Stochastic models and control for electrical power line temperature

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***Abstract*— In this paper we consider the evolution of the temperature of a power line under stochastic exogenous factors such as ambient temperature. We present a solution to the resulting stochastic heat equation and we propose a number of control algorithms designed to maximize delivered power under chance constraints used to limit the probability that a line exceeds its critical temperature.**

I. INTRODUCTION

When a power line overheats it becomes exposed to a number of risk factors. If the overheating is severe the physical/mechanical attributes of the line may be compromised, rendering it unusable. Under less severe overheating the line may sag, thus bringing it into proximity with other objects, and thereby potentially causing a contact or arc which will trip the line. If overheating is determined, the line will be protectively tripped (be taken out of service). In each of these cases we see a situation where a line is no longer available; as a result, power previously flowing on that line will be transferred to other lines, in a complex fashion (i.e. following laws of physics) possibly congesting those lines and causing them to overheat, as well. In a failure scenario of a transmission system, this sequence of events may result in a *cascade* resulting in a large-scale blackout. The Northeast U.S.-Canada blackout of 2013 produced precisely this type of event, see [10].

The temperature of a power line primarily depends on the amount of (active, or real) power flowing on that line (we refer the reader to [2], [3] or [9] for background on power engineering). However, high-voltage power lines are uninsulated and exposed to numerous exogenous factors, such as in particular wind and ambient temperature, among many. All of these factors can and do influence power line temperature. IEEE Standard 738 [7] amounts to a deterministic codification of the impact of a very large number of such factors, starting from a base model that relies on the classical heat equation. Even though this is a very thorough approach, an examination of the standard highlights the potential for mis-estimation due to erroneous, missing or variable data. The previously mentioned report [10] describes instances during the 2013 cascade where incorrect calibration of a power line lead to unexpected tripping which contributed to system instability. A somewhat more nuanced analysis of power line temperature based on the heat equation is given in [1].

In this paper we model the aggregated impact of uncertainty in exogenous factors using a generic stochastic model. In a previous work [6] we focused on time-dependent stochasticity of exogenous factors. In this paper instead we assume that randomness is primarily of a spatial nature and ignore the time component. This is a reasonable model given the scope of current control practice (with various levels of control applied every fifteen minutes and even more frequently) and the length of typical lines in a transmission system (short lines may measure 50 miles, and long lines much more). A comprehensive model that accounts for short-term exogenous variability over a large spatial domain may prove challenging; though in future work we may address this point. We compute a general solution to the resulting stochastic heat equation, and suggest several control mechanisms relying on so-called "chance constraints" to maximize delivered power while maintaining an acceptable level of risk.

II. FORMULATION

We now focus on a particular power line on the time domain $[0, \tau]$. The line is modeled as a one-dimensional object parameterized by x, $0 \leq x \leq L$. Let

- $I = I(t)$ denote the current of that line at time t, with the dependence on t highlighted so as to allow for control actions.
- $T(x,t)$ the temperature at x at time t.

The *heat equation* states:

$$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2} + \alpha I^2(t) - \nu(T(x,t) - T^{ext}(x,t)), \quad (1)$$

where $\kappa \geq 0$, $\alpha \geq 0$ and $\nu \geq 0$ are (line dependent) constants, and $T^{ext}(x,t)$ is the ambient temperature at (x,t). In order to account for stochasticity, we model $T^{ext}(x,t) = G(\boldsymbol{h(x)})$ where $\boldsymbol{h(x)}$ denotes a random variable at x, with distribution that is either known or can be estimated (in what follows boldface will be used to denote uncertain quantities) and $G(\cdot) > 0$. We thus obtain

$$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2} + \alpha I^2(t) - \nu(T(x,t) - G(\boldsymbol{h(x)})). \quad (2)$$

We will further assume $\kappa = 0$. This is consistent with the use of the heat equation in [7], [1]; it is justified by noting that propagation in the time domain is much faster than in the spatial domain. [We will acount for the randomness of exogenous conditions in the spatial domain in an average,

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or aggregated manner made precise below]. We therefore obtain:

$$\frac{\partial T(x,t)}{\partial t} = \alpha I^2 - \nu(T(x,t) - G(\boldsymbol{h(x)})). \quad (3)$$

Integrating on both sides with respect to x, and dividing by L, we have

$$\frac{1}{L}\int_0^L \frac{\partial T(x,t)}{\partial t}dx \;=\; \alpha I^2(t) - \frac{\nu}{L}\int_0^L T(x,t)dx$$
$$+\frac{\nu}{L}\int_0^L G(\boldsymbol{h(x)})dx. \quad (4)$$

Denoting by $\boldsymbol{H(t)}$ average internal temperature along the line at time t, by R the average ambient temperature along the line, i.e.,

$$\boldsymbol{H(t)} \triangleq \frac{1}{L}\int_0^L T(x,t)dx, \quad \boldsymbol{R} \triangleq \frac{1}{L}\int_0^L G(\boldsymbol{h(x)})dx,$$

we therefore have

$$\frac{d\boldsymbol{H(t)}}{dt} = \frac{d}{dt}\frac{1}{L}\int_0^L T(x,t)dx = \frac{1}{L}\int_0^L \frac{\partial T(x,t)}{\partial t}dx.$$

Then (4) becomes:

$$\frac{d\boldsymbol{H(t)}}{dt} = \alpha I^2(t) - \nu \boldsymbol{H(t)} + \nu \boldsymbol{R}, \quad (5)$$

with solution

$$\boldsymbol{H(t)} \;=\; \quad (6)$$
$$\int_0^t e^{-\nu(t-s)}(\alpha I^2(s) + \nu \boldsymbol{R})ds + Ce^{-\nu t}$$
$$= \int_0^t e^{-\nu(t-s)}\alpha I^2(s)ds + \boldsymbol{R}(1-e^{-\nu t}) + Ce^{-\nu t},$$

where

$$C = \boldsymbol{H(0)} = \frac{1}{L}\int_0^L T(x,0)dx.$$

We note that the quantity $\boldsymbol{R}$ is *not* observed – however we can aasume that its distribution can be estimated). We are interested in control schemes that vary $I(t)$ in response to observed conditions. As criterion for stability, we will enforce the chance-constraint [5], [8]

$$P\left(\max_{t\in[0,\tau]} \boldsymbol{H(t)} > k\right) \le \epsilon, \quad (7)$$

where $k > 0$ is a given limit and $\epsilon > 0$ is small.

III. Constant $I(t), t \in [0, \tau]$

The case where $I(t)$ is constant in the time window of interest is of special interest because of its simplicity. We are interested in computing those values $\breve{I}$ such that setting $I(t) = \breve{I}$ for $0 \le t \le \tau$ satisfies (7). From the closed-form solution above we obtain

$$\boldsymbol{H(t)} = (\alpha/\nu \breve{I}^2 + \boldsymbol{R})(1-e^{-\nu t}) + Ce^{-\nu t} \quad (8)$$

from which it follows that $H'(t) > 0$ for $\breve{I}$ large enough (and negative or zero otherwise). Assuming, thus, that $\boldsymbol{H(0)} < k$, it follows therefore that (7) is equivalent to

$$P(\boldsymbol{H(\tau)} > k) \le \epsilon. \quad (9)$$

Using (8) this implies

$$\breve{I}^2 \;\le\; \frac{\nu}{\alpha}\frac{k - Ce^{-\nu\tau} - r_\epsilon(1-e^{-\nu\tau})}{1-e^{-\nu\tau}}, \quad (10)$$

where r_ϵ is the ϵ-quantile of $\boldsymbol{R}$. As discussed above we assume that the distribution of $\boldsymbol{R}$ is known, and consequently the bound in (10) is computable.

For future use, we denote by $L(\tau, k)$ the right-hand side of expression (10).

Now consider an entire grid where several lines are considered to be thermally stressed. We can then compute the upper bound on the current for each such line that is implied by computation in (10). These values can then be used in e.g. generator dispatching at time $t = 0$.

IV. Adaptive Control

As an enhancement to the analysis above, we consider a setup where additional information becomes available at some point in the time window $[0, \tau]$; this information can then be used to improve the line ratings obtained as in e.g. (10). Define the random variable

$$\boldsymbol{W} \triangleq \boldsymbol{R}(1-e^{-\nu\tau}),$$

and to simplify we assume that W is discretely distributed, $P(\boldsymbol{W} = w_i) = p_i, i = 1, 2, \ldots, n$ with a *k*nown distribution (i.e., the distribution of $\boldsymbol{R}$ is known). We are interested in a control scheme with the following characteristics

1 At time $\tau = 0$, we compute values I_1, and $I_{2,i}$ for $i = 1, 2, \ldots, n$. These values are used as follows
2 In the time window $[0, \tau/2]$ current is set to the constant value I_1.
3 At time $\tau/2$, we observe the value of $\boldsymbol{R}$ and thus of $\boldsymbol{W}$. Assuming $\boldsymbol{W} = w_i$ then current is set, in the interval $[\tau/2, \tau]$, to the constant value $I_{2,i}$.

The values I_1, and $I_{2,i}$, $1 \le i \le m$ are computed according to the following criteria

(a) $P(\boldsymbol{H(\tau)} > k) < \epsilon$.
(b) $I_1 \le L(\tau/2, k)$.
(c) Where $F : \mathbb{R}^2 \to \mathbb{R}_+$ is coordinate-wise monotonely increasing, we want to maximize, in expectation

$$\sum_{i=1}^n F(I_1, I_{2,i})p_i \quad (11)$$

We now discuss these modeling features. Note that as per the analysis in the previous section, for any choice of the values I_1, $I_{2,i}$ we will always have that $\boldsymbol{H(t)}$ is monotonely increasing or decreasing for $t \in [0, \tau/2]$ and also for $t \in [\tau/2, \tau]$. Thus, our control scheme may possibly result in a realization where e.g. $\boldsymbol{H(\tau/2)} > k$. However, (a) guarantees that even if this were to be the case, the line temperature will have reached a safe value by time τ. And of course (b) helps ensure that the probabilty of this event is small.

Regarding item (c), many examples are reasonable. For example, we could set $F(I_1, I_2) = \pi_1 I_1 + \pi_2 I_2$ where $\pi_1, \pi_2 \ge 0$. Or, $F(I_1, I_2) = \pi_1 I_1^2 + \pi_2 I_2^2$. Below we discuss several cases.

We will now cast the choice of the values I_1, $I_{2,i}$ as an optimization problem. To do so, suppose that $\boldsymbol{W} = w_i$. Then, using (6),

$$\boldsymbol{H(\tau)} = v_1 I_1^2 \ + \ v_2 I_2^2(i) \ + \ w_i \ + \ Ce^{-\nu\tau}, \quad (12)$$

where

$$v_1 \ \triangleq \ \int_0^{\tau/2} e^{-\nu(\tau-s)}\alpha \ ds \quad (13)$$

and

$$v_2 \ \triangleq \ \int_{\tau/2}^{\tau} e^{-\nu(\tau-s)}\alpha \ ds. \quad (14)$$

Define

$$z_1 \triangleq v_1 I_1^2, \quad z_2(i) \triangleq v_2 I_2^2(i), \quad 1 \le i \le n, \quad (15)$$

and, for $1 \le i \le n$

$$\bar{k}_i \triangleq k - Ce^{-\nu\tau} - w_i.$$

Using this notation we have that, when $\boldsymbol{W} = w_i$

$$\boldsymbol{H(\tau)} \ > \ k \quad \text{is equivalent to:} \quad z_1 + z_2(i) > \bar{k}_i.$$

Moreover, let us define

$$\tilde{F}(z_1, z_2) \triangleq F(\sqrt{z_1/v_1}, \sqrt{z_2(i)/v_2}), \quad (16)$$

which is simply recasting function F in terms of the z variables. It follows that we can write our optimal control problem as

$$\begin{aligned}
\mathcal{P}_1 : \quad \max \quad & \sum_{i=1}^{n} \tilde{F}(z_1, z_2(i)) p_i, \\
\text{s.t.} \quad & \sum_{i=1}^{n} \mathbb{I}\{z_1 + z_2(i) > \bar{k}_i\} p_i \ \le \ \epsilon \quad (17) \\
& z_1 \ \le \ v_1^2 L_1^2(\tau/2) \quad (18) \\
& z_1 + z_2(i) \ \le \ u_i, \quad \forall i, \quad (19) \\
& z_1 \ge 0, \quad z_2(i) \ge 0 \, \forall i. \quad (20)
\end{aligned}$$

In (17), $\mathbb{I}$ is the indicator function; the square roots in the objective and (18) arise from our definition of the z variables. Constraint (19) models a reasonable requirement: that the line temperature at time τ not exceed an absolute maximum limit, with probability 1. Of course, this constraint may render the problem above infeasible – however assuming that $\boldsymbol{H(0)}$ is "safe" the problem *will* be feasible (if necessary by setting $z_1 = z_2(i) = 0$ for all i) assuming realistic $\boldsymbol{R}$. We assume that $u_i > \bar{k}_i$.

Lemma 1: Let z_1^*, $z_2^*(i)$ $(1 \le i \le n)$ be an optimal solution to problem $\mathcal{P}_1$. Then, for each $1 \le i \le n$

$$z_1^* + z_2^*(i) \ = \ \bar{k}_i \text{ or } u_i.$$

Proof. Suppose that for some i, $z_1^* + z_2^*(i) < k_i$. Then increasing $z_2^*(i)$ maintains feasibility, and the monotonicity asumption on F implies that the objective improve. The same reasoning applies if $u_i < z_1^* + z_2^*(i) < k_i$. ■

Using this observation we can simplify the optimization problem. Define, for $1 \le i \le n$ a binary variable y_i such that

$$y_i = \begin{cases} 0 & \text{when } z_1 + z_2(i) = \bar{k}_i \\ 1 & \text{when } z_1 + z_2(i) = u_i \end{cases} \quad (21)$$

Then the above optimization problem can be recast as:

$$\begin{aligned}
& \mathcal{P}_2 : \\
& \max \sum_{i=1}^{n} \tilde{F}(z_1, \bar{k}_i - z_i) p_i (1 - y_i) + \tilde{F}(z_1, u_i - z_i) p_i y_i \\
& \text{s.t.} \sum_{i=1}^{n} u_i p_i y_i \ \le \ \epsilon \\
& 0 \ \le \ z_1 \ \le \ \min\{v_1^2 L_1^2(\tau/2) \, , \, \min_i \{u_i\}\} \quad (22) \\
& y_i \ = \ 0 \text{ or } 1, \ \text{all } i.
\end{aligned}$$

Problem $\mathcal{P}_2$ is a nonlinear, binary optimization problem. We are interested in methodologies and special cases that render it practicable. We will describe a general approximation method that should prove effective when the parameter n is small. We also describe a provably good approximation algorithm for the case where n is large, which however only applies in a special case of the function F.

As an aside, the issue of the magnitude of n concerns several practical questions, primarily with regards with how accurate a representation of the random variable $\boldsymbol{R}$ can be constructed in "real time". Adequate sensorization of power lines should help in this regard, however there is a larger issue of how data uncertainty can arise in this context (e.g., the spatial distribution of exogenous temperatures over a small time window).

A. General approach for small n

Suppose that in problem $\mathcal{P}_2$ we were to *fix* the variable z_1 to a value $\hat{z}_1$ satisfying (22). The remaining part of problem $\mathcal{P}_2$ has the following general structure:

$$\begin{aligned}
& \mathcal{P}_2(\hat{z}_1) : \\
& \qquad \max \sum_{i=1}^{n} \tilde{f}_i(\hat{z}_1) y_i \\
& \qquad \text{s.t.} \sum_{i=1}^{n} u_i p_i y_i \ \le \ \epsilon, \\
& \qquad y_i \ = \ 0 \text{ or } 1, \ \text{all } i.
\end{aligned}$$

Problem $\mathcal{P}_2(\hat{z}_1)$ is a (binary) *knapsack* problem. Knapsack problems are NP-hard – however in this case we are dealing with small n. In fact, it is fair to say that knapsack problems are the easiest of the NP-hard problems (and, commercial mixed-integer program solvers dispatch them with ease even for largish n).

These observations suggest the following (grid-like) approach:

(1) Enumerate equally spaced values of $\hat{z}_1$ between the two bounds in 22.
(2) For each enumerated value, solve $\mathcal{P}_2(\hat{z}_1)$.

While suffering from an enumerative component, this approach does have the attribute of handling any objective function F in the definition of our problem.

A separate issue regarding the small n case concerns the robustness of the computed answers with respect to e.g. the (necessarily estimated) parameters p_i and w_i. Using a small n has the effect of accumulating more probability mass into fewer values, with a resulting increase in numerical sensitivity (to the choices for the p_i and w_i).

B. Large n

Suppose now that n is large. As stated above we expect that even in this case a good mixed-integer programming solver should be able to solve the problems $\mathcal{P}_2(\hat{z}_1)$. Nevertheless, we would like to discuss a case where a solution with sound theoretical foundation can be found.

Recall the formula (12) for $\boldsymbol{H(\tau)}$ as well as (15), and note that $z_1 + z_2(i)$ appears in $\boldsymbol{H(\tau)}$ and thus, in the chance constraint (17). Consider the special case where

$$F(I_1, I_2) \;=\; v_1^2 I_1^2 \;+\; v_2^2 I_2^2 \tag{23}$$

for all I_1, I_2. Below we will discuss the implications of this assumption. Using (23), it will follow that (using (16),

$$\tilde{F}(z_1, z_2) \quad = \quad z_1 + z_2, \quad \text{for all } z_1, z_2. \tag{24}$$

Thus, problem $\mathcal{P}_2$ can be equivalently restated as:

$$\begin{aligned}
\mathcal{P}_3 : \qquad & \max \sum_{i=1}^{n} \tilde{F}(\bar{k}_i) p_i (1 - y_i) + \tilde{F}(u_i) p_i y_i \\
& \text{s.t.} \sum_{i=1}^{n} u_i p_i y_i \;\le\; \epsilon \\
& 0 \;\le\; z_1 \;\le\; \min\{v_1^2 L_1^2(\tau/2)\, , \, \min_i\{u_i\}\} \qquad (25) \\
& y_i \;=\; 0 \text{ or } 1, \text{ all } i.
\end{aligned}$$

We can see that constraint (25) is redundant. In short, $\mathcal{P}_3$ can be rewritten in the form:

$$\begin{aligned}
\mathcal{P}'_3 : \qquad & \max \sum_{i=1}^{n} f_i \, y_i \\
& \text{s.t.} \sum_{i=1}^{n} q_i y_i \;\le\; \epsilon \\
& y_i \;=\; 0 \text{ or } 1, \text{ all } i,
\end{aligned}$$

for appropriate quantities $f_i > 0$ and $q_i > 0$; this constitutes a standard $0 - 1$ knapsack problem.

In comparing this approach to that used for the small n case, we can see that see that we have simplified the problem (no enumeration over the $\hat{z}_1$ values). Of course, we do have to solve the possibly large knapsack problem $\mathcal{P}_3$. As we discussed before, this should prove routine (and very fast) even for n in the hundreds, if not more.

However, from a theoretical perspective, more can be said. The following result can be read from those in [4].

Theorem 2: Consider a $0 - 1$ knapsack problem on N variables

$$\begin{aligned}
\mathcal{K} : \qquad & \max \sum_{j=1}^{N} p_j x_j \\
& \text{s.t.} \sum_{j=1}^{n} a_j x_j \le b, \\
& x_j \;=\; 0 \text{ or } 1, \text{ all } j.
\end{aligned}$$

For each fixed tolerance $0 < \delta < 1$ there is a linear program LP with the following properties

- The number of variables and constraints in LP is $O(N^2)$.
- The x_j are among the variables of LP.
- The solution of LP, together with a simple rounding for the x_j variables yields a (binary) solution for $\mathcal{K}$ that is guaranteed to have value within tolerance δ of the optimum for $\mathcal{K}$.

Now we comment on (23). We would argue that this is a "reasonable" functional form for $F(I_1, I_2)$ in that it amounts to a weighted sum. Of course the weights are not flexibly chosen. Nevertheless, note (see (13), (14)) that $v_2 > v_1$. Thus, (23) places more emphasis on what happens in the time interval $[\tau/2, \tau]$. We would argue that this is a reasonable approach, in the sense that we focus in the later time interval, where, coincidentally, we are able to make more precise decisions since randomness has been resolved.

C. Robustness of the solutions

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