

Strong Formulations for Convex Functions over Non-Convex Domains

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Introduction

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- F is a mixed-integer set
- F is constrained in a nasty way, e.g.

$$x_1 - 3 \sin(x_2) + 2 \cos(x_3) = 4$$

Or,

Convex constraint:

$$Q(x) \leq q, \quad \text{and} \quad x \in F,$$

- $Q(x)$ convex, especially: convex quadratic
- F nonconvex

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Examples: lattice-free sets, geometry

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3. Add to the formulation an inequality $\mathbf{a}z + \alpha^T \mathbf{x} \geq \alpha_0$ valid for

$$\{(x, z) : x \in \mathbb{R}^n - S, z \geq Q(x)\}$$

but violated by (x^*, z^*) .

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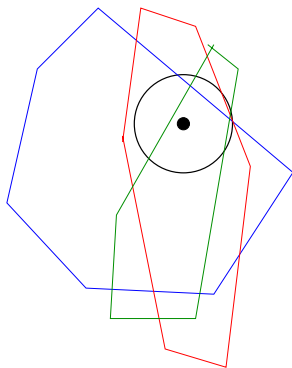
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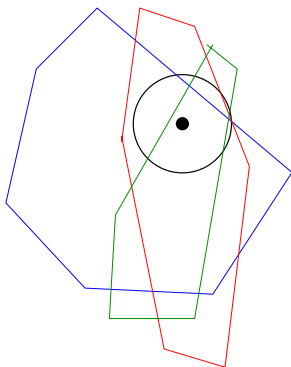
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(application: X-ray lithography; see Ahmadi (2010))

(yes, it is NP-hard)

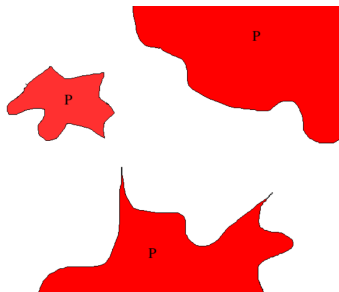




- Typical values for d (dimension): less than 20; usually even smaller
- Typical values for K (number of polyhedra): possibly hundreds, but often less than 50
- Very hard problem

First problem setting

- Let $Q(x)$ be a **strongly convex** function on \mathbb{R}^d ,
- Let $P \subset \mathbb{R}^d$ be closed and such that $\mathbb{R}^d - P \neq \emptyset$ is unbounded.



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Want to produce a linear inequality description for:

$$\left\{ (x, q) \in \mathbb{R}^{d+1} : Q(x) \leq q, \quad x \in \mathbb{R}^d - \text{int}(P) \right\}.$$

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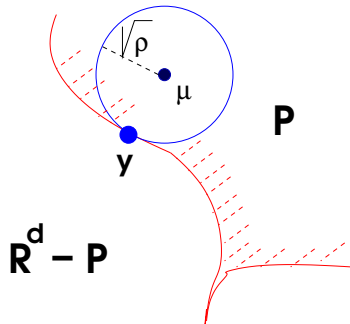
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How can we use the structure of P to strengthen the inequality?

Definition:

Given $y \in \partial P$, say P is **locally flat** at y if
 $\exists \mathcal{B}(\mu, \sqrt{\rho}) \subseteq P$ with $\|\mu - y\|^2 = \rho$ and $\rho > 0$.



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- Inequality is tight at $(y, Q(y))$, and cuts-off points $(x, Q(x))$ and $x \in \text{int}(P)$.
- Largest possible α : “lifted first-order inequality”.

$$\left\{ (x, q) \in \mathbb{R}^{d+1} : Q(x) \leq q, x \in \mathbb{R}^d - \text{int}(P) \right\}$$

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Theorem.

Any linear inequality valid for S is dominated by a lifted first-order inequality. More precisely,

$$\text{conv} \left\{ (x, q) \in \mathbb{R}^{d+1} : Q(x) \leq q, x \in \mathbb{R}^d - \text{int}(P) \right\} =$$
$$\left\{ (x, q) \in \mathbb{R}^{d+1} : \text{LFO} + \text{FO} \right\}$$

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How do we make this computationally practicable?

First problem setting

- Let $Q(x)$ is a **positive definite** quadratic on \mathbb{R}^d ,
- $P = \{x \in \mathbb{R}^d : a_i^T x \leq b_i, 1 \leq i \leq m\}$ full dimensional

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change in coordinates \rightarrow

$$S \doteq \left\{ (x, q) \in \mathbb{R}^{d+1} : \sum_{j=1}^d x_j^2 \leq q, x \in \mathbb{R}^d - \text{int}(P) \right\}.$$

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$$q \geq 2y^T x - \|y\|^2 - \alpha (a_i^T x - b_i)$$

for $\alpha > 0$ appropriately chosen.

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- **Theorem:** Let $(\hat{x}, \hat{q}) \in \mathbb{R}^{d+1}$ with $\hat{v} \in \text{int}(P)$.

We can compute a lifted first-order inequality maximally violated by (\hat{x}, \hat{q}) , by solving m linearly constrained convex quadratic programs on $O(d)$ variables.

When does a point

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violate a lifted first-order inequality

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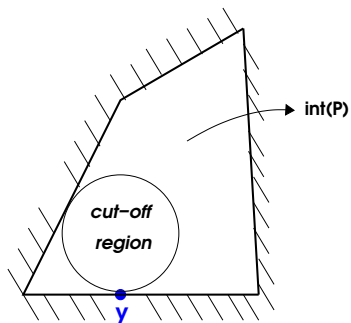
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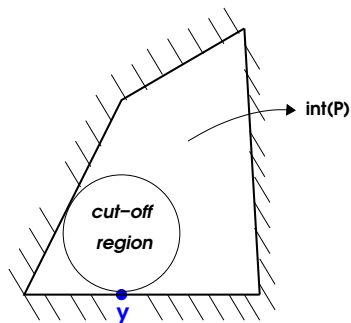
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This describes the interior of a **ball**, which must be contained in $\text{int}(P)$

Geometrical characterization

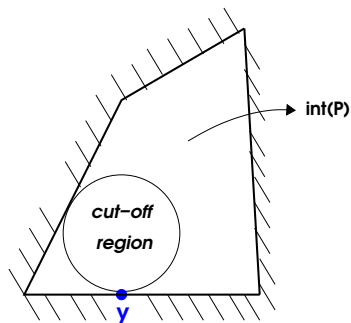


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$$\text{conv}(S) = \text{conv}(Q_1 \cup Q_2 \cup \dots \cup Q_m),$$

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Separation:

solve SOCP, use SOCP “Farkas Lemma”, get linear cut



Second setting: separating across a quadratic set

For $\mathbf{A} \succ \mathbf{0}$, polynomially separable linear inequality description for:

$$\{(x, q) \in \mathbb{R}^{d+1} : \sum_{j=1}^d x_j^2 \leq q, \quad x^T A x - 2c^T x + b \geq 0\}$$

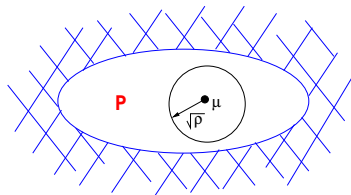
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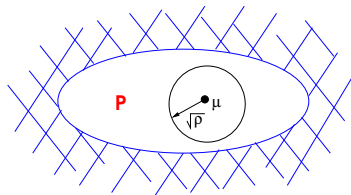


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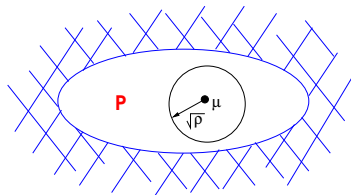
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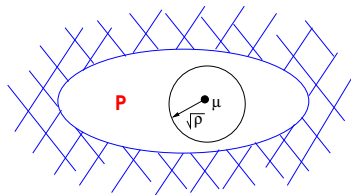
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$$\rho - (\hat{q} - 2\mu^T \hat{x} + \mu^T \mu) = \rho - \|\hat{x} - \mu\|^2 - \hat{q} + \|\hat{x}\|^2$$



Separation problem:

$$\begin{aligned} \min_{\mu, \rho} \quad & \|\mu\|^2 - \rho - 2\bar{x}^T \mu \\ \text{Subject to:} \quad & \{x : \|x - \mu\|^2 \leq \rho\} \subseteq P \end{aligned}$$

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Theorem:

Optimal choices for μ and ρ are given by:

$$\hat{\mu} = \hat{\theta}b + (I - \hat{\theta}A)\bar{x}$$

and

$$\hat{\rho} = \|\hat{\mu}\|^2 - 2\bar{x}^T \hat{\mu} + \|\bar{x}\|^2 - \hat{\theta}(\bar{x}^T A \bar{x} - 2b^T \bar{x} + c).$$

$$\text{Here, } \hat{\theta} = \frac{1}{\lambda_{\max} A}.$$

Separating across general quadratics

$$\Pi \doteq \{(x, w, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} : z \geq x^T Q x + q^T x, \quad w \leq x^T A x\}$$

$(A \succ 0, Q \succ 0)$.

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Linear transformation $\rightarrow \Pi$ is the set of points $\in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ s.t.

$$z \geq \|x\|^2 + q^T x, \quad w \leq x^T \Lambda x \quad (\Lambda \succ 0).$$

Write $P \doteq \{(x, w) \in \mathbb{R}^d \times \mathbb{R} : x^T \Lambda x - w \leq 0\}$, and for $\mu \in \mathbb{R}^d, \nu \in \mathbb{R}$,

$$M(\mu, \nu) \doteq \{(x, w) \in \mathbb{R}^d \times \mathbb{R} : \lambda_{\max} \|x - \mu\|^2 + (\nu - w) \leq 0\}.$$

Then

$x \in \mathbb{R}^d - \text{int}(P)$ iff $x \in \mathbb{R}^d - \text{int}(M(\mu, \nu))$, for all μ, ν with $M(\mu, \nu) \subseteq P$.

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- $P = \{(x, w) \in \mathbb{R}^d \times \mathbb{R} : x^T \Lambda x - w \leq 0\}$,
- $M(\mu, \nu) = \{(x, w) \in \mathbb{R}^d \times \mathbb{R} : \lambda_{\max} \|x - \mu\|^2 + (\nu - w) \leq 0\}$.

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So, valid inequality for any μ, ν with $M(\mu, \nu) \subseteq P$:

$$\lambda_{\max} \|\mu\|^2 - \lambda_{\max} (2\mu + q)^T x + (\nu - w) + \lambda_{\max} z \geq 0$$

Separation problem, given $(\bar{x}, \bar{w}) \in \text{int}(P)$

$$\begin{aligned} & \bar{w} - \nu - \lambda_{\max} \|\mu\|^2 + 2\lambda_{\max} \bar{x}^T \mu + 2\lambda_{\max} q^T \bar{x} \\ \text{subject to: } & \nu + \min_x \{ \lambda_{\max} \|x - \mu\|^2 - x^T \Lambda x \} \geq 0. \end{aligned}$$

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$$\Pi \doteq \{ (x, w, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} : z \geq x^T Q x + q^T x, \quad w \leq x^T A x \}$$

and transformed it into:

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Theorem. Eigenspace not necessary for poly-time separation (only max eigenvalue of A).

Example: $f(x) \doteq 2(x_1x_2 + x_1x_3 + x_2x_3)$ over $[0, 1]^3$.

McCormick relaxation gives zero lower bound on $f(\bar{x})$, where

$$\bar{x} = (1/2, 1/2, 1/2)^T.$$

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→ $x \rightarrow (0, 0, \frac{2}{\sqrt{3}}(x_1 + x_2 + x_3))^T$

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McCormick relaxation gives zero lower bound on $f((1/2, 1/2, 1/2)^T)$.

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\rightarrow **Paraboloid cut:**

$$p_1^2 + p_2^2 + (p_3 - 2\alpha\sqrt{3})^2 + \epsilon \geq 2p_1^2 + 2p_2^2 + \frac{1}{2}p_3^2.$$

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$$p_1^2 + p_2^2 + (p_3 - 2\alpha\sqrt{3})^2 + \epsilon \geq 2p_1^2 + 2p_2^2 + \frac{1}{2}p_3^2.$$

or,

$$f(x) \geq 8\alpha(x_1 + x_2 + x_3) - 12\alpha^2 - \epsilon$$

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When valid?

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$$f(x) \geq 8\alpha(x_1 + x_2 + x_3) - 12\alpha^2 - \epsilon$$

- $\alpha = \epsilon = 1/10$, valid when $x_1 + x_2 + x_3 \geq 3/2$.

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get: $f(x) \geq \frac{4}{5}(x_1 + x_2 + x_3) - \frac{11}{50}$.

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get: $f(x) \geq \frac{4}{5}(x_1 + x_2 + x_3) - \frac{11}{50}$. At $(.5, .5, .5)^T$ get .98

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get: $f(x) \geq 4(x_1 + x_2 + x_3) - 5$.

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Disjunction:

either

$$(x, f) \in \left\{ (x, f) : x \in [0, 1]^3, \sum_j x_j \geq 3/2, f \geq \frac{4}{5}(x_1 + x_2 + x_3) - \frac{11}{50} \right\},$$

or

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Proves: $f(x) \geq 1.245714(x_1 + x_2 + x_3) - 1.5571435$

valid throughout,

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$f(x) \geq \frac{4}{5}(x_1 + x_2 + x_3) - \frac{11}{50}$ valid for $x_1 + x_2 + x_3 \geq 3/2$

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valid throughout, at $(.5., .5., .5)^T$ get .3114275.