Using Integer Programming to Train Neural Networks

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Bordeaux July 2018

Empirical Risk Minimization problem

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Given:

- $\boldsymbol{D}$ data points $\boldsymbol{(x_i, y_i)}, \quad \boldsymbol{i = 1, \ldots, D}$
- $\boldsymbol{x_i} \in \mathbb{R}^{\boldsymbol{n}}, \; \boldsymbol{y_i} \in \mathbb{R}^{\boldsymbol{m}}$ all i
- A "loss" function $\boldsymbol{\ell} : \mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{m}} \rightarrow \mathbb{R}$ (not necessarily convex)

Compute $\boldsymbol{f} : \mathbb{R}^{\boldsymbol{n}} \rightarrow \mathbb{R}^{\boldsymbol{m}}$ to solve

$$\min_f \frac{1}{D} \sum_{i=1}^{D} \ell(f(x^i), y^i) \qquad (+ \text{ optional regularizer } \Phi(f))$$

$$\boldsymbol{f} \in \boldsymbol{F} \quad \text{(some class)}$$

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Prototype problem:

$$\begin{aligned} &\min \ c^T x \\ \text{s.t. } &f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ &x \in [0,1]^n \end{aligned}$$

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  A vertex for each variable and an edge whenever two variables appear in the same $\boldsymbol{f_i}$
- **Tree-width**

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An extension of work in B. and Muñoz 2015, SIOPT 2018.

Suppose:

the intersection graph has tree-width ω and $\boldsymbol{f_i}$ has Lipschitz constant $\boldsymbol{\mathcal{L}_i} \leq \boldsymbol{\mathcal{L}}$.

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Optimality and feasibility errors $\boldsymbol{O(\epsilon)}$

Application to ERM problem

$$\min_{f \in F} \frac{1}{D} \sum_{i=1}^{D} \ell(f(x^i), y^i)$$

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Linearize objective using epigraph trick

$$\min_{f \in F} \frac{1}{D} \sum_{i=1}^{D} L_i$$

$$L_i \geq \ell(f(x^i), y^i) \quad 1 \leq i \leq D$$

Function parameterization

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Examples:

- **Neural Networks with k layers.** $f(x) = T_{k+1} \circ \sigma \circ T_k \circ \sigma \ldots \circ \sigma \circ T_1(x)$, each $\boldsymbol{T_j}$ affine, **parameterized** by its coefficients.
- **Linear Regression.** $f(x) = Ax + b$ with ℓ_2-loss.
- **Binary Classification.** Varying f architectures and cross-entropy loss: $\ell(p, y) = -y \log(p) - (1 - y) \log(1 - p)$

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We assume $F = \{f(x, \theta) \text{ for } \theta \in \Theta \subseteq [-1, 1]^N\}$

We now apply the LP approximation result to:

$$\min_{\theta \in \Theta} \frac{1}{D} \sum_{i=1}^{D} L_i$$
$$L_i \geq \ell(f(x^i, \theta), y^i) \quad 1 \leq i \leq D$$

Let θ^* be an optimal solution. For every $\mathbf{0} < \epsilon < \mathbf{1}$ there is an **LP** of size

$$\boldsymbol{O\left((\mathcal{L}/\epsilon)^{2N+2}\,(N+D)\log(\mathcal{L}/\epsilon)\right)}$$

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$$\frac{1}{D} \sum_{i=1}^{D} \ell(f(x^i, \hat{\theta}), y^i) \leq \frac{1}{D} \sum_{i=1}^{D} \ell(f(x^i, \theta^*), y^i) + \epsilon$$

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$\mathcal{L}$ is an upper bound on the Lipschitz constant of $g^i(\theta) \doteq \ell(f(x^i, \theta), y^i)$.

Proof sketch:

$$\min_{\theta \in \Theta} \frac{1}{D} \sum_{i=1}^{D} L_i, \qquad \text{s.t.} \quad L_i \geq \ell(f(x^i, \theta), y^i) \quad 1 \leq i \leq D$$

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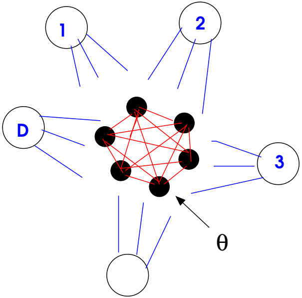

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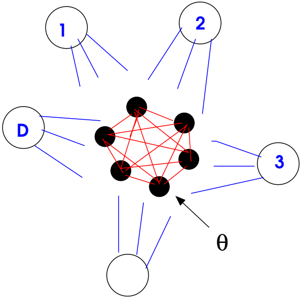

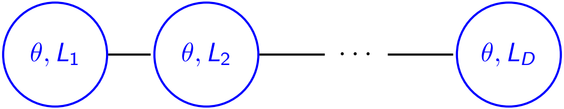

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As per Arora Basu Mianjy Mukherjee ICLR '18

The setup:

- D data points (x_i, y_i), $1 \leq i \leq D$, $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}$

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Theorem (Arora et al 2018).
If k = 1 (one “hidden layer”) there is an exact algorithm of complexity

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Theorem (Arora et al 2018).
If k = 1 (one "hidden layer") there is an exact algorithm of complexity

$$O\left(2^w D^{nw} \text{poly}(D, n, w)\right)$$

Polynomial in the size of the data set, for fixed $\boldsymbol{n}, \boldsymbol{w}$

Our result: if the entries of A_i, b_i are required to be in $[-1, 1]$, for any $\boldsymbol{k}, \boldsymbol{n}, \boldsymbol{w}, \epsilon$ there is an LP of size

$$\boldsymbol{O\left((w/\epsilon)^{poly(n,k,w)}\left(poly(n,k,w)+D\right)\log(w/\epsilon)\right)}$$

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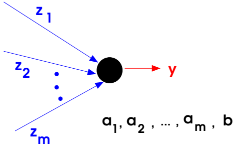
$$\boldsymbol{O\left((w/\epsilon)^{poly(n,k,w)}\,(poly(n,k,w)+D)\log(w/\epsilon)\right)}$$

- $poly(n,k,w) =$ quadratic in w, in k, linear in n
- Treewidth **independent** of D
- Number of variables **linear** in D

The Arora-Blum setup (Binarized Neural Networks)

Activation units:



With $\boldsymbol{z} \in \{\mathbf{0}, \mathbf{1}\}^{\boldsymbol{m}}$,

$$y = \begin{cases} 1, & \text{if } a^T z > b \\ 0, & \text{otherwise.} \end{cases}$$

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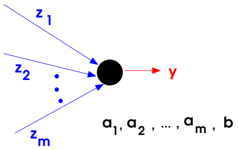

With $z \in \{0,1\}^m$,

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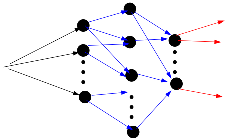
Network with n binary inputs, m binary outputs, k layers



Binarized Neural Networks, 2

Training data: Set of D pairs (x^i, y^i), $1 \leq i \leq D$

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Problem: Compute the activation function at each node to

$$\min \ \frac{1}{D} \sum_{i=1}^{D} \ell(f(x^i), y^i)$$

(f = network function)

- When $\ell \in$ (absolute value, 2-norm squared) NP-hard if $k = 3$ and $D = n$, $m = 1$
- But we are interested in the case D very large compared to n
- And also other loss functions, e.g. smooth convex

Our result on Binarized Neural Networks

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ERM problem: Compute the activation function at each node to

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Theorem. When $\ell \in$ (absolute value, 2-norm squared) there is an LP of encoding length

$$O(2^{poly(k,n,m)}(poly(k,n,m) + D))$$

that solves the ERM problem exactly in absolute case, and within $O(\epsilon)$ additive error in the 2-norm case.

Extensions