow \text{EXTRACT-MIN}(Q)$ 
7           $f[z] \leftarrow f[x] + f[y]$ 
8           $\text{INSERT}(Q, z)$ 
9  return  $\text{EXTRACT-MIN}(Q)$  ▷ Return the root of the tree.
```

# Proving Huffman is optimal

**Theorem** Let  $C$  be an alphabet in which each character  $c \in C$  has frequency  $f[c]$ . Let  $x$  and  $y$  be two characters in  $C$  having the lowest frequencies. Then there exists an optimal prefix code for  $C$  in which the codewords for  $x$  and  $y$  have the same length and differ only in the last bit.

## **Proof**

**Idea:** Take a tree  $T$  representing an arbitrary optimal prefix code and modify it to make a tree representing another optimal prefix code such that the characters  $x$  and  $y$  appear as sibling leaves of maximum depth in the new tree. If we can do this, then their codewords will have the same length and differ only in the last bit.

# Proving Huffman is optimal

**Theorem** Let  $C$  be an alphabet in which each character  $c \in C$  has frequency  $f[c]$ . Let  $x$  and  $y$  be two characters in  $C$  having the lowest frequencies. Then there exists an optimal prefix code for  $C$  in which the codewords for  $x$  and  $y$  have the same length and differ only in the last bit.

## Details:

- Let  $a$  and  $b$  be two characters that are sibling leaves of maximum depth in  $T$ . (wlog,  $f[a] \leq f[b]$  and  $f[x] \leq f[y]$ .)
- $f[x] \leq f[a]$  and  $f[y] \leq f[b]$ , since  $f[x]$  and  $f[y]$  are the two lowest leaf frequencies.
- Exchange the positions in  $T$  of  $a$  and  $x$  to produce a tree  $T'$ .
- Exchange the positions in  $T'$  of  $b$  and  $y$  to produce a tree  $T''$ .

## Proof Continued

- Let  $a$  and  $b$  be two characters that are sibling leaves of maximum depth in  $T$ . (wlog,  $f[a] \leq f[b]$  and  $f[x] \leq f[y]$ .)
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- Exchange the positions in  $T$  of  $a$  and  $x$  to produce a tree  $T'$ .
- Exchange the positions in  $T'$  of  $b$  and  $y$  to produce a tree  $T''$ .

Now look at the difference between  $B(T)$  and  $B(T')$

$$\begin{aligned} B(T) - B(T') &= \sum_{c \in C} f(c)d_T(c) - \sum_{c \in C} f(c)d_{T'}(c) \\ &= f[x]d_T(x) + f[a]d_T(a) - f[x]d_{T'}(x) - f[a]d_{T'}(a) \\ &= f[x]d_T(x) + f[a]d_T(a) - f[x]d_T(a) - f[a]d_T(x) \\ &= (f[a] - f[x])(d_T(a) - d_T(x)) \\ &\geq 0, \end{aligned}$$

Reasons for last inequality:

- $f[a] - f[x]$  is nonnegative because  $x$  is a minimum-frequency leaf,
- $d_T(a) - d_T(x)$  is nonnegative because  $a$  is a leaf of maximum depth in  $T$ .

## Proof Finished

Now look at the difference between  $B(T)$  and  $B(T')$

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### Conclusions:

- $B(T') \leq B(T)$
- By same argument –  $B(T'') \leq B(T')$
- Conclusions
- $B(T'') \leq B(T)$ , which completes the proof. which the lemma follows.