# Stochastic models and control for power line temperature

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- 2003 event: critical temperatures estimates were sometimes incorrect.

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- It also relies on the heat equation for a "static" calculation.
- Note: power lines can be more than 100 miles long.
- How can we account for data uncertainty, errors, unavailability?

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- *Heat equation:*

$$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2} + \alpha I^2(t) - \nu(T(x,t) - T^{ext}(x,t)),$$

where $\kappa \geq 0$, $\alpha \geq 0$ and $\nu \geq 0$ are (line dependent) constants, and $T^{ext}(x,t)$ is the ambient temperature at $(x,t)$

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$\mathbf{h(x)}$ = a random variable, at $x$.

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**The goal:** algorithm- and data-driven estimates for "safe" current/temperature limits

## Back to the stochastic heat equation

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Recall: $x \in [0, L]$, $t \in [0, \tau]$

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Integrate and divide by $L$, get

$$\frac{1}{L}\int_0^L \frac{\partial T(x,t)}{\partial t} dx \;=\; \alpha I^2(t) - \frac{\nu}{L}\int_0^L T(x,t)dx + \frac{\nu}{L}\int_0^L G(\mathbf{h(x)})dx.$$

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$$\mathbf{H(t)} \triangleq \frac{1}{L}\int_0^L T(x,t)dx \quad \text{(average internal line temperature at t)}$$

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$$\mathbf{H(t)} = \int_0^t e^{-\nu(t-s)}\alpha I^2(s)ds + \mathbf{R}(1 - e^{-\nu t}) + Ce^{-\nu t},$$

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(c) What about performance?

We want to maximize:

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**$\mathbf{F(I_1, I_2)}$ :** a monotonely increasing function of $I_1,\ I_2$

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**Recall:**

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\text{s.t.} & P(\mathbf{H}(\boldsymbol{\tau}) > k) \leq \epsilon \\
 & \mathbf{H}(\boldsymbol{\tau}) \ \leq \ u \quad (k < u < \text{ critical temp}) \\
 & I_1 \ \leq \ L(\tau/2, k) \\
 & \text{other constraints.}
\end{array}
$$

**Recall:**

$$
\begin{aligned}
\mathbf{H}(\boldsymbol{\tau}) \quad &= \quad \int_0^{\tau} e^{-\nu(\tau - s)} \alpha I^2(s) ds + \mathbf{R}(1 - e^{-\nu\tau}) + Ce^{-\nu\tau}, \\
&= \quad v_1 \ I_1^2 \ + \ v_2 \ I_2^2(i) \ + \ r_i(1 - e^{-\nu\tau}) \ + \ Ce^{-\nu\tau} \qquad \textbf{in state i}
\end{aligned}
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$$

So chance constraint is of the form:

$$
\sum_{i=1}^{n} \mathbb{I}\{v_1 \ I_1^2 \ + \ v_2 \ I_2^2(i) > u \ - \ r_i(1 - e^{-\nu\tau}) \ - \ Ce^{-\nu\tau}\} p_i \ \leq \ \epsilon.
$$

$$\begin{array}{ll} \max & \displaystyle\sum_{i=1}^{n} F(I_1, I_{2,i}) p_i \\ \text{s.t.} & P(\mathbf{H}(\boldsymbol{\tau}) > k) \le \epsilon \\ & \mathbf{H}(\boldsymbol{\tau}) \ \le \ u \quad (k < u < \text{ critical temp}) \\ & I_1 \ \le \ L(\tau/2, k) \\ & \text{other constraints.} \end{array}$$

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So chance constraint s of the form:

$$\sum_{i=1}^{n} \mathbb{I}\{\underbrace{v_1 \ I_1^2}_{z_1} \ + \ \underbrace{v_2 \ I_2^2(i)}_{z_2(i)} > \underbrace{u \ - \ r_i(1 - e^{-\nu\tau}) \ - \ Ce^{-\nu\tau}}_{w_i}\} p_i \ \le \ \epsilon.$$

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \tilde{F}(z_1, z_2(i))\; p_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} \mathbb{I}\{z_1 + z_2(i) > w_i\} p_i \;\le\; \epsilon \\
& z_1 + z_2(i) \;\le\; u_i \quad (w_i < u_i) \\
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**Lemma:** At optimality,

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**Lemma:** At optimality,

$$z_1 \; + \; z_2(i) \; = \; w_i \text{ or } u_i, \quad \text{all } i$$

$\longrightarrow$ Use **binary** variable

$$y_i = \begin{cases} 0 & \text{when} \quad z_1 + z_2(i) = w_i \\ 1 & \text{when} \quad z_1 + z_2(i) = u_i \end{cases}$$

## Continuous knapsack problem

$$\max \sum_{i=1}^{n} \tilde{F}(z_1, w_i - z_i) p_i (1 - y_i) + \tilde{F}(z_1, u_i - z_i) p_i y_i$$

$$\text{s.t.} \sum_{i=1}^{n} u_i p_i y_i \ \leq \ \epsilon$$

$$0 \ \leq \ z_1 \ \leq \ \bar{k}$$

$$y_i \ = \ 0 \text{ or } 1, \quad \text{all } i.$$

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$$\max_{z_1 \in [0,\bar{k}]} M(z_1)$$

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**Practicable!**