NP-complete Partitioning Problems

Subset Sum: Given a list of t positive integers $S = \{x_1, x_2, \ldots, x_t\}$ and an integer B, is there a subset $S' \subseteq S$ s.t. $\sum_{x_i \in S'} x_i = B$.

- **Yes instance:** $S = \{1, 2, 5, 7, 8, 10, 11\}, B = 22$.
- **No instance:** $S = \{4, 10, 11, 12, 15\}, B = 28$.

Note: It is still NP-complete if $B = \sum_i x_i/2$

3-Partition Given a list of $3t$ positive integers $S = \{x_1, x_2, \ldots, x_{3t}\}$ with $\sum_{x_i \in S} x_i = tB$, and each x_i satisfying $B/4 < x_i < B/2$, can you partition S into t groups of size 3, such that each group sums to exactly B.

- **Yes instance:** $S = \{26, 26, 27, 28, 29, 29, 31, 33, 39, 40, 45, 47\}$
- **No instance:** $S = \{26, 26, 27, 28, 29, 29, 31, 33, 38, 40, 45, 48\}$ (I think)

$P||C_{\max}$

Problem: Given n jobs with processing times p_j, schedule them on m machines so as to minimize the makespan.

Decision version: Given n jobs with processing times p_j and a number D, can you schedule them on m machines so as to complete by time D.

Sample inputs:

- **Jobs are $\{1,2,5,7,8,10,11\}$, 2 machines, $D = 22$.**
- **Jobs are $S = \{4,4,10,11,12,15\}$, 3 machines $D = 20$.**

Reduction: Subset sum reduces to $P||C_{\max}$.

Idea of reduction: Given a subset sum instance, create a 2-machine instance of $P||C_{\max}$, with $p_j = x_j$ and $D = B$. Now there is a feasible schedule iff there is a subset summing to B.

$1|r_j|L_{\max}$

Reduction: **Reduce 3-partition to $1|r_j|L_{\max}$.**

3-Partition **Given a list of $3t$ positive integers $S = \{x_1, x_2, \ldots, x_{3t}\}$ with $\Sigma_{x_i \in S}\, x_i = tB$, can you partition S into t groups of size 3, such that each group sums to exactly B.**

Given a 3-partition instance, we will creat a $1|r_j|L_{\max}$ instance in the following way:

Jobs: $n = 4t - 1$ **jobs,** $t - 1$ **of which are dummy jobs**

Dummy Jobs:

j	r_j	p_j	d_j
1	**B**	**1**	**B+1**
2	**2B +1**	**1**	**2B+2**
3	**3B + 2**	**1**	**3B+3**
⋮	⋮	⋮	⋮
$t-1$	$(t-1)B + (t-2)$	**1**	$(t-1)B + (t-1)$

Real Jobs:

- **indexed** t **through** $4t - 1$ **.**
- **All have** $r_j = 0$

- **All have** $d_j = tb + (t-1)$
- $p_j = x_{j-(t-1)}$

Proof

Dummy Jobs:

j	r_j	p_j	d_j
1	**B**	**1**	**B+1**
2	**2B +1**	**1**	**2B+2**
3	**3B + 2**	**1**	**3B+3**
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t-1$	$(t-1)B+(t-2)$	**1**	$(t-1)B+(t-1)$

Real Jobs:

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Idea of Proof: **Argue that there is a schedule with** $L_{\max} = 0$ **iff the partition instance is yes.**