Tightening simple mixed-integer sets with guaranteed bounds

Daniel Bienstock and Benjamin McClosky
Columbia University
New York

July 2008, revised June 2010, November 2010, December 2010

Abstract

In this paper we study how to reformulate knapsack sets and simple mixed integer sets in order to obtain provably tight, polynomially large formulations.

1 Introduction

In this paper we consider 0/1 knapsack sets and certain simple fixed-charge network flow sets. The study of such sets is relevant in that a popular approach for solving general mixed-integer programs consists of selecting a subset of constraints with particular structure (such as a single-node fixed-charge flow problem) and tightening that part of the formulation through the use, for example, of classical cutting-plane families (see e.g. [16], [13]). A question of interest is in what sense the resulting stronger formulation is provably good.

Motivated by questions posed in [20], and extending the study initiated in [3], we show how the use of appropriate disjunctions [1] leads to provably tight, yet polynomially large, formulations for several simple sets. Unlike the use of familiar disjunctive cuts ([2], [18] and [9], [11]) the disjunctions we employ are 'combinatorial', or 'structural', that is to say, they depend on the structure of the problem at hand. Previous work [6] has shown how structural disjunctions can lead to provably good approximations of combinatorial polyhedra (also see [14], [4], [5], [19]); we expect that many other results of this type are possible.

1.0.1 Minimum knapsack

In Section 2 we consider the "minimum" 0/1 knapsack problem,

$$\begin{aligned} \boldsymbol{\mathcal{KMIN}}: \qquad v^Z &= \min \sum_{j=1}^{N} c_j x_j, \\ s.t. \qquad \sum_{j=1}^{N} w_j x_j &\geq b, && (1) \\ x &\in \{0\,,\,1\}^N, && (2) \end{aligned}$$

where $c_j > 0$ and $0 < w_j \leq b$ for $1 \leq j \leq N$. We denote by v^* the value of the LP relaxation of $\boldsymbol{\mathcal{KMIN}}$.

In [8], Carr, Fleischer, Leung and Phillips consider the so-called knapsack-cover inequalities, which are different from cover inequalities. Let $A \subseteq \{1, \ldots, N\}$ and write $b(A) = b - \sum_{j \in A} w_j$. The knapsack-cover inequality corresponding to A is $\sum_{j \notin A} \min\{w_j, b(A)\} x_j \geq b(A)$. Since knapsack-covers are valid inequalities, a lower bound for v^Z is provided by $v^{(2)}$, where

$$\begin{aligned}
v^{(2)} \quad &= \quad \min \sum_{j=1}^{N} c_j x_j, \\
s.t. \quad & \quad \sum_{j \notin A} \min\{w_j, b(A)\} x_j \;\geq\; b(A), \quad \forall\, A && (3) \\
& \quad 0 \;\leq\; x_j \;\leq\; 1, \quad \forall\, 1 \leq j \leq N. && (4)
\end{aligned}$$

In [8] it is shown that $v^{(2)} \geq v^Z/2$, and that using the ellipsoid method [12] one can obtain in polynomial time a fractional vector satisfying (1), (3) and (4), of cost at most $v^{(2)}$, and which can be rounded to an integral solution feasible for $\mathcal{KMIN}$ while at most doubling the cost. Thus this provides a polynomial-size relaxation for $\mathcal{KMIN}$ with a multiplicative gap of at most 2. Furthermore, they provide an algorithm not relying on the ellipsoid method such that for any $0 < \epsilon < 1$ one can estimate $v^{(2)}$ within a multiplicative factor of $1 + \epsilon$ in time polynomial in N and ϵ^{-1}. Carnes and Shmoys [7] present a primal-dual, 2-approximation algorithm for $\mathcal{KMIN}$ that relies on knapsack-cover inequalities, and extend their techniques to other problems, such as capacitated single-item lot-sizing.

In this paper we show how a simple disjunction provides a polynomially large linear programming relaxation to $\mathcal{KMIN}$ whose value $\bar{v}$ satisfies $v^Z < 2\bar{v}$. In fact, we show

Theorem 1.1 *For each $0 < \epsilon < 1$ there is a linear programming relaxation to $\mathcal{KMIN}$ with $O\left((1/\epsilon)^{O(1/\epsilon^2)}\, N^2\right)$ variables and constraints, whose value $v(\epsilon)$ satisfies $v^Z < (1+\epsilon)v(\epsilon)$.*

1.0.2 Single-node, fixed-charge sets

In Section 3 we consider an optimization problem of the form

$$\begin{aligned}
\boldsymbol{\mathcal{FXN}}\textbf{ :} \qquad v^Z \quad &= \quad \min \sum_{j=1}^{n} \left(f_j x_j \,+\, c_j y_j\right), \\
& \quad \sum_{j \in \delta^+} y_j \quad - \quad \sum_{j \in \delta^-} y_j \quad = \quad b, && (5) \\
& \quad 0 \leq y_j \leq u_j\, x_j \quad \forall 1 \leq j \leq n, && (6) \\
& \quad x_j = 0 \text{ or } 1, \quad \forall 1 \leq j \leq n. && (7)
\end{aligned}$$

Here, the sets δ^+, δ^- partition the indices $1 \leq j \leq n$. We assume that c, f, u are nonnegative vectors. Many practical problems arising in logistics, network design, finance, and other applications frequently include such "one-node" fixed-charge flow systems as subproblems. As a result, these systems have been a motivating factor for several classical families of valid inequalities for mixed-integer programs (see [15]). We have:

Theorem 1.2 *Let $0 < \epsilon < 1$. There is a linear programming relaxation $\mathcal{F}(\epsilon)$ to $\mathcal{FXN}$ with $O\left((1/\epsilon)^{O(1/\epsilon^2)}\, n^3\right)$ variables and constraints, such that from any extreme point solution to $\mathcal{F}(\epsilon)$ we can obtain a mixed-integer solution for $\mathcal{FXN}$ while increasing cost by at most a factor of $(1+\epsilon)$.*

1.0.3 Maximum knapsack

In Section 4 we consider the "maximum" 0/1 knapsack problem,

$$\begin{aligned}
\mathcal{KMAX}: \qquad v^Z \;&=\; \max \sum_{j=1}^{N} p_j x_j, \\
s.t. \qquad \sum_{j=1}^{N} w_j x_j \;&\leq\; b, \qquad (8)\\
x \;&\in\; \{0\,,\,1\}^N, \qquad (9)
\end{aligned}$$

where $p_j > 0$ and $0 < w_j \leq b$ for $1 \leq j \leq N$. We denote by v^* the value of the LP relaxation of $\mathcal{KMAX}$. A simple observation is that $v^* \leq 2v^Z$ (see Section 1.2). A relevant question is at what cost and to what degree can this bound be improved.

In [20] Van Vyve and Wolsey ask whether, given an instance of $\mathcal{KMAX}$, and $0 < \epsilon \leq 1$, there is a formulation of the form $Ax + A'x' \leq b'$, such that

(a) For each vector $x \in \{0\,,\,1\}^N$ with $\sum_{j=1}^{N} w_j x_j \;\leq\; b$ there exists x' such that $Ax + A'x' \leq b'$,

(b) The number of variables x' and rows of A and A' is polynomial in N and/or ϵ^{-1}, and

(c) For every $w \in R_+^N$,

$$\max\left\{w^T x \;:\; Ax + A'x' \;\leq\; b'\right\} \quad\leq\quad (1+\epsilon)v^Z.$$

In [3] we provided a partial answer to this question: there *is* a formulation satisfying (a)-(c) which has polynomially many variables and constraints *for each fixed* ϵ (also see [19]). This formulation amounts to a multi-term disjunction, which, although polynomial, is complex from a practical perspective. Related results are described in [10].

The result in [3] motivates several questions, in particular:

1. Is there a formulation achieving (a)-(c) but restricted to the original space of variables?
2. How about achieving (c), but restricting to the original space of variables *and* allowing exponentially many constraints, so long as these are polynomially separable?
3. In fact, what can be achieved in polynomial time? Is there a "simple" relaxation involving polynomially separable inequalities, whose value $\hat{v}$ satisfies $\hat{v}/v^Z < \theta$ for some $\theta < 2$?

We note that Van Vyve [19] has shown that the formulation that incorporates all valid inequalities with integer coefficients with values in $\{0, 1, \ldots, \lceil N/\epsilon \rceil\}$ proves an LP/IP ratio at most $1+\epsilon$, for any $0 < \epsilon < 1$ (an open problem is whether one can separate in polynomial time over such a system of inequalities). As a counterpart to this result, we can ask the following question. Suppose we pick a *fixed* integer $k > 0$, and we strengthen the LP relaxation of $\mathcal{KMAX}$ with all valid inequalities of the form $\sum_j \alpha_j x_j \;\leq \beta$, where the α_j take values in $\{0, 1, \ldots, k\}$. Is it true that the value of the resulting linear program is at most

$$(1 + f(k))v^Z,$$

where $f(k) \to 0$ as $k \to +\infty$? The answer to this question is (perhaps, not surprisingly) negative. We show that for each $k > 0$, if N is large enough there is an example of $\mathcal{KMAX}$ where, after adding all valid inequalities with left-hand coefficients in $\{0, 1, \ldots, k\}$, the value of the linear program remains arbitrarily close to $2v^Z$. In fact, this result holds even with $k = N^{1-\pi}$ with $\pi > 0$ arbitrarily small. This is discussed in Section 4.1. At the same time, in Section 4.2 we show that using a *single* (polynomially separable) disjunction, one obtains a relaxation whose value is at most $\left(1 + \frac{\sqrt{19}-2}{3}\right) v^Z$. Thus, question 3 above does have a positive answer.

However, the disjunction used in Section 4.2 depends on the structure of the objective coefficients p_j and is therefore not quite in the "a priori strengthening" spirit of the question of Van Vyve and Wolsey. Further, the examples in Section 4.1 have "large" constraint coefficients, that is to say we have $w_j \approx b$ for some j. One might consider such examples "artificial" and wonder what happens if we insist that the ratios w_j/b be bounded strictly away from 1.

These issues are taken up in Section 4.3. Given a subset $S \subseteq \{1, 2, \ldots, N\}$ with $w_i + w_j > b$ for each pair of distinct indices $i, j \in S$, the *clique* inequality [15]

$$\sum_{j \in S} x_j \leq 1$$

is valid for $\mathcal{KMAX}$. It can be seen that one can separate over the clique inequalities in polynomial time; in fact there is at most a linear number of maximal cliques. Let v^ω denote the value of the linear program obtained by augmenting the continuous relaxation of $\mathcal{KMAX}$ with all clique inequalities. In Section 4.3 we prove:

Theorem 1.3 *For each constant* $0 \leq \psi < 1$ *there exists* $\epsilon = \epsilon(\psi) > 0$ *satisfying the following property. For* N *large enough, if* $w_j \leq \psi b$ *for* $1 \leq j \leq N$*, then* $v^\omega \leq (2 - \epsilon)v^Z$.

1.1 Remarks on disjunctive representations of convex hulls

Here we present some standard concepts related to disjunctions. See [1].

Definition 1.4 *Let* $P = \{x \in \mathbb{R}^N \,:\, Ax \leq b\}$ *be a polyhedron. The* homogenized *version of* P *is the cone* $\breve{P} \doteq \{(x, \alpha) \in \mathbb{R}^N \times \mathbb{R}_+ \,:\, Ax - \alpha b \leq 0\}$.

Definition 1.5 *Let* $P^i \subseteq \mathbb{R}^N$*,* $1 \leq i \leq m$*, be polyhedra. The* sum *of the* P^i *is the polyhedron* $\sum_i P^i \doteq \{x \in \mathbb{R}^N \,:\, \exists x^i \in P^i, 1 \leq i \leq m, \text{ with } x = \sum_{i=1}^m x^i\}$.

Remark: Suppose that for $1 \leq i \leq m$, $P^i = \{x \in \mathbb{R}^N \,:\, A^i x \leq b^i\}$, and $\breve{P}^i$ is the homogenized version of P^i. Then the sum of the $\breve{P}^i$ is the cone consisting of all $(x, \alpha) \in \mathbb{R}^N \times \mathbb{R}_+$ such that for $1 \leq i \leq m$ there exists $x^i \in \mathbb{R}^N$ and $\alpha^i \geq 0$ with:

$$\begin{aligned} A^i x^i \quad &\leq \quad \alpha^i b, \quad \alpha^i \geq 0, \quad 1 \leq i \leq m && (10) \\ (x, \alpha) \quad &= \quad \sum_i (x^i, \alpha^i). && (11) \end{aligned}$$

If we insist that $\alpha = 1$, we obtain $conv(\bigcup_i P^i)$. Thus there is a representation for $conv(\bigcup_i P^i)$ involving the $N + 1$ constraints (11) (in addition to (10), used to represent the $\breve{P}^i$).

1.2 Remarks on solutions to knapsack LPs

Consider an instance of the maximum knapsack problem $\max\{p^T x \,:\, w^T x \leq b, \quad x \in \{0\,,\,1\}^N\}$. Let x^* be an extreme point optimal solution for the continuous relaxation of the problem. Then, assuming $p_1/w_1 \geq p_2/w_2 \ldots \geq p_N/w_N$, x^* has the following structure: there is an index k s.t. $x^*_j = 1$ for $1 \leq j < k$; $x^*_k = w_k^{-1}(b - \sum_{j=1}^{k-1} w_j x_j)$, and $x^*_j = 0$ for $j > k$. The index k is such that $0 < x^*_k \leq 1$. In fact in every extreme point there is at most one fractional coordinate. This fact amounts to folklore; similar versions apply to the minimum knapsack problem and related problems.

In fact note that if we "round down" the vector x^* in the preceding paragraph, we obtain a 0/1 vector feasible for the knapsack problem. And, since $w_k \leq b$, we obtain a different 0/1 vector feasible for the knapsack problem by setting $x_k = 1$ and $x_j = 0$ for $j \neq k$. At least one of the two 0/1 vectors thus constructed achieves at least half the objective value of x^* – in other words, if $w_j \leq b$ for all j then the LP/IP ratio is at most 2. Again, this seems to be a folklore fact.

2 Minimum knapsack problem

In this section we consider problem $\mathcal{KMIN}$ and prove Theorem 1.1. Our construction is inspired by that in [21] and [5] in the context of the set-covering problem, and it relies on disjunctive inequalities. Prior to our main proof, we will first show a simpler result in order to motivate our approach. In what follows we will assume without loss of generality that

$$c_1 \geq c_2 \geq \ldots \geq c_N. \tag{12}$$

For $1 \leq h \leq N$, let P^h denote the polyhedron defined by:

$$\sum_{j=1}^{N} w_j x_j \;\geq\; b, \tag{13}$$

$$x_1 = x_2 = \cdots = x_{h-1} = 0, \;\; x_h = 1, \tag{14}$$

$$0 \leq x_j \leq 1, \;\; h < j \leq N. \tag{15}$$

and write

$$M \;=\; conv\left(P^1 \cup P^2 \cup \cdots P^N\right), \tag{16}$$

$$\bar{v} \;=\; \min\left\{\, c^T x \,:\, x \in M \right\}. \tag{17}$$

Note that M is the projection to $\mathbb{R}^N$ of the feasible set for a system of $O(N^2)$ linear constraints in $O(N^2)$ variables. We have that $x \in M$ for any 0/1 vector x that satisfies (13) and therefore $\bar{v} \leq v^Z$.

Lemma 2.1 *$v^Z < 2\bar{v}$.*

Proof. Let $\bar{x}$ be a solution to the linear program (17). It follows that there exist reals λ_h such that

$$0 \leq \lambda_h, \; (1 \leq h \leq N), \quad \sum_{h=1}^{N} \lambda_h = 1,$$

and, for each $1 \leq h \leq N$ with $\lambda_h > 0$, a vector $x^h \in P^h$, such that

$$\bar{x} \;=\; \sum_{h \,:\, \lambda_h > 0} \lambda_h \, x^h.$$

For $1 \le h \le N$ write

$$v^h \;=\; \min\Big\{\, c^T x \;:\; x \in P^h \Big\}. \tag{18}$$

Suppose $\lambda_h > 0$. It is straightforward to see that there is an optimal solution to (18) with at most one fractional variable. By rounding up this variable we obtain a feasible 0/1 solution to the original min-knapsack problem. We therefore have by (12) and (14)

$$v^Z - v^h \;\; < \;\; c_h \;\; \le \;\; c^T x^h, \tag{19}$$

where the second inequality follows since $x^h_h = 1$ again by (14). Consequently, writing

$$\Lambda \;=\; \{1 \le h \le n \;:\; \lambda_h > 0\}\,,$$

we have

$$\begin{aligned}
v^Z - \bar{v} \;\; &= \;\; \sum_{h \in \Lambda} \lambda_h v^Z \;\; - \;\; \sum_{h \in \Lambda} \lambda_h \, c^T x^h & (20)\\
&\le \;\; \sum_{h \in \Lambda} \lambda_h v^Z \;\; - \;\; \sum_{h \in \Lambda} \lambda_h \, v^h & (21)\\
&= \;\; \sum_{h \in \Lambda} \lambda_h \left(v^Z \; - \; v^h \right) & (22)\\
&< \;\; \sum_{h \in \Lambda} \lambda_h \, c^T x^h \;\; = \;\; \bar{v}. & (23)
\end{aligned}$$

■

2.1 Proof of Theorem 1.1.

We begin with a technical result. Recall that we assume $c_j > 0$ and $0 < w_j \le b$ for $1 \le j \le N$.

Lemma 2.2 *Let $H \ge 1$ be an integer. Suppose $S \subseteq \{1, 2, \ldots, N\}$, and let $0 \le \bar{x}_j \le 1$ ($j \in S$) be given values. Let $c^{max} = \max_{j \in S}\{c_j\}$, $c^{min} = \min_{j \in S}\{c_j\}$.*

(a) Suppose first that

$$\sum_{j \in S} \bar{x}_j \;=\; H.$$

Then there exist 0/1 values $\hat{x}_j$ ($j \in S$) such that

$$\begin{aligned}
\sum_{j \in S} w_j \hat{x}_j \;\; &\ge \;\; \sum_{j \in S} w_j \bar{x}_j, \quad \textit{and} & (24)\\
\sum_{j \in S} c_j \hat{x}_j \;\; &\le \;\; \left(1 - \frac{1}{H} + \frac{c^{max}}{H c^{min}}\right) \sum_{j \in S} c_j \bar{x}_j. & (25)
\end{aligned}$$

(b) Suppose next that

$$\sum_{j \in S} \bar{x}_j \;\ge\; H. \tag{26}$$

Then there exist $0/1$ *values* $\hat{x}_j$ $(j \in S)$ *satisfying (24) and*

$$\sum_{j \in S} c_j \hat{x}_j \ \leq \ \left(1 + \frac{c^{max}}{H c^{min}}\right) \sum_{j \in S} c_j \bar{x}_j. \tag{27}$$

Proof. (a) Let $\breve{x}$ be an extreme point solution to the linear program

$$\begin{aligned} \min \quad & \sum_{j \in S} c_j x_j \\ \text{s.t.} \quad & \sum_{j \in S} w_j x_j \ \geq \sum_{j \in S} w_j \bar{x}_j && (28) \\ & \sum_{j \in S} x_j \ = \ H && (29) \\ & 0 \leq x_j \leq \ 1, \quad \ \forall j \in S. && (30) \end{aligned}$$

Consequently at most two of the values $\breve{x}_j$, $j \in S$, are fractional; but since H is integral either zero or *exactly* two $\breve{x}_j$ are fractional. Thus, we can assume that there are indices $i, k \in S$ with

$$\begin{aligned} & 0 < \breve{x}_i < 1, \quad 0 < \breve{x}_k < 1, \quad \text{and} && (31) \\ & \breve{x}_i + \breve{x}_k = 1. && (32) \end{aligned}$$

Suppose $w_i \geq w_k$. Then we can set $\hat{x}_i = 1$, $\hat{x}_k = 0$, and $\hat{x}_j = \breve{x}_j$ for all other j, thereby obtaining a $0/1$ vector $\hat{x}$ which satisfies (28) while increasing cost by at most

$$c_i \ - \ c_i(\breve{x}_i) - c_k(\breve{x}_k) \ = \ (c_i - c_k)(1 - \breve{x}_i) \ \leq \ c^{max} \ - \ c^{min}.$$

Hence

$$\frac{\sum_{j \in S} c_j \hat{x}_j - \sum_{j \in S} c_j \breve{x}_j}{\sum_{j \in S} c_j \breve{x}_j} \ \leq \ \frac{c^{max} \ - \ c^{min}}{\sum_{j \in S} c_j \breve{x}_j} \ \leq \ \frac{c^{max} \ - \ c^{min}}{H c^{min}}, \tag{33}$$

as desired.

(b) Proceeding in a way similar to (a) (using, instead of (29), $\sum_{j \in S} x_j \geq H$), it can be assumed that either zero, one or two of the $\breve{x}_j$ $(j \in S)$ are fractional. If two are fractional the result is implied by (a). If there is only one fractional $\breve{x}_j$ then rounding up $\breve{x}$ provides a $0/1$ vector $\hat{x}$ that is feasible while increasing the cost by at most c^{max}. Hence

$$\frac{\sum_{j \in S} c_j \hat{x}_j - \sum_{j \in S} c_j \breve{x}_j}{\sum_{j \in S} c_j \breve{x}_j} \ \leq \ \frac{c^{max}}{H \, c^{min}}, \tag{34}$$

as desired. ■

Let $0 < \epsilon < 1$. Define K as the smallest integer such that $(1+\epsilon)^{-K} \leq \epsilon$. Without loss of generality, ϵ is small enough that $K \approx \log(1/\epsilon)/\epsilon$. Write $J = \lceil 1 + 1/\epsilon \rceil$.

In what follows we still assume the ordering (12).

Definition 2.3 *A* **signature** *is an integral* K*-vector* σ *such that* $0 \leq \sigma_i \leq J$ *for* $i = 1, 2, \ldots, K$.

Let $1 \leq h \leq N$. For $k = 1, 2, \ldots, K$, let

$$S^{h,k} \;=\; \Big\{ j \,:\, c_h(1+\epsilon)^{-(k-1)} \;\geq\; c_j \;>\; c_h(1+\epsilon)^{-k} \;\text{ and }\; j > h \Big\}.$$

[Note: the "and" is redundant when $k > 1$.] For each $1 \leq h \leq N$, and each signature σ, define

$$\boldsymbol{P^{h,\sigma}} \;=\; \boldsymbol{\{}\;\; x \in [0,1]^N \;: \quad \sum_{j=1}^{N} w_j x_j \;\geq\; b, \tag{35}$$

$$x_1 = x_2 = \cdots = x_{h-1} = 0, \;\; x_h = 1, \tag{36}$$

$$\sum_{j \in S^{h,k}} x_j \;=\; \sigma_k, \quad \forall k \;\text{ such that } \sigma_k < J, \tag{37}$$

$$\sum_{j \in S^{h,k}} x_j \;\geq\; J, \quad \forall k \;\text{ such that } \sigma_k = J \tag{38}$$

$$\boldsymbol{\}.}$$

Note: the sets $S^{h,k}$ partition the variables x_j whose cost c_j lie between c_h and ϵc_h; all variables in each set have "nearly" the same cost. In the set $P^{h,\sigma}$ the signature σ *counts* the number of x_j that take value 1 in each $S^{h,k}$. Thus every feasible solution for $\mathcal{KMIN}$ belongs to some $P^{h,\sigma}$.

Lemma 2.4 *For each h and σ with $P^{h,\sigma} \neq \emptyset$, there is a 0/1 vector $\hat{x}^{h,\sigma}$ feasible for $\mathcal{KMIN}$ such that $c^T \hat{x}^{h,\sigma} \;\leq\; (1+\epsilon) \min\left\{c^T x \,:\, x \in P^{h,\sigma}\right\}$. As a result, $v^Z \;\leq\; (1+\epsilon) \min\left\{c^T x \,:\, x \in P^{h,\sigma}\right\}$.*

Proof. Let $\bar{x} \in P^{h,\sigma}$. Set $c^{max} = \max_{j \in S^{h,k}}\{c_j\}$, $c^{min} = \min_{j \in S^{h,k}}\{c_j\}$, and define $\hat{x}^{h,\sigma}$ as follows. First, for each k such that $\sigma_k > 0$, we obtain the values $\hat{x}_j^{h,\sigma}$ for each $j \in S^{h,k}$ by applying Lemma 2.2 with $S = S^{h,k}$; note that when $\sigma_k < J$, then we have

$$1 - \frac{1}{\sigma_k} + \frac{c^{max}}{\sigma_k c^{min}} \;\leq\; \left(1 - \frac{1}{\sigma_k}\right)\frac{c^{max}}{c^{min}} + \frac{c^{max}}{\sigma_k c^{min}} \;=\; \frac{c^{max}}{c^{min}} \;\leq\; 1+\epsilon,$$

by construction of the sets $S^{h,k}$. And if $\sigma_k = J$, we also have

$$\left(1 + \frac{c^{max}}{J\, c^{min}}\right) \;\;\leq\;\; 1+\epsilon, \tag{39}$$

by our choice for J. If on the other hand $\sigma_k = 0$ we set $\hat{x}_j^{h,\sigma} = 0$ for every $j \in S^{h,k}$.

Finally, define $T^h \;=\; \{j \,:\, c_j \leq (1+\epsilon)^{-K} c_h\}$. Thus the $S^{h,k}$, together with T^h, partition $1, \cdots, n$. The problem

$$\min \quad \sum_{j \in T^h} c_j x_j \tag{40}$$

$$s.t. \quad \sum_{j \in T^h} w_j x_j \;\geq\; \sum_{j \in T^h} w_j \bar{x}_j \tag{41}$$

$$0 \leq x_j \leq 1, \quad \forall\, j \in T^h, \tag{42}$$

is a knapsack problem, and hence it has an optimal solution y with at most one fractional variable. We set $\hat{x}_j^{h,\sigma} = \lceil x_j \rceil$ for each $j \in T^h$; thereby increasing cost (from y) by less than $(1+\epsilon)^{-K} c_h \le \epsilon\, c_h$ by definition of K.

In summary,

$$
\begin{aligned}
c^T \hat{x}^{h,\sigma} - c^T \bar{x} &= \sum_{k\,:\,\sigma_k > 0} \left(\sum_{j \in S^{h,k}} c_j \hat{x}_j^{h,\sigma} - \sum_{j \in S^{h,k}} c_j \bar{x}_j \right) + \sum_{j \in T^h} c_j (\hat{x}_j^{h,\sigma} - \bar{x}_j) && (43)\\
&\le \epsilon \sum_{k\,:\,\sigma_k > 0} \sum_{j \in S^{h,k}} c_j \bar{x}_j \; + \; \epsilon\, c_h && (44)\\
&\le \epsilon\, c^T \bar{x}. && (45)
\end{aligned}
$$

Here, (44) follows from Lemma 2.2, and by definition of the sets $S^{h,k}$, and (45) follows from the fact that $\bar{x}_h = 1$, by definition of $P^{h,\sigma}$. ■

Consider the polyhedron

$$
Q = conv \left(\bigcup_{h,\sigma} P^{h,\sigma} \right). \tag{46}
$$

Note that there are at most $(J+1)^K N = O\left((1/\epsilon)^{O(1/\epsilon^2)} N \right)$ polyhedra $P^{h,\sigma}$, and that each $P^{h,\sigma}$ is described by a system with $O(K+N)$ constraints in N variables. Thus, Q is the projection to $\mathbb{R}^N$ of the feasible set for a system with at most

$$
O\left(\left(\frac{1}{\epsilon} \right)^{O(1/\epsilon^2)} N^2 \right) \text{ constraints in } O\left(\left(\frac{1}{\epsilon} \right)^{O(1/\epsilon^2)} N^2 \right) \text{ variables.} \tag{47}
$$

Furthermore, any 0/1 vector x that is feasible for the knapsack problem satisfies $x \in P^{h,\sigma}$ for some h and σ; in other words, Q constitutes a valid relaxation to the knapsack problem.

Lemma 2.5 $v^Z \le (1+\epsilon) \min\{c^T x \,:\, x \in Q\}$.

Proof. Let $\tilde{x} \in Q$. Then there exist reals $\lambda_{h,\sigma}$ (for each $1 \le h \le N$ and signature σ) such that

$$
0 \le \lambda_{h,\sigma}, \ \forall\, h \text{ and } \sigma, \quad \text{and} \quad \sum_h \sum_\sigma \lambda_{h,\sigma} = 1,
$$

and, for each h and σ with $\lambda_{h,\sigma} > 0$, a vector $x^{h,\sigma} \in P^{h,\sigma}$, such that

$$
\tilde{x} = \sum_{h,\sigma\,:\,\lambda_{h,\sigma} > 0} \lambda_{h,\sigma}\, x^{h,\sigma}.
$$

Let $\Lambda = \{(h,\sigma) \,:\, \lambda_{h,\sigma} > 0\}$. Then

$$\begin{aligned}
v^Z - c^T \tilde{x} &= \sum_{(h,\sigma)\in\Lambda} \lambda_{h,\sigma}\, v^Z \;-\; \sum_{(h,\sigma)\in\Lambda} \lambda_{h,\sigma}\; c^T x^{h,\sigma} && (48)\\
&= \sum_{(h,\sigma)\in\Lambda} \lambda_{h,\sigma}\, \left(v^Z \;-\; c^T x^{h,\sigma}\right) && (49)\\
&\leq \sum_{(h,\sigma)\in\Lambda} \lambda_{h,\sigma}\, \left(\epsilon\, c^T x^{h,\sigma}\right) \quad \text{(by Lemma 2.4)} && (50)\\
&= \epsilon\, c^T \tilde{x}, && (51)
\end{aligned}$$

as desired. ■

3 Single node, fixed-charge sets

In this section we consider problem as $\mathcal{FXN}$ given in the introduction; for convenience its formulation is repeated here:

$$\begin{aligned}
v^Z \;=\; \min \sum_{j=1}^{n} \left(f_j x_j \,+\, c_j y_j\right), && (52)\\
\sum_{j\,\in\,\delta^+} y_j \;-\; \sum_{j\,\in\,\delta^-} y_j \;=\; b, && (53)\\
0 \leq y_j \leq u_j\, x_j \quad \forall j, && (54)\\
x_j = 0 \text{ or } 1, \quad \forall j. && (55)
\end{aligned}$$

For $1 \leq j \leq n$, write $\kappa_j = f_j + c_j u_j$. We assume that the arcs have been labeled so that

$$\kappa_j \;\geq\; \kappa_{j+1}, \quad \text{for} \;\; 1 \leq j < n.$$

As discussed in the introduction, we assume that the vectors c, u, f are all nonnegative. We say that a vector (x, y) is *efficient* if it is a mixed-integer extreme-point feasible solution to $\mathcal{FXN}$ and $y_j > 0$ whenever $x_j = 1$, for $1 \leq j \leq n$. Note that under the assumption $f \geq 0$, any feasible instance of $\mathcal{FXN}$ has an efficient optimal solution.

In the following discussion, Lemmas 3.1 and 3.2 are used to set up the disjunctions that we will use to prove Theorem 1.2. Suppose (x, y) is feasible for $\mathcal{FXN}$. An arc j with $x_j = 1$ is *1-tight* if $y_j = u_j x_j$; and if $x_j = 1$ but $0 < y_j < u_j x_j$ we say j is *slack*. The following result is routine.

Lemma 3.1 *In any extreme point solution to (53)-(55) there is at most one slack arc.* ■

Denote by Π the set of integer pairs (i, h) with $i \neq h$, and $0 \leq i \leq n$, $1 \leq h \leq n + 1$. For each $(i, h) \in \Pi$, consider the polyhedron $D^{i,h} \subseteq \mathbb{R}^n_+ \times \mathbb{R}^n_+$ given by:

$$\sum_{j\,\in\,\delta^+} y_j \;-\; \sum_{j\,\in\,\delta^-} y_j \;=\; b, \qquad (56)$$

$y_j = x_j = 0,\;\; \forall\, j \neq i$ with $1 \leq j < h,$ (57)

if $1 \leq i$ **then** $\;x_i = 1\;$ and $\;0 \leq y_i \leq u_i,$ (58)

if $h \leq n$ **then** $x_h = 1$ and $0 \leq x_j \leq 1,\;\; y_j = u_j x_j,\;\; \forall j$ with $h \leq j \leq n$ and $j \neq i.$ (59)

As a consequence of Lemma 3.1 we have:

Lemma 3.2 *Let $(\hat{x}, \hat{y})$ be an efficient optimal solution to $\mathcal{FXN}$. Then there exists a pair $(i, h) \in \Pi$ such that $(\hat{x}, \hat{y}) \in D^{i,h}$.*

Proof. If $(\hat{x}, \hat{y}) = (0, 0)$ then set $i = 0$ and $h = n+1$. Otherwise, we have $\hat{x} \neq 0$. If there is a unique index j with $\hat{x}_j = 1$, then set $i = j$ and $h = n + 1$, and we are done. Otherwise there is at least one 1-tight arc (because of efficiency and Lemma 3.1); let h be the minimum index of a 1-tight arc. Set i to the index of the slack arc, if such an arc exists, and set $i = 0$ otherwise. ∎

Remarks on the set $D^{i,h}$:

(a) If $i = 0$ and $h = n + 1$, then by (57) we have $y_j = x_j = 0$ for all j; thus the polyhedron is empty unless $b = 0$.

(b) If $1 \leq i$ and $h = n + 1$, then $y_j = x_j = 0$ $\forall j \neq i$; thus in order to satisfy (56) y_i must take value b (if $i \in \delta^+$) or $-b$ (if $i \in \delta^-$).

(c) Consider a *mixed-integer* point $(x, y) \in D^{i,h}$. If $h \leq n$ then h is 1-tight. If $i \geq 1$, then i can be slack. If $h \leq n$ and either (1) $i = 0$, or (2) i not 1-tight and $1 \leq i < h$, or (3) $h < i$, then h is the minimum index of a 1-tight arc.

(d) In constructing $D^{i,h}$, the only variables *not* fixed are y_i (if $1 \leq i$) and y_j, x_j, for $h < j \leq n$ with $j \neq i$. In the second case we have $y_j = u_j x_j$. Thus, $D^{i,h}$ may be restated as the set of solutions to a system of the form

$$\rho_i y_i + \sum_{j \in \delta^+ \setminus i \,:\, j > h} u_j x_j - \sum_{j \in \delta^- \setminus i \,:\, j > h} u_j x_j = b - \rho_h u_h, \tag{60}$$

$$0 \leq y_i \leq u_i, \tag{61}$$

$$0 \leq x_j \leq 1, \quad h < j \leq n \text{ and } j \neq i. \tag{62}$$

Here, the term $\rho_i y_i$ is omitted if $i = 0$, and otherwise $\rho_i = 1$ if $i \in \delta^+$ and $\rho_i = -1$ if $i \in \delta^-$; similarly with ρ_h.

Now we turn to solving $\mathcal{FXN}$. If $b = 0$, then $x = y = 0$ is optimal for $\mathcal{FXN}$ since we assume $f, c \geq 0$, so in what follows we will assume $b \neq 0$. Let $0 < \epsilon < 1$. In order to obtain a formulation for $\mathcal{FXN}$ with MIP/LP ratio at most $1 + \epsilon$, we apply the technique used in Section 2.1.

As before, we define K as the smallest integer such that $(1+\epsilon)^{-K} \leq \epsilon$ and $J = \lceil 1 + 1/\epsilon \rceil$. Given a pair $(i, h) \in \Pi$ with $h \leq n$, then for $k = 1, 2, \ldots, K$ let

$$S^{i,h,k} = \left\{ j \,:\, \kappa_h (1+\epsilon)^{-(k-1)} \geq \kappa_j > \kappa_h (1+\epsilon)^{-k} \text{ with } j > h \text{ and } j \neq i \right\}. \tag{63}$$

For each pair $(i, h) \in \Pi$ with $h \leq n$ and each signature σ (as per Definition 2.3) define the polyhedron $D^{i,h,\sigma}$ as the intersection of $D^{i,h}$ with the set of vectors satisfying the following constraints:

$$\sum_{j \in S^{i,h,k}} x_j = \sigma_k, \quad \forall k \text{ such that } \sigma_k < J, \tag{64}$$

$$\sum_{j \in S^{i,h,k}} x_j \geq J, \quad \forall k \text{ such that } \sigma_k = J. \tag{65}$$

Lemma 3.3 *For each $(h,i) \in \Pi$ with $h \le n$ and σ with $D^{i,h,\sigma} \neq \emptyset$, there is a vector $(\hat{x}^{i,h,\sigma}, \hat{y}^{i,h,\sigma})$ feasible for $\mathcal{FXN}$ such that $c^T\hat{x}^{i,h,\sigma} + f^T\hat{y}^{i,h,\sigma} \le (1+\epsilon)\min\left\{c^Tx + f^Ty \,:\, (x,y) \in D^{i,h,\sigma}\right\}$.*

Proof: Let $(\bar{x},\bar{y})$ be an optimal extreme point solution to $\min\left\{c^Tx + f^Ty \,:\, (x,y) \in D^{i,h,\sigma}\right\}$. Consider an index k with $0 < \sigma_k < J$ (the case of an index k with $\sigma_k \ge J$ is similarly handled and will be skipped). We will show how to obtain $\{(\hat{x}_j, \hat{y}_j) \,:\, j \in S^{i,h,k}\}$ satisfying

$$\hat{x}_j = 0 \text{ or } 1, \qquad 0 \le \hat{y}_j \le u_j\hat{x}_j, \quad \forall\, j \in S^{i,h,k} \tag{66}$$

$$\sum_{j \,\in\, \delta^+ \cap S^{i,h,k}} \hat{y}_j \;-\; \sum_{j \,\in\, \delta^- \cap S^{i,h,k}} \hat{y}_j \;=\; \bar{b}^k \;\doteq\; \sum_{j \,\in\, \delta^+ \cap S^{i,h,k}} \bar{y}_j \;-\; \sum_{j \,\in\, \delta^- \cap S^{i,h,k}} \bar{y}_j, \tag{67}$$

$$\sum_{j \in S^{i,h,k}} (c_j\hat{y}_j + f_j\hat{x}_j) \;\le\; (1+\epsilon) \sum_{j \in S^{i,h,k}} (c_j\bar{y}_j + f_j\bar{x}_j). \tag{68}$$

This will yield the Lemma, since the sets $S^{i,h,k}$ partition $\{j \,:\, j > h \text{ and } j \neq i\}$.

As per Remark (d) above (see equations (60)-(62)) it follows that $\{\bar{x}_j \,:\, j \in S^{i,h,k}\}$ must be an *extreme point* of a set of the form

$$\sum_{j \,\in\, \delta^+ \cap S^{i,h,k}} u_jx_j \;-\; \sum_{j \,\in\, \delta^- \cap S^{i,h,k}} u_jx_j \;=\; \breve{b}^k, \tag{69}$$

$$\sum_{j \in S^{i,h,k}} x_j \;=\; \sigma_k, \tag{70}$$

$$0 \le x_j \le 1 \quad \forall\, j \in S^{i,h,k}, \tag{71}$$

for appropriate $\breve{b}^k$. Thus $\bar{x}$ is either integral, or there exist indices $p \neq q$ in $S^{i,h,k}$ with $0 < \bar{x}_p < 1$, $0 < \bar{x}_q < 1$ and $\bar{x}_p + \bar{x}_q = 1$.

Assume first that $p \in \delta^+$ and $q \in \delta^-$. Also assume that $\bar{y}_q \le \bar{y}_p$ (the other case is symmetric). Then, setting

$$\hat{x}_p \leftarrow 1, \qquad \hat{y}_p \leftarrow \bar{y}_p - \bar{y}_q, \tag{72}$$

$$\hat{x}_q \leftarrow 0, \qquad \hat{y}_q \leftarrow 0, \tag{73}$$

and $(\hat{y}_j, \hat{x}_j) = (\hat{y}_j, \hat{x}_j)$ for all other indices $j \in S^{i,h,k}$, we satisfy (66) and (67). Moreover, the cost change is (using $c \ge 0$ and (59))

$$f_p(1-\bar{x}_p) - c_p\bar{y}_q - f_q\bar{x}_q - c_q\bar{y}_q \;<\; \kappa_p(1-\bar{x}_p) - \kappa_q\bar{x}_q = (\kappa_p - \kappa_q)(1-\bar{x}_p) \tag{74}$$

$$< \; (\kappa_p - \kappa_q) \tag{75}$$

$$\le \; \kappa_h(1+\epsilon)^{-k}\epsilon \qquad \text{(by (63))} \tag{76}$$

$$\le \; \epsilon\; \sigma_k^{-1} \sum_{j \in S^{i,h,k}} \kappa_j\bar{x}_j \qquad \text{(by (63) and (64))} \tag{77}$$

$$= \; \epsilon\; \sigma_k^{-1} \sum_{j \in S^{i,h,k}} (c_j\bar{y}_j + f_j\bar{x}_j), \tag{78}$$

which is (68), as desired.

Assume next that $p, q \in \delta^+$ or $p, q \in \delta^-$. Without loss of generality, assume that $u_p \geq u_q$. Then we set

$$\hat{x}_p \leftarrow 1, \qquad \hat{y}_p \leftarrow \bar{y}_p + \bar{y}_q, \tag{79}$$

$$\hat{x}_q \leftarrow 0, \qquad \hat{y}_q \leftarrow 0. \tag{80}$$

This satisfies (66), because $\bar{y}_q = u_q \bar{x}_q = u_q(1 - \bar{x}_p) \leq u_p(1 - \bar{x}_p)$. Now an analysis similar to that leading to equation (78) proves (68). ■

In summary, as a corollary to Lemma 3.3, we obtain

Corollary 3.4 *If* $v^Z \leq (1 + \epsilon) \min \left\{ c^T x + f^T y \,:\, (x, y) \in conv\left(\bigcup_{i,h,\sigma} D^{i,h,\sigma} \right) \right\}$.

To complete this section, we note that each system $D^{i,h}$ requires $O(n)$ constraints and variables; furthermore there are $O(n^2)$ such systems. Thus, overall, we need $O\left((1/\epsilon)^{O(1/\epsilon^2)}\, n^3 \right)$ variables and constraints so as to formulate $conv\left(\bigcup_{i,h,\sigma} D^{i,h,\sigma} \right)$, as claimed in Theorem 1.2.

4 Maximum knapsack

In this section we will present the results on the maximum knapsack problem. First we will show that using valid inequalities with "small" coefficients then the LP/IP ratio can remain arbitrarily close to 2. Then we we will discuss our use of disjunctions, and finally we will provide our analysis of clique inequalities in the case that the coefficients w_j are not "large".

4.1 Valid inequalities with small coefficients

Lemma 4.1 *Let* $0 < \delta < \pi < 1$. *Consider the knapsack instance with* $N = n + 1$ *where*

$$p_1 = p_2 = \ldots = p_n = 1,\; p_{n+1} = n, \tag{81}$$

$$w_1 = w_2 = \ldots = w_n = 1,\; w_{n+1} = n^2 - \lfloor n^{1-\pi+\delta} \rfloor, \tag{82}$$

$$b = n^2. \tag{83}$$

Consider the point $\hat{x}$ *with* $\hat{x}_j = 1 - n^{-\delta}$ *for* $1 \leq j \leq n + 1$. *Then for* n *large enough,* $\hat{x}$ *is feasible for the continuous relaxation of* $\boldsymbol{\mathcal{KMAX}}$ *and satisfies each valid inequality*

$$\sum_{j=1}^{n+1} \alpha_j x_j \leq \beta \tag{84}$$

for the knapsack polytope defined by (81)-(83), where $\alpha_j \in \{0, 1, \ldots, \lfloor n^{1-\pi} \rfloor\}$ *for* $1 \leq j \leq n + 1$.

Proof. The first assertion follows because

$$n(1 - n^{-\delta}) + (n^2 - \lfloor n^{1-\pi+\delta} \rfloor)(1 - n^{-\delta}) < n + n^2 - n^{2-\delta} < n^2, \tag{85}$$

for n large enough, since $\delta < 1$. To prove the second, consider an inequality (84), and let B denote the sum of the $\lfloor n^{1-\pi+\delta} \rfloor$ largest α_j chosen among the indices $1 \leq j \leq n$. Then, without loss of generality,

$$\beta = \max \left\{ \sum_{j=1}^{n} \alpha_j \,,\, B + \alpha_{n+1} \right\}.$$

If fewer than $n^{1-\pi+\delta}$ coefficients α_j with $1 \leq j \leq n$ are positive then $\sum_{j=1}^{n+1} \alpha_j \hat{x}_j < B + \alpha_{n+1}$ and we are done. In the other case

$$\sum_{j=1}^{n+1} \alpha_j \hat{x}_j \; < \; \sum_{j=1}^{n} \alpha_j \left(1 - n^{-\delta}\right) + n^{1-\pi} \; \leq \; \sum_{j=1}^{n} \alpha_j \; - \; n^{-\delta} n^{1-\pi+\delta} + n^{1-\pi} \; = \; \sum_{j=1}^{n} \alpha_j,$$

as desired. ■

4.2 Using disjunctions

Let $r = \frac{\sqrt{19}-2}{3}$. In this section we describe a simple disjunction which is guaranteed to result in an LP value at most $(1+r)v^Z \approx 1.79 v^Z$. Without loss of generality, assume that the optimal solution to the continuous relaxation of $\mathcal{KMAX}$ has value 2. Thus, $v^Z \geq 1$.

Suppose $p_j \geq 2/(1+r)$ for some j; since $w_j \leq b$ the solution with $x_j = 1$ and $x_i = 0$ for all $i \neq j$ is feasible and we are done. We assume therefore that $p_j < 2/(1+r)$ for all j. Define

$$\Omega \; = \; \{ j \, : \, p_j \geq r \}, \;\text{ and }\; \tilde{w} \; = \; \min\{ w_j \, : \, j \in \Omega \}.$$

Let $j^* \in \Omega$ be such that $w_{j^*} = \tilde{w}$ (if $\Omega = \emptyset$ j^* will be irrelevant). We have that

$$\mathcal{KMAX} \;\; \subseteq \;\; conv(\mathcal{L}^2 \cup \mathcal{L}^1 \cup \mathcal{L}^0), \tag{86}$$

where $\mathcal{L}^i$, $0 \leq i \leq 2$ are the following convex polyhedra. First, $\mathcal{L}^2$ is the set of solutions to the system:

$$\sum_{j=1}^{N} w_j x_j \; \leq \; b, \tag{87}$$

$$\sum_{j \in \Omega} x_j \; \geq \; 2, \tag{88}$$

$$0 \leq x_j \leq 1, \quad \forall \, j. \tag{89}$$

Similarly, $\mathcal{L}^1$ is the set of solutions to the system:

$$\sum_{j=1}^{N} w_j x_j \; \leq \; b, \tag{90}$$

$$\sum_{j \in \Omega} x_j \; = \; 1, \tag{91}$$

$$x_j \; = \; 0, \quad \forall \, j \notin \Omega \text{ with } w_j + \tilde{w} > b, \tag{92}$$

$$0 \leq x_j \leq 1, \quad \forall \, j. \tag{93}$$

Finally, $\mathcal{L}^0$ is the set of solutions to the system:

$$\sum_{j=1}^{N} w_j x_j \; \leq \; b, \tag{94}$$

$$x_j \; = \; 0 \;\; \forall \, j \in \Omega, \tag{95}$$

$$0 \leq x_j \leq 1, \quad \forall \, j. \tag{96}$$

It is clear that (86) holds. Further, we can separate from $conv(\mathcal{L}^2 \cup \mathcal{L}^1 \cup \mathcal{L}^0)$ in polynomial time. Now, if $\mathcal{L}^2 \neq \emptyset$, there exist distinct $i(1), i(2) \in \Omega$ with $w_{i(1)} + w_{i(2)} \leq b$ (e.g. the two indices in Ω with smallest w_j). In that case, $v^Z \geq 2r$, and so the LP to IP ratio is at most

$$\frac{2}{2r} \leq 1 + r. \tag{97}$$

In what follows we will assume $\mathcal{L}^2 = \emptyset$, and show that

$$\max \left\{ \sum_j p_j x_j \, : \, x \in \mathcal{L}^k \right\} \;\leq\; (1+r)v^Z \quad \text{for } k \,=\, 0, 1, \tag{98}$$

as desired.
Consider first $k = 1$. Let $\hat{x}$ be an optimal solution to $\max \left\{ \sum_j p_j x_j \, : \, x \in \mathcal{L}^1 \right\}$, and suppose that $\sum_j p_j \hat{x}_j > 1 + r$ (if not, we are done since $v^Z \geq 1$). Since

$$\sum_{j \in \Omega} p_j \hat{x}_j \leq \max_{j \in \Omega} p_j \tag{99}$$

(by (91)), and

$$\sum_{j \in \Omega} w_j \hat{x}_j \geq \tilde{w} \tag{100}$$

(by (91) and the definition of $\tilde{w}$), we have, writing $\Gamma = \{ j \notin \Omega, \, : \, w_j + \tilde{w} \leq b \}$, and using (92),

$$\sum_{j \in \Gamma} p_j \hat{x}_j \geq 1 + r - \frac{2}{1+r}, \text{ and} \tag{101}$$

$$\sum_{j \in \Gamma} w_j \hat{x}_j \leq b - \tilde{w}. \tag{102}$$

Hence there is a set $S \subseteq \Gamma$ with

$$\sum_{j \in S} p_j \geq \frac{1}{2} \left(1 + r - \frac{2}{1+r} \right), \text{ and} \tag{103}$$

$$\sum_{j \in S} w_j \leq b - \tilde{w}. \tag{104}$$

Therefore, setting $x_j = 1$ if $j \in S \cup \{j^*\}$, and $x_j = 0$ otherwise, yields a feasible solution to the knapsack problem with value at least

$$\frac{1}{2} \left(1 + r - \frac{2}{1+r} \right) + r \;\geq\; \frac{2}{1+r} \tag{105}$$

as a simple calculation shows, as desired.

Next we consider $\mathcal{L}^0$. Clearly, $\max \left\{ \sum_j p_j x_j \, : \, x \in \mathcal{L}^0 \right\}$ is simply the continuous relaxation of a knapsack problem. As pointed out in Section 1.2, there is an optimal solution $\tilde{x}$ to this problem with the following structure: for some set S, $\tilde{x}_j = 1$ for all $j \in S$; $0 < x_k < 1$ for at most one additional index k (note that $k \notin \Omega$, by (95)), and $x_j = 0$ otherwise. Hence, the value of the relaxation is strictly less than

$$v^Z + \max_{j \notin \Omega} \{p_j\} < v^Z + r \leq (1+r)v^Z,$$

as desired. ■

4.3 Knapsacks with small coefficients

In this section we prove Theorem 1.3. Let $0 \le \psi < 1$ be given. Consider the linear programming relaxation of $\mathcal{KMAX}$,

$$\begin{aligned} v^* &= \max \sum_{j=1}^{N} p_j x_j, \\ s.t. \quad & \sum_{j=1}^{N} w_j x_j \ \le \ b, \qquad (106) \\ & x \in [0\,,\,1]^N. \qquad (107) \end{aligned}$$

We obtain an optimal solution x^* to this linear program as in Section 1.2; this is repeated here for convenience. Assume without loss of generality that $p_1/w_1 \ge p_2/w_2 \ge \ldots \ge p_N/w_N$. Then, for some integer $n \ge 1$, we have

$$\begin{aligned} x_j^* &= 1, \ \text{ for } \ 1 \le j \le n, \\ x_{n+1}^* &= \frac{b - \sum_{j=1}^n w_j}{w_{n+1}}, \\ x_j^* &= 0, \ \text{ for } \ n+1 < j \le N. \end{aligned}$$

If $x_{n+1}^* = 0$ or 1 then $v^Z = v^*$ and there is nothing left to prove. We will assume $0 < x_{n+1}^* < 1$.

Write $\lambda = \max\{\psi, 1/2\}$ and $\kappa = \kappa(\psi) = 2\lambda - 1$; thus $0 \le \kappa < 1$ and $\kappa(1/2) = 0$. We choose

$$\epsilon \ = \ \epsilon(\psi) \ = \ \min\left\{ \frac{1-\kappa}{4(1+\kappa)}\,,\, \frac{1}{137} \right\}.$$

Note that $0 < \epsilon < 1 - \kappa$. Further, $\epsilon \to 0^+$ as $\psi \to 1^-$, and monotonically so when $\psi \ge 1/2$. In other words, our upper bound on the LP/IP ratio is at most $2 - 1/137$ for $\psi \le 1/2$, and converges monotonically from below to 2 as ψ approaches 1. This simply restates that the fact (observed above) that "large" coefficients produce more difficult knapsacks.

Our proof of Theorem 1.3 will proceed in a number of steps. We will assume by contradiction that $v^\omega > (2-\epsilon)v^Z$. Without loss of generality, we will assume that the p_j have been scaled so that $v^* = 2$ (and thus $v^Z > 1$). Likewise, we will assume the w_j have been scaled so that $b = 2$. We next prove some structural results (Lemma 4.2 through Lemma 4.5) that follow from these assumptions.

Lemma 4.2 *(a)* $\max\{\ \sum_{j=1}^n p_j\ ,\ \max_j\{p_j\}\ \} < 1 + \epsilon$. *(b)* $\min\{\ \sum_{j=1}^n p_j\ ,\ p_{n+1}\ \} > 1 - \epsilon$. *(c)* $x_{n+1}^* > 1 - 2\epsilon$.

Proof. (a) Assume that $p_k \ge 1 + \epsilon$ for some k. The solution with $x_k = 1$ and $x_j = 0$ for all other j is feasible, and thus

$$\frac{v^*}{v^Z} \ \le \ \frac{2}{1+\epsilon} \ \le \ 2 - \epsilon, \qquad (108)$$

(since $0 \le \epsilon \le 1$) a contradiction. Similarly, $\sum_{j=1}^n p_j < 1 + \epsilon$. To prove (b), note that

$$2 \ = \ \sum_{j=1}^n p_j \ + \ x_{n+1}^*\, p_{n+1} \ < \ \sum_{j=1}^n p_j \ + \ 1 + \epsilon, \qquad (109)$$

yielding the bound on $\sum_{j=1}^{n} p_j$. The bound on p_{n+1} is similarly obtained from the first equation in (109). This equation also yields

$$x_{n+1}^* = \frac{2 - \sum_{j=1}^{n} p_j}{p_{n+1}} > \frac{1-\epsilon}{1+\epsilon} > 1 - 2\epsilon, \tag{110}$$

thereby proving (c). ■

Write $\Delta = w_{n+1} - x_{n+1}^* w_{n+1}$. By definition of κ, $\lambda b = 2\lambda = 1 + \kappa$, so by Lemma 4.2(c), $\Delta < 2\epsilon\ w_{n+1} \leq 2\epsilon\lambda b = 2\epsilon(1+\kappa)$. Also, note that $2 = \sum_{j=1}^{n} w_j \,+\, x_{n+1}^* w_{n+1}$, so

$$\sum_{j=1}^{n} w_j \,-\, \Delta \,=\, b - x_{n+1}^* w_{n+1} \,-\, \Delta \,=\, b - w_{n+1} \geq 0. \tag{111}$$

Define

$$j^* = \min\left\{ i\, :\, \sum_{j=1}^{i} w_j \,\geq\, \sum_{j=1}^{n} w_j \,-\, \Delta \right\}.$$

Lemma 4.3 *(a)* $\sum_{j=1}^{j^*-1} p_j < 2\epsilon$. *(b)* $\sum_{j=j^*+1}^{n} p_j < 2\epsilon$.

Proof. (a) The vector x with $x_1 = \ldots = x_{j^*-1} = 1$, $x_{n+1} = 1$, and $x_j = 0$ for all other j, is feasible, by definition of j^*, which implies $\sum_{j=1}^{j^*-1} p_j \,+\, p_{n+1} < 1 + \epsilon$. Together with Lemma 4.2 this yields the desired result. (b) The vector x with $x_{j^*+1} = \ldots = x_n = 1$, $x_{n+1} = 1$, and $x_j = 0$ for all other j, is feasible, because (since $w_{n+1} \leq \lambda b = 1 + \kappa$),

$$\sum_{j=j^*+1}^{n} w_j \,+\, w_{n+1} \leq \Delta \,+\, w_{n+1} \leq 2\epsilon(1+\kappa) \,+\, 1 + \kappa \,=\, (1+\kappa)(1+2\epsilon) \,\leq\, 2, \tag{112}$$

by definition of ϵ. Hence, we must have $\sum_{j=j^*+1}^{n} p_j \,+\, p_{n+1} < 1 + \epsilon$ and we conclude as in (a). ■

Corollary 4.4 $p_{j^*} \geq 1 - 5\epsilon$ *and* $w_{j^*} + w_{n+1} > 2$.

Proof. Lemma 4.2 (b) and Lemma 4.3 yield the bound on p_{j^*}. If $w_{j^*} + w_{n+1} \leq 2$ then

$$\frac{v^*}{v^Z} \leq \frac{2}{2 - 6\epsilon} < 2 - \epsilon,$$

a contradiction. ■

Lemma 4.5 *(a)* $w_{n+1} > 1 - \epsilon$. *(b)* $w_{j^*} < 1 + 3\epsilon$.

Proof. (a) By our indexing of variables in non-increasing order of values p_j/w_j,

$$\frac{p_1 + \ldots + p_n}{w_1 + \ldots w_n} \,\geq\, \frac{p_{n+1}}{w_{n+1}}, \tag{113}$$

and thus, since $b = 2$,

$$w_{n+1} \,\geq\, \frac{1-\epsilon}{1+\epsilon}(w_1 + \ldots + w_n) \,>\, \frac{1-\epsilon}{1+\epsilon}(2 - w_{n+1}), \tag{114}$$

from which the result follows.

(b) This follows from

$$w_{j^*} \leq w_1 + \ldots + w_n \;=\; 2 - x^*_{n+1} w_{n+1} \;<\; 2 - (1-2\epsilon)(1-\epsilon) \;<\; 1 + 3\epsilon.$$

■

In what follows we consider an arbitrary vector $\hat{x}$ that satisfies $\sum_j w_j \hat{x}_j \leq b$, all clique inequalities, and $0 \leq \hat{x}_j \leq 1$, $\forall j$. We will assume that $\sum_j p_j \hat{x}_j > (2-\epsilon)v^Z$, and show that this leads to a contradiction, thereby proving Theorem 1.3. Define

$$P = \{1 \leq j \leq N \,:\, j > n+1 \text{ and } w_j + w_{j^*} \leq 2\}.$$

Lemma 4.6 *If $\sum_{j \in P} \hat{x}_j p_j > 60\,\epsilon$, there exists $T \subseteq P$ with*

$$\sum_{j \in T} w_j \;\leq\; 2 - w_{j^*} \hat{x}_{j^*}, \quad \textit{and} \tag{115}$$

$$\sum_{j \in T} p_j \;>\; 30\epsilon. \tag{116}$$

Proof. Consider the maximum knapsack problem

$$\tilde{\mathcal{K}}: \quad \max\{\sum_{j \in P} p_j x_j \,:\, \sum_{j \in P} w_j x_j \leq 2 - w_{j^*}\hat{x}_{j^*},\; x \in \{0,1\}^P\}.$$

By construction, the restriction of $\hat{x}$ to indices in P is feasible for the continuous relaxation of $\tilde{\mathcal{K}}$, and has objective larger than 60ϵ. Since the LP/IP ratio for a knapsack problem is not larger than 2, there is a 0/1 vector feasible for $\tilde{\mathcal{K}}$ with objective at least 30ϵ. ■

Lemma 4.7 $\sum_{j \in P} \hat{x}_j p_j \leq 60\,\epsilon$.

Proof. Assume by contradiction that $\sum_{j \in P} \hat{x}_j p_j > 60\epsilon$. The proof will construct a subset $A \subseteq P$ such that

$$w_{j^*} + \sum_{j \in A} w_j \;\leq\; 2, \quad \text{and} \tag{117}$$

$$\sum_{j \in A} p_j \;>\; 6\epsilon. \tag{118}$$

This will provide a contradiction, since in this case setting $x_j = 1$ for every $j \in A \cup \{j^*\}$ yields a feasible solution, with objective (by Corollary 4.4) at least $1+\epsilon$ and thus

$$\frac{v^\omega}{v^Z} \leq \frac{2}{1+\epsilon} < 2 - \epsilon.$$

Let α be defined as follows. If $w_{j^*} \leq 1$, then $\alpha = w_{j^*}$, whereas if $w_{j^*} > 1$ then $\alpha = 2 - w_{j^*} = 1 - (w_{j^*} - 1)$. Note that in either case

$$\alpha \;\leq\; w_{j*}, \tag{119}$$

$$w_{j*} - \alpha \;<\; 6\epsilon, \quad \text{and} \tag{120}$$

$$w_{j*} - \alpha \;<\; \alpha \;\leq\; 2 - w_{j*}. \tag{121}$$

Here (119) holds by construction, and (120) and (121) hold by Lemma 4.5 (b).

Let T be as in Lemma 4.6, and write $T = \{i(1), i(2), \cdots, i(|T|)\}$. For $k = 1, \cdots, |T|$, define a_k and b_k as follows:

$$a_1 = 0, \tag{122}$$

$$b_k = \sum_{j=1}^{k} w_{i(j)}, \quad k = 1, \cdots, |T|, \tag{123}$$

$$a_{k+1} = b_k, \quad k = 1, \cdots, |T| - 1. \tag{124}$$

For $k = 1, \cdots, |T|$, let $I_k = [a_k, b_k]$. Refer to Figure 1.
Then we can partition the intervals I_k into at most 5 disjoint classes (some of which may be empty),

(1) Set of intervals I_k with $b_k \leq \alpha$,

(2) Set of intervals I_k with $a_k \geq w_{j*}$,

(3) Set of intervals I_k with $a_k > \alpha$ and $b_k < w_{j*}$,

(4) An interval I_k with $a_k \leq \alpha$ and $b_k > \alpha$,

(5) Possibly one interval I_k with $a_k < w_{j*}$ and $b_k > w_{j^*}$.

Classes (1), (3) and (5) may be empty. Let A be the class with largest sum of p_j – thus A satisfies (118). We need to show that (117) is satisfied. This is the case if A corresponds to class (1) by the second inequality in (121), or if it corresponds to class (2) since in that case $w_{j^*} + \sum_{j \in A} w_j \leq \sum_{j \in T} w_j \leq 2$ by construction of T. In cases that A corresponds to class (4) or (5), (117) is also satisfied by definition of P. Finally, (117) follows in case (3) again using (121). ■

Lemma 4.3 and Lemma 4.7 imply that

$$\sum_{j=1,\, j \neq j^*}^{n} \hat{x}_j p_j \;+\; \sum_{j \in P} \hat{x}_j p_j \;\leq 64\,\epsilon, \tag{125}$$

In the rest of this section we will show that the remaining terms in $\sum_j \hat{x}_j p_j$ amount to less than $(3/2 + 7/2\ \epsilon) v^Z$. Thus, overall

$$\sum_j \hat{x}_j p_j \;\leq\; 64\epsilon + \left(\frac{3}{2} + \frac{7}{2}\epsilon\right) v^Z \;\leq\; \left(\frac{3}{2} + \frac{135}{2}\epsilon\right) v^Z \;\leq\; (2-\epsilon) v^Z,$$

(by our choice of ϵ) which is the desired contradiction. Note that the remaining terms consist of

- index j^*, and
- indices $j \neq j^*$ with $w_j + w_{j^*} > 2$. Let $\mathcal{I}$ be the set of such indices j.

Our approach will be to upper-bound the sum of remaining terms by the value of a linear program, whose constraints will primarily amount to clique inequalities, restricted to variables x_j with $j \in \mathcal{I} \cup \{j^*\}$.

We partition $\mathcal{I}$ into

$$\mathcal{S} = \{j \in \mathcal{I} \,:\, w_j \,\leq\, 1\} \;\text{ and }\; \mathcal{L} = \{j \in \mathcal{I} \,:\, w_j \,>\, 1\}\,.$$

Lemma 4.8 *Suppose $\mathcal{S} = \emptyset$. Then $\hat{x}_{j^*}\, p_{j^*} \, + \, \sum_{j \in \mathcal{I}} \hat{x}_j p_j \, \leq \, 1 + \epsilon$.*

Proof. By definition of $\mathcal{I}$ and $\mathcal{L}$,

$$\sum_{j \in \mathcal{I} \cup \{j^*\}} x_j \, \leq \, 1,$$

is a clique inequality, and the result follows by Lemma 4.2(a). ■

The remainder of the proof handles the case $\mathcal{S} \neq \emptyset$, and consequently, by definition of $\mathcal{I}$, $w_{j^*} > 1$. Note that for each $j \in \mathcal{I}$ we have $j > n$ and $w_j > 1 - 3\epsilon$ (this by Lemma 4.5(b)). Thus, since $\sum_{j=1}^{N} w_j \hat{x}_j \leq 2$, we also have

$$\sum_{j \, \in \mathcal{I} \cup \{j^*\}} \hat{x}_j < 2/(1 - 3\epsilon) < 2 + 7\epsilon. \tag{126}$$

Also note that if $j \in \mathcal{S}$,

$$p_j \, \leq \, p_j / w_j \, \leq \, p_{j^*} / w_{j^*} \, < \, p_{j^*}. \tag{127}$$

Definition:

- $s(1) = \text{argmax}\{p_j \, : \, j \in \mathcal{S}\}$,
- $\mathcal{L}^1 = \{j \in \mathcal{L} \, : \, w_j + w_{s(1)} > 2\}$, and
- $\mathcal{L}^2 = \mathcal{L} - \mathcal{L}^1$.

Lemma 4.9 *Suppose $|\mathcal{S}| = 1$. Then $\hat{x}_{j^*}\, p_{j^*} \, + \, \sum_{j \in \mathcal{I}} \hat{x}_j p_j \, \leq \, 1 + \epsilon$.*

Proof. Let $\mathcal{S} = \{i\}$. Then $\mathcal{L}^1 = \{j \in \mathcal{L} \, : \, w_i + w_j > 2\}$. The following are clique inequalities:

$$x_{j^*} \, + \sum_{j \in \mathcal{L}^1} x_j \, + \sum_{j \in \mathcal{L}^2} x_j \, \leq \, 1, \tag{128}$$

$$x_{j^*} \, + \sum_{j \in \mathcal{L}^1} x_j \, + \, x_i \, \leq \, 1, \tag{129}$$

and thus, $\hat{x}_{j^*} p_{j^*} \, + \, \hat{x}_i p_i \, + \, \sum_{j \in \mathcal{L}} \hat{x}_j p_j$ is upper-bounded by the value of the linear program

$$\max \, \left\{ p_{j^*} x_{j^*} \, + \, p_i x_i \, + \sum_{j \in \mathcal{L}} p_j x_j \, : \, \text{ s.t. (128)-(129), each variable in } [0,1] \right\}. \tag{130}$$

We conclude that

$$\hat{x}_{j^*}\, p_{j^*} \, + \, \hat{x}_i p_i \, + \sum_{j \in \mathcal{L}} \hat{x}_j p_j \, \leq \, \max \left\{ \max_{k \in \mathcal{L}^1 \cup \{j^*\}} w_k \, , \, w_i \, + \max_{k \in \mathcal{L}^2} \{w_k\} \right\} \, \leq \, v^Z, \tag{131}$$

where the last inequality follows because by definition of $\mathcal{L}^2$, there is an integer feasible solution to $\mathcal{KMAX}$ of value precisely $w_i + \max_{k \in \mathcal{L}^2}\{w_k\}$ (and clearly there is one of value $\max_{k \in \mathcal{L}^1 \cup \{j^*\}} w_k$). ■

In the remainder of the proof we will assume $|\mathcal{S}| \geq 2$. Consider the linear program

$$\theta = \max \sum_{j \in \mathcal{L}^1 \cup \{j^*\}} p_j x_j + \sum_{j \in \mathcal{L}^2} p_j x_j + \sum_{j \in \mathcal{S}} p_j x_j \tag{132}$$

Subject to:

$$(\bar{\alpha}): \quad \sum_{j \in \mathcal{L}^1 \cup \{j^*\}} x_j + \sum_{j \in \mathcal{L}^2} x_j \leq 1, \tag{133}$$

$$(\bar{\beta}): \quad \sum_{j \in \mathcal{L}^1 \cup \{j^*\}} x_j + x_{s(1)} \leq 1, \tag{134}$$

$$(\bar{\gamma}): \quad \sum_{j \in \mathcal{L}^1 \cup \{j^*\}} x_j + \sum_{j \in \mathcal{L}^2} x_j + \sum_{j \in \mathcal{S}} x_j \leq 2 + 7\epsilon, \tag{135}$$

$$x \geq 0. \tag{136}$$

Here, (133) and (134) are clique inequalities, and (135) is the same as (126). Thus,

$$\sum_{j \in \mathcal{L} \cup j^*} \hat{x}_j p_j + \sum_{j \in \mathcal{S}} \hat{x}_j p_j \leq \theta.$$

In the above formulation, we have indicated the names of the dual variables. Next, define:

- $h = \mathrm{argmax}\{p_j \,:\, j \in \mathcal{L}^1 \cup \{j^*\}\}$.
- $s(2) = \mathrm{argmax}\{p_j \,:\, j \in \mathcal{S} - s(1)\}$.

Lemma 4.10 *Suppose* $\left\{j \in \mathcal{L} \,:\, p_j > p_{s(2)}\right\} \subseteq \mathcal{L}^1$. *Then* $\theta \leq \left(\frac{3}{2} + \frac{7}{2}\epsilon\right) v^Z$.

Proof. By construction (and (127)), $p_h > p_{s(1)} \geq p_{s(2)} \geq p_j$ for each $j \in \mathcal{L}^2$. Thus, the following vector is a dual feasible solution to the LP (132)-(136):

$$\bar{\alpha} = 0, \quad \bar{\beta} = p_h - \frac{p_{s(1)} + p_{s(2)}}{2}, \quad \bar{\gamma} = \frac{p_{s(1)} + p_{s(2)}}{2}. \tag{137}$$

The value of this dual feasible solution is

$$p_h + (1 + 7\epsilon)\frac{p_{s(1)} + p_{s(2)}}{2} \leq \left(\frac{3}{2} + \frac{7}{2}\epsilon\right) \max\{\, p_h \,,\, p_{s(1)} + p_{s(2)} \,\}. \tag{138}$$

This conclude s the proof, since we have an integer feasible solution to $\boldsymbol{\mathcal{KMAX}}$ by setting $x_h = 1$ (and all other $x_j = 0$), and another by setting $x_{s(1)} = x_{s(2)} = 1$ and all other $x_j = 0$. ■

Lemma 4.11 *Suppose there exists* $k \in \mathcal{L}^2$ *with* $p_k > p_{s(2)}$. *Then* $\theta \leq \left(\frac{3}{2} + \frac{7}{2}\epsilon\right) v^Z$.

Proof. Without loss of generality we can assume $k = \mathrm{argmax}\{p_j \,:\, j \in \mathcal{L}^2\}$. As previously, $p_h > p_{s(1)}$. Also, since $k \in \mathcal{L}$, $w_k > 1$. Since $\sum_{j=1}^n w_j \leq 2$, and $w_{j^*} > 1$, we therefore have $k > n$ and so

$$p_k/w_k \leq p_{j^*}/w_{j^*}. \tag{139}$$

Further, $s(1) \in \mathcal{I}$ implies $w_{j^*} + w_{s(1)} > 2$. But since $k \in \mathcal{L}^2$, $w_k + w_{s(1)} \leq 2$. Consequently, $w_{j^*} > w_k$, and using (139) we have

$$p_k < p_{j^*} \leq p_h.$$

As a result, if $p_k \le p_{s(1)}$, the following is a dual feasible solution to the LP (132)-(136):

$$\bar{\alpha} = 0, \quad \bar{\beta} = p_h - \frac{p_{s(1)} + p_k}{2}, \quad \bar{\gamma} = \frac{p_{s(1)} + p_k}{2}; \tag{140}$$

and if $p_k > p_{s(1)}$, the following vector is dual feasible:

$$\bar{\alpha} = p_h - \frac{p_{s(1)} + p_k}{2}, \quad \bar{\beta} = 0, \quad \bar{\gamma} = \frac{p_{s(1)} + p_k}{2}. \tag{141}$$

In either case, the value of the solution is

$$p_h + (1 + 7\epsilon)\frac{p_{s(1)} + p_k}{2} \le \left(\frac{3}{2} + \frac{7}{2}\epsilon\right) \max\{\, p_h \,,\, p_{s(1)} + p_k \,\}. \tag{142}$$

This concludes the proof, since we have an integer feasible solution to $\mathcal{KMAX}$ by setting $x_h = 1$ (and all other $x_j = 0$), and another by setting $x_{s(1)} = x_k = 1$ and all other $x_j = 0$. ■

Acknowledgment. We thank Bruce Shepherd for suggesting the questions in Section 4 during a visit to McGill University which was supported by his NSERC grant.

References

[1] E. Balas, Disjunctive Programs: Cutting Planes from Logical Conditions, in O.L. Mangasarian et al., eds., *Nonlinear Programming* **2** (1975) Academic Press, 279 – 312.

[2] E. Balas, S. Ceria and G. Cornuéjols, A lift-and-project cutting plane algorithm for mixed 0-1 programs, *Mathematical Programming* **58** (1993), 295 – 324.

[3] D. Bienstock, Approximate formulations for 0-1 knapsack sets, *Operations Research Letters* **36** (2008), 317–320.

[4] D. Bienstock and N.Özbay, Tree-width and the Sherali-Adams operator, *Discrete Optimization* **1** (2004) 13-22.

[5] D. Bienstock and M. Zuckerberg, Subset algebra lift operators for 0-1 Integer Programming, *SIAM J. Optimization* **15** (2004) 63-95.

[6] D. Bienstock and M. Zuckerberg, Approximate fixed-rank closures of covering problems *Math. Programming* **105** (2006), 9 – 27.

[7] T. Carnes and D. Shmoys, Primal-Dual Schema for Capacitated Covering Problems, *Proc. IPCO 2008*.

[8] R.D. Carr, L.K. Fleischer, V.J. Leung, C.A. Phillips, Strengthening Integrality Gaps for Capacitated Network Design and Covering Problems, Proc. 2000 SODA,

[9] W. Cook and S. Dash, On the matrix-cut rank of polyhedra, *Mathematics of Operations Research* **26** (2001), 19 – 30.

[10] A.M. Frieze and M.R.B. Clarke, Approximation algorithms for the m-dimensional knapsack problem: Worst-case and probabilistic analyses, *Eur. J. of Oper. Research* **15** (1984), 100 – 109.

[11] M.X. Goemans and L. Tunçel, When does the positive semidefiniteness constraint help in lifting procedures?, *Mathematics of Operations Research* **26** (2001), 796 – 815.

[12] M.Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, (1988).

[13] Z. Gu, G. Nemhauser and M. Savelsbergh, Lifted flow cover inequalities for mixed 0/1 integer programs, *Math. Programming* **85** (1999), 439 –467.

[14] A.N. Letchford, On disjunctive cuts for combinatorial optimization, *J. of Comb. Opt.* **5** (2001) 299 –315.

[15] G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization*, Wiley, New York (1988).

[16] M. Padberg, T. Van Roy, and L. Wolsey, Valid linear inequalities for fixed charge problems, *Operations Research* **33** (1985), 842 – 861.

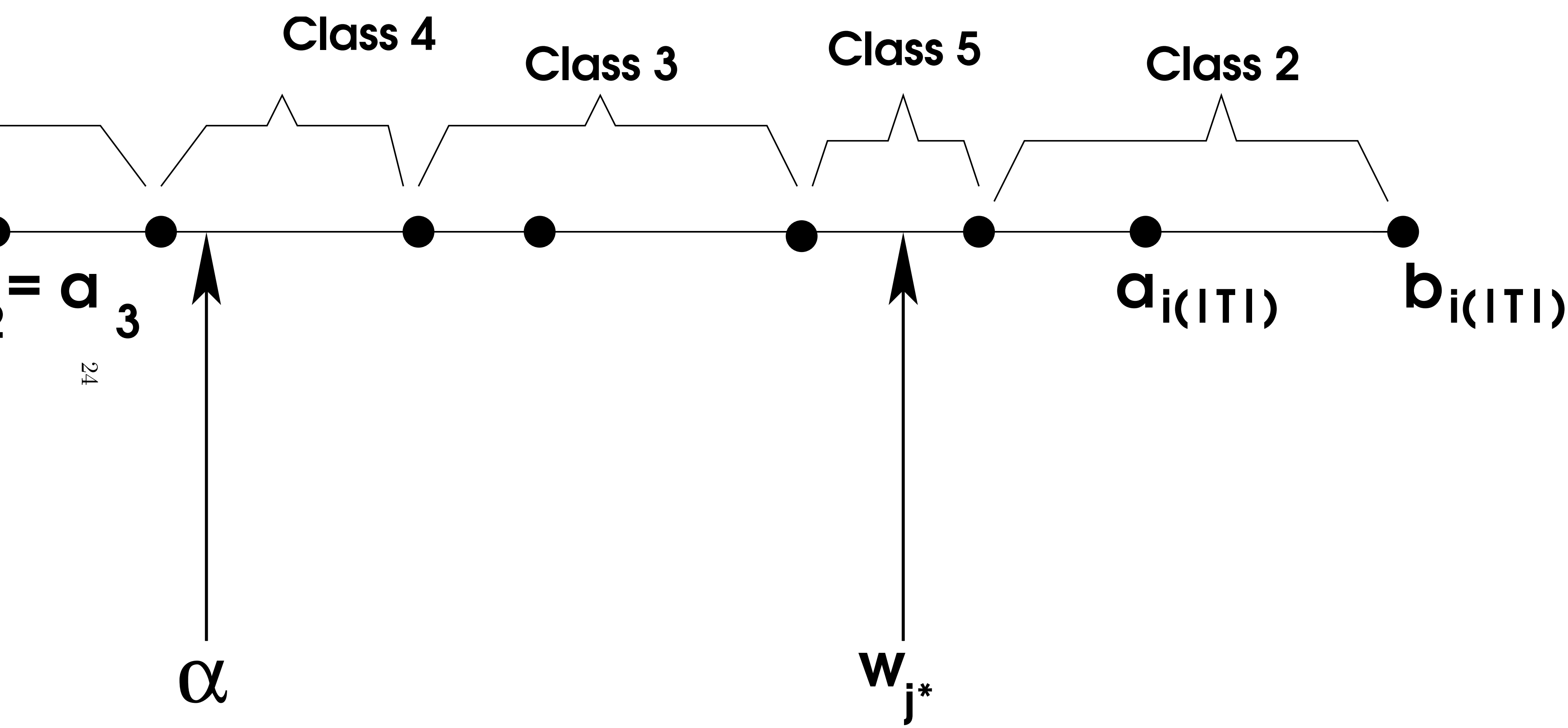
[17] N. Robertson and P.D. Seymour, Graph Minors II: Algorithmic aspects of tree-width, *J. Algorithms* **7** (1986), 309-322.

[18] S. Sherali and W. Adams, A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems, *SIAM J. on Discrete Mathematics* **3** (1990), 411 – 430.

[19] M. Van Vyve, Polynomial relaxation schemes for knapsack, manuscript, February 2005.

[20] M. Van Vyve and L.A. Wolsey, Approximate extended formulations, *Math. Program.* **105** (2006) 501 –522.

[21] M. Zuckerberg, *A Set Theoretic Approach to Lifting Procedures for 0,1 Integer Programming PhD dissertation*, Columbia University (2003).


Class 4
Class 3
Class 5
Class 2
= a 3
a i(|T|)
b i(|T|)
α
w j*