Chance-constrained optimization problems in the operation of the power grid

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Michigan 2013

Part I: Robust Optimal Power Flow with Uncertain Renewables

with Michael Chertkov (LANL) and Sean Harnett (Columbia)

THE ENERGY CHALLENGE

Wind Energy Bumps Into Power Grid's Limits

Mike Groll/Associated Press

The Maple Ridge Wind farm near Lowville, N.Y. It has been forced to shut down when regional electric lines become congested.

By MATTHEW L. WALD
Published: August 26, 2008

When the builders of the Maple Ridge Wind farm spent $320 million to put nearly 200 wind turbines in upstate New York, the idea was to get paid for producing electricity. But at times, regional electric lines have been so congested that Maple Ridge has been forced to shut down even with a brisk wind blowing.

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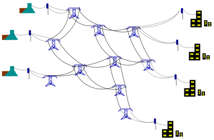
Optimal power flow (economic dispatch, tertiary control)



- Used periodically to handle the next time window (e.g. 15 minutes, one hour)
- Choose generator outputs
- Minimize cost (quadratic)
- Satisfy demands, meet generator and network constraints
- **Constant load (demand) estimates for the time window**

OPF:

$$\min \ c(p) \quad \text{(a quadratic)}$$

s.t.

$$B\theta = p - d \tag{1}$$

$$|y_{ij}(\theta_i - \theta_j)| \ \leq \ u_{ij} \quad \text{for each line } ij \tag{2}$$

$$P_g^{min} \ \leq \ p_g \ \leq \ P_g^{max} \quad \text{for each bus } g \tag{3}$$

Notation:

$p \ = \$ vector of generations $\ \in \mathcal{R}^n, \quad d \ = \$ vector of loads $\ \in \mathcal{R}^n$

$B \ \in \mathcal{R}^{n \times n}, \quad$ (bus susceptance matrix)

$$\forall i,j : \quad B_{ij} = \begin{cases} -y_{ij}, & ij \ \in \ \mathcal{E} \text{ (set of lines)} \\ \sum_{k;\{k,j\} \in \mathcal{E}} y_{kj}, & i = j \\ 0, & \text{otherwise} \end{cases}$$

$$\min \; c(p) \quad \text{(a quadratic)}$$

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How does OPF handle short-term fluctuations in **demand** (d)?
Frequency control:

- Automatic control: primary, secondary
- The output of special generators varies up or down **proportionally** to **aggregate** change

How does OPF handle short-term fluctuations in renewable output?
Answer: Same mechanism, now used to handle aggregate wind power change

CIGRE -International Conference on Large High Voltage Electric Systems '09

- Large unexpected fluctuations in wind power can cause additional flows through the transmission system (grid)
- Large power deviations in renewables must be balanced by other sources, which may be far away
- Flow reversals may be observed – control difficult
- A solution – expand transmission capacity! Difficult (expensive), takes a long time
- Problems **already observed** when renewable penetration high

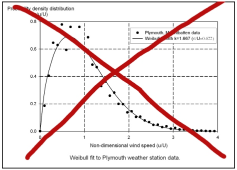
CIGRE -International Conference on Large High Voltage Electric Systems '09

- "Fluctuations" – 15-minute timespan
- Due to turbulence ("storm cut-off")
- Variation of the same order of magnitude as mean
- Most problematic when renewable penetration starts to exceed $20 - 30\%$
- Many countries are getting into this regime

Wind model?

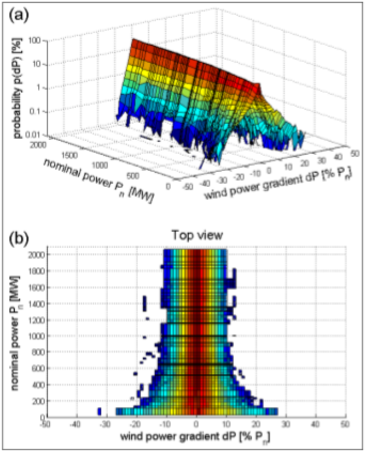
Need to model variation in wind power between dispatches

Wind at farm attached to bus i of the form $\mu_i + \mathbf{w_i}$ – Weibull distribution?



Wind model

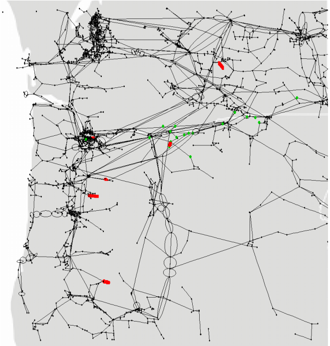
From CIGRE report, aggregated over Germany:



Experiment

Bonneville Power Administration data, Northwest US

- data on wind fluctuations at planned farms
- with standard OPF, 7 lines exceed limit $\geq 8\%$ of the time



Line limits and line tripping

If power flow in a line exceeds its limit, the line becomes compromised and may 'trip'. But process is complex and time-averaged:

- Thermal limit is most common
- Thermal limit may be in terms of terminal equipment, not line itself
- Wind strength and wind direction contributes to line temperature
- In medium-length lines ($\sim$ 100 miles) the line limit is due to voltage drop, not thermal reasons
- In long lines, it is due to phase angle change (stability), not thermal reasons
- In 2003 U.S. blackout event, many critical lines tripped due to thermal reasons, but well short of their line limit

Line trip model

summary: exceeding limit for too long is bad, but complicated

want: "fraction time a line exceeds its limit is small"

proxy: prob(violation on line i) $< \epsilon$ for each line i

Control

For each generator i, two parameters:

- $\overline{p_i}$ = mean output
- α_i = response parameter

Real-time output of generator i:

$$p_i = \overline{p}_i - \alpha_i \sum_j \Delta\omega_j$$

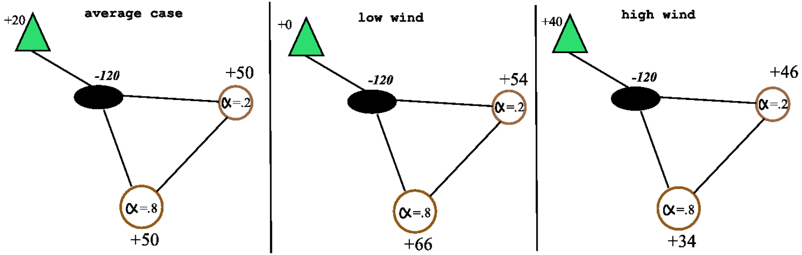
where $\Delta\omega_j$ = change in output of renewable j (from mean).

$$\sum_i \alpha_i = 1$$

$\sim$ primary + secondary control

Set up

control



Computing line flows

wind power at bus i: $\mu_i + \mathbf{w}_i$

DC approximation

- $B\boldsymbol{\theta} = \overline{p} - d$
  $+(\mu + \mathbf{w} - \alpha \sum_{i \in G} \mathbf{w}_i)$
- $\boldsymbol{\theta} = B^{+}(\bar{p} - d + \mu) + B^{+}(I - \alpha e^{T})\mathbf{w}$
- flow is a linear combination of bus power injections:

$$\mathbf{f_{ij}} = y_{ij}(\boldsymbol{\theta}_i - \boldsymbol{\theta}_j)$$

Computing line flows

$$\mathbf{f_{ij}} = y_{ij} \left((B_i^+ - B_j^+)^T (\bar{p} - d + \mu) + (A_i - A_j)^T \mathbf{w} \right),$$

$$A = B^+(I - \alpha e^T)$$

Given distribution of wind can calculate moments of line flows:

- $E\mathbf{f_{ij}} = y_{ij}(B_i^+ - B_j^+)^T(\bar{p} - d + \mu)$
- $var(\mathbf{f_{ij}}) := s_{ij}^2 \geq y_{ij}^2 \sum_k (A_{ik} - A_{jk})^2 \sigma_k^2$
(assuming independence)
- and higher moments if necessary

Chance constraints to deterministic constraints

- chance constraint: $P(\mathbf{f_{ij}} > f_{ij}^{max}) < \epsilon_{ij}$ **and** $P(\mathbf{f_{ij}} < -f_{ij}^{max}) < \epsilon_{ij}$
- from moments of $\mathbf{f_{ij}}$, can get conservative approximations using e.g. Chebyshev's inequality

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- from moments of $\mathbf{f_{ij}}$, can get conservative approximations using e.g. Chebyshev's inequality
- for Gaussian wind, can do better, since $\mathbf{f_{ij}}$ is Gaussian :

$$|E\mathbf{f_{ij}}| + var(\mathbf{f_{ij}})\phi^{-1}(1 - \epsilon_{ij}) \leq f_{ij}^{max}$$

Formulation:
Choose mean generator outputs and control to minimize expected cost, with the probability of line overloads kept small.

$$\min_{\overline{p},\alpha} \mathbb{E}[c(\overline{p})]$$

$$\text{s.t.} \sum_{i \in G} \alpha_i = 1, \ \alpha \geq 0$$

$$B\delta = \alpha, \delta_n = 0$$

$$\sum_{i \in G} \overline{p}_i + \sum_{i \in W} \mu_{\mathbf{i}} = \sum_{i \in D} d_i$$

$$\overline{f}_{ij} = y_{ij}(\overline{\theta}_i - \overline{\theta}_j),$$

$$B\overline{\theta} = \overline{p} + \mu - d, \ \overline{\theta}_n = 0$$

$$s_{ij}^2 \geq y_{ij}^2 \sum_{k \in W} \sigma_{\mathbf{k}}^{\mathbf{2}} (B_{ik}^+ - B_{jk}^+ - \delta_i + \delta_j)^2$$

$$|\overline{f}_{ij}| + s_{ij}\phi^{-1}(1 - \epsilon_{ij}) \leq f_{ij}^{max}$$

Data errors?

$$s_{ij}^2 \geq y_{ij}^2 \sum_{k \in W} \sigma_{\mathbf{k}}^{\mathbf{2}} (B_{ik}^{+} - B_{jk}^{+} - \delta_i + \delta_j)^2$$

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(the $\overline{f}_{ij}$ implicitly incorporate the $\boldsymbol{\mu}_{\mathbf{i}}$)

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(the $\overline{f}_{ij}$ implicitly incorporate the $\boldsymbol{\mu}_{\mathbf{i}}$)

What if the $\boldsymbol{\mu}_{\mathbf{i}}$ or the $\boldsymbol{\sigma}_{\mathbf{k}}$ are incorrect? ... What happens to

$$Prob(\mathbf{f_{ij}} > u_{ij})?$$

Let the *correct* parameters be $\tilde{\mu}_i$, $\tilde{\sigma}_i$ for each farm i.

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Theorem: Suppose there are parameters $M > 0$, $V > 0$ such that

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for all i. Then:

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Can we *guarantee* that $Prob(f_{ij} > f_{ij}^{max})$ is small even under data errors?

Polyhedral data error model:

$$|\tilde{\sigma}_i^2 - \sigma_i^2| \le \gamma_i \ \forall i, \quad \sum_i \frac{|\tilde{\sigma}_i^2 - \sigma_i^2|}{\gamma_i} \le \Gamma.$$

Ellipsoidal data error model:

$$(\tilde{\sigma}^2 - \sigma^2)^T A (\tilde{\sigma}^2 - \sigma^2) \ \le \ b$$

Here $A \succeq 0$ and $b > 0$ are parameters.

chance constraints

Nominal case:

chance constraints

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Robust case: $\max_{\mathcal{E}} \left\{ |E\,\mathbf{f_{ij}}| + var(\mathbf{f_{ij}})\phi^{-1}(1-\epsilon_{ij}) \right\} \le f_{ij}^{max}$

($\mathcal{E}$ **:** data error model)

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Theorem. The robust problem is a convex optimization problem and can be solved in polynomial time in the polyhedral and ellipsoidal data cases.

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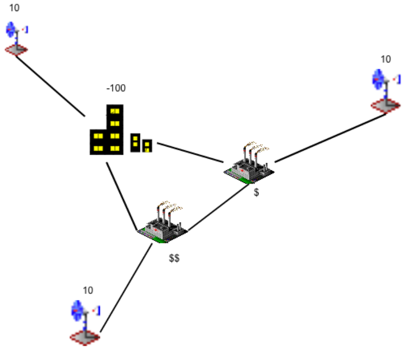
Theorem. The robust problem is a convex optimization problem and can be solved in polynomial time in the polyhedral and ellipsoidal data cases.

An "ambiguous chance-constrained problem"

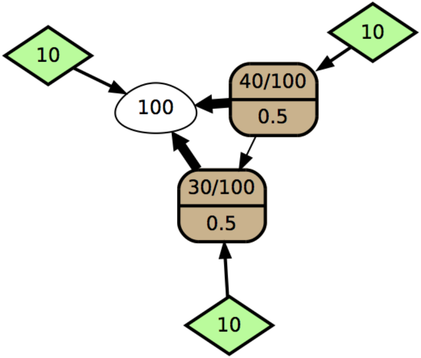
Toy example

1. What if no line limits?
2. What if tight limit on line connecting generators?



Answer 1

What if no line limits?

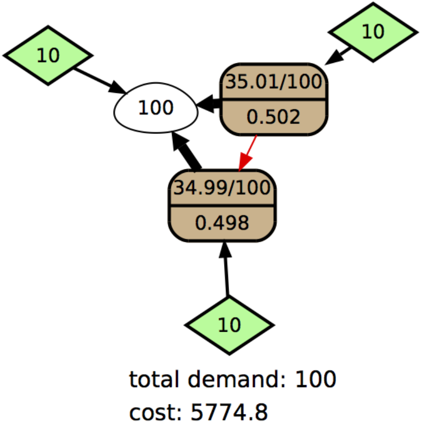


total demand: 100

cost: 5720

Answer 2

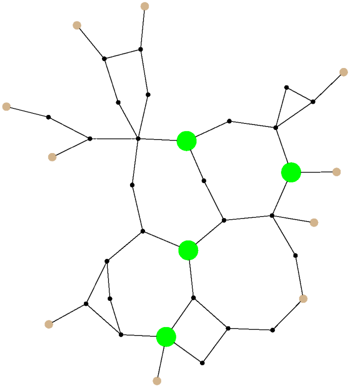
What if small limit on line connecting generators?



Experiment

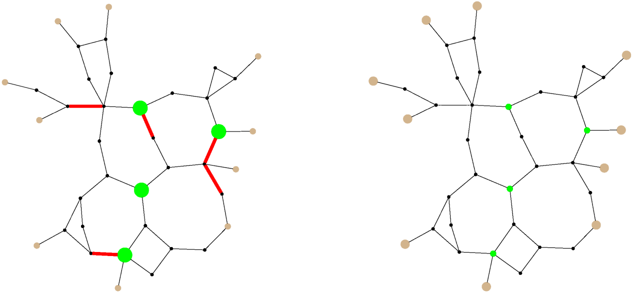
How much wind penetration can we handle?
And how much money does this save?

39-bus New England system from MATPOWER



Experiment

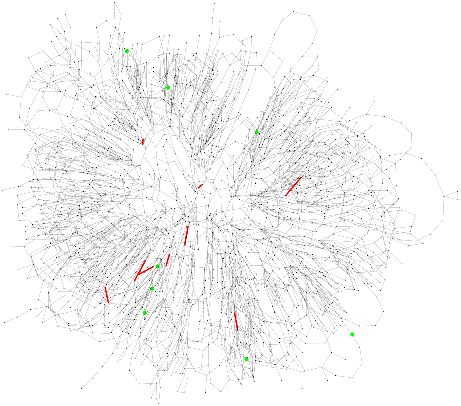
'standard' OPF solution with 10% buffer on line limits
feasible only up to 5% penetration (right)



Cost 1,275,000 – almost 5 times greater than chance-constrained

Big cases

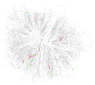
Polish system - winter 2003-04 evening peak



Big cases

Polish 2003-2004 winter peak case

- 2746 buses, 3514 branches, 8 wind sources
- 5% penetration and $\sigma = .3\mu$ each source



CPLEX: the optimization problem has

- 36625 variables
- 38507 constraints, 6242 conic constraints
- 128538 nonzeros, 87 dense columns

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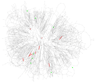

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Big cases

CPLEX:

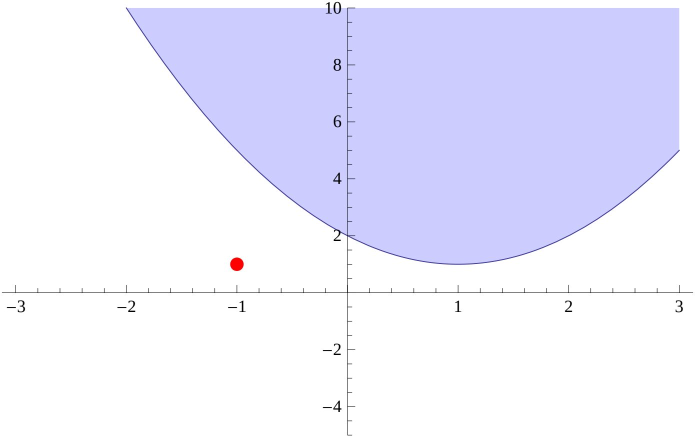
- total time on 16 threads = 3393 seconds
- "optimization status 6"
- solution is wildly infeasible

Gurobi:

- time: 31.1 seconds
- "Numerical trouble encountered"

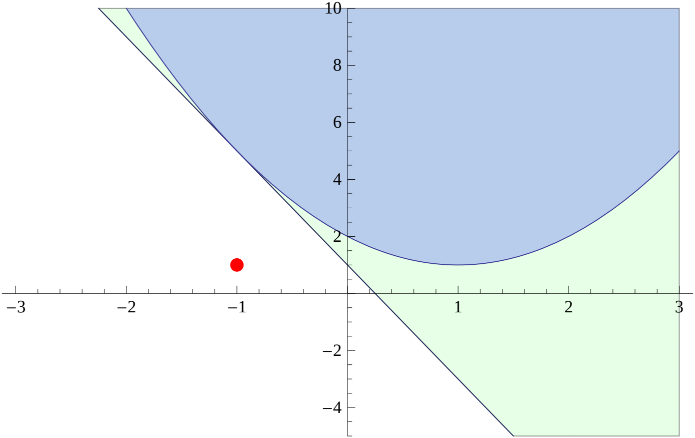
Cutting-plane method

Candidate solution violates conic constraint



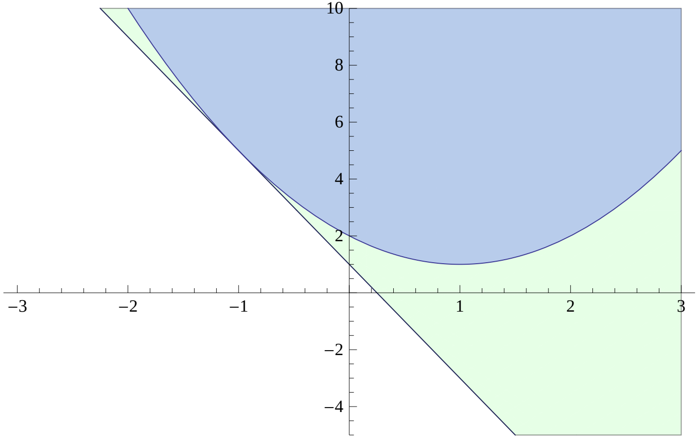
Cutting-plane method

Separate: find a linear constraint also violated



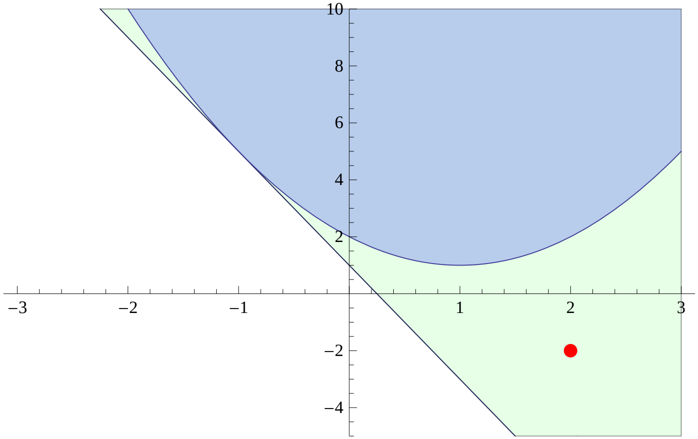
Cutting-plane method

Solve again with linear constraint



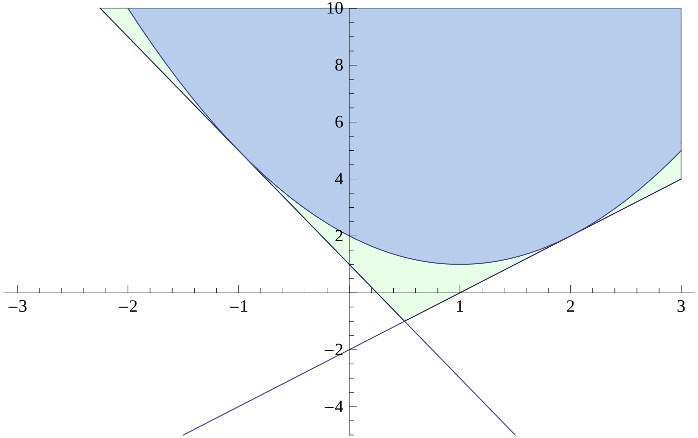
Cutting-plane method

New solution still violates conic constraint



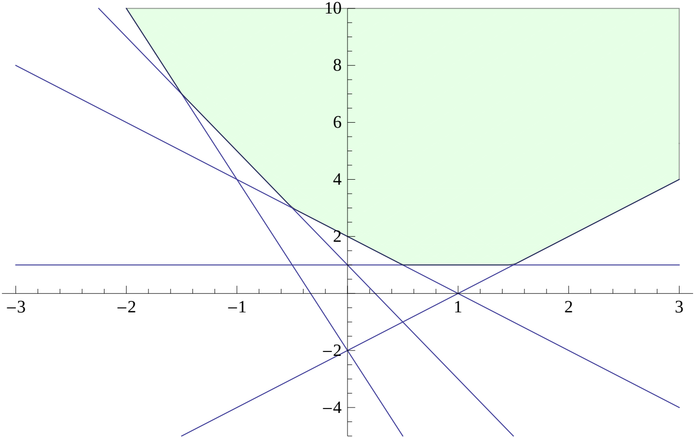
Cutting-plane method

Separate again



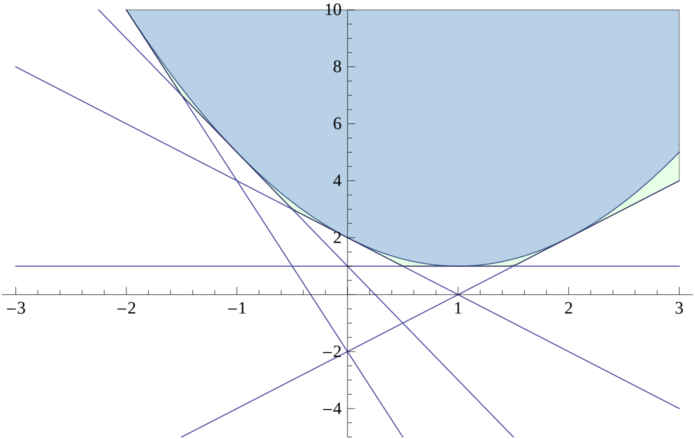
Cutting-plane method

We might end up with many linear constraints



Cutting-plane method

... which approximate the conic constraint



conic constraint:

$$\sqrt{x_1^2 + x_2^2 + \ldots + x_k^2} = \|x\|_2 \le y$$

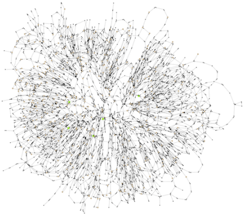
candidate solution:

$$(x^*, y^*)$$

cutting-plane (linear constraint):

$$\|x^*\|_2 + \frac{x^{*T}}{\|x^*\|_2}(x - x^*) = \frac{x^{*T} x}{\|x^*\|_2} \le y$$

Polish 2003-2004 case
CPLEX: "opt status 6"
Gurobi: "numerical trouble"



Example run of cutting-plane algorithm:

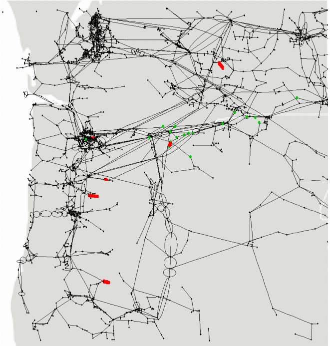
Iteration	Max rel. error	Objective
1	1.2e-1	7.0933e6
4	1.3e-3	7.0934e6
7	1.9e-3	7.0934e6
10	1.0e-4	7.0964e6
12	8.9e-7	7.0965e6

Total running time: 32.9 seconds

Back to motivating example

BPA case

- standard OPF: cost 235603, 7 lines unsafe $\geq$ 8% of the time
- CC-OPF: cost 237297, every line safe $\geq$ 98% of the time
- run time $=$ 9.5 seconds (one cutting plane!)



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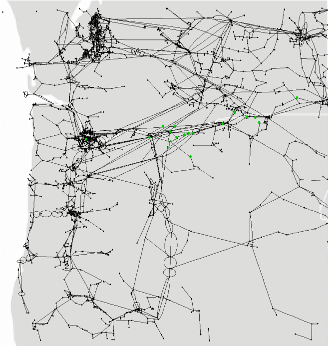

Part II: Modeling line temperature

with Jose Blanchet and Juan Li (Columbia)

Background

- 2003 North American blackout: initiated by several line trips

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- If the line overheats enough, it may sag and experience a contact/arc, which will cause a trip
- If overheating is detected, and is deemed risky, the line will may be preemptively tripped

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- What is risky?

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- What is risky? What is a critical temperature?

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- If the line overheats enough, it may sag and experience a contact/arc, which will cause a trip
- If overheating is detected, and is deemed risky, the line will may be preemptively tripped
- What is risky? What is a critical temperature?
- 2003 event: critical temperatures estimates were sometimes incorrect.

IEEE Standard 738

- A comprehensive method for determining the temperature of a power line,

IEEE Standard 738

- A comprehensive method for determining the temperature of a power line, as a function of current and pause physical properties of the conductor

IEEE Standard 738

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- It attempts to account for:

IEEE Standard 738

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IEEE Standard 738

- A comprehensive method for determining the temperature of a power line, as a function of current and pause physical properties of the conductor .
- It attempts to account for: wind, and ambient temperature, day of the year, latitude and longitude, angle between wind and conductor,

IEEE Standard 738

- A comprehensive method for determining the temperature of a power line, as a function of current and pause physical properties of the conductor .
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- It also relies on the heat equation for a "static" calculation.
- Note: power lines can be more than 100 miles long.
- How can we account for data uncertainty, errors, unavailability?

The heat equation on a 1-dimensional line

- Line modeled as one-dimensional object parameterized by x, $0 \leq x \leq L$.
- Time domain: $[0, \tau]$

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- *Heat equation:*

$$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2} + \alpha I^2(t) - \nu(T(x,t) - T^{ext}(x,t)),$$

where $\kappa \geq 0$, $\alpha \geq 0$ and $\nu \geq 0$ are (line dependent) constants, and $T^{ext}(x,t)$ is the ambient temperature at (x,t)

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$$\frac{\partial T(x,t)}{\partial t} = \alpha I^2 - \nu(T(x,t) - G(\mathbf{h}(\mathbf{x})).$$

$\mathbf{h}(\mathbf{x}) \ = \$ a random variable, at x.

This paper: stochasticity in the spatial domain (x)

CDC '13: stochasticity in the time domain (t)

This paper: stochasticity in the spatial domain (x)

CDC '13: stochasticity in the time domain (t)

The goal: algorithm- and data-driven estimates for "safe" current/temperature limits

Back to the stochastic heat equation

$$\frac{\partial T(x,t)}{\partial t} = \alpha I^2 - \nu(T(x,t) - G(\mathbf{h}(\mathbf{x})).$$

Recall: $x \in [0, L]$, $t \in [0, \tau]$

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Integrate and divide by L, get

$$\frac{1}{L}\int_0^L \frac{\partial T(x,t)}{\partial t}dx \ = \ \alpha I^2(t) - \frac{\nu}{L}\int_0^L T(x,t)dx + \frac{\nu}{L}\int_0^L G(\mathbf{h}(\mathbf{x}))dx.$$

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$$\frac{1}{L}\int_0^L \frac{\partial T(x,t)}{\partial t} dx \;=\; \frac{d}{dt}\frac{1}{L}\int_0^L T(x,t)dx \;=\; \frac{d\mathbf{H}(\mathbf{t})}{dt}.$$

$$\mathbf{H}(\mathbf{t}) \triangleq \frac{1}{L}\int_0^L T(x,t)dx \quad \text{(average internal line temperature at t)}$$

Back to the stochastic heat equation

$$\frac{\partial T(x,t)}{\partial t} = \alpha I^2 - \nu(T(x,t) - G(\mathbf{h}(\mathbf{x})).$$

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$$\mathbf{R} \triangleq \frac{1}{L}\int_0^L G(\mathbf{h}(\mathbf{x}))dx \quad \text{(average ambient temperature,}$$

Back to the stochastic heat equation

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Once more

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$$\mathbf{H(t)} = \int_0^t e^{-\nu(t-s)}\alpha I^2(s)ds + \mathbf{R}(1-e^{-\nu t}) + Ce^{-\nu t},$$

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Control goal: make $I(t)$ "large",

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Control goal: make $I(t)$ "large", but with $P\left(\max_{t \in [0,\tau]} \mathbf{H(t)} > k\right) \leq \epsilon$

Constant control: $l(t) = \bar{l}$, for all $t \in [0, \tau]$

Constant control: $I(t) = \bar{I}$, for all $t \in [0, \tau]$

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Constant current $\Rightarrow \mathbf{H(t)} = (\frac{\alpha}{\nu} \bar{I}^2 + \mathbf{R})(1 - e^{-\nu t}) + C e^{-\nu t}$

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Solution:

$$\bar{I}^2 \quad \leq \quad \frac{\nu}{\alpha} \frac{k - Ce^{-\nu\tau} - \rho_\epsilon(1 - e^{-\nu\tau})}{1 - e^{-\nu\tau}}$$

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Simplification:

$\mathbf{R}$ is a discrete random variable. $P(\mathbf{R} = r_i) = p_i, i = 1, 2, \ldots, n$ (known).

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3. At time $\tau/2$, we **observe the value** of $\mathbf{R}$. Assuming $\mathbf{R} = r_i$, then for all $t \in [\tau/2, \tau]$, we set $I(t) = I_{2,i}$.

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(a) $P(\mathbf{H}(\boldsymbol{\tau}) > k) < \epsilon$.

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(c) What about performance?

We want to maximize:

- "Total" current: $\frac{\tau}{2}\ I_1 \ + \frac{\tau}{2}\ I_{2,i} \quad ?$

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$\mathbf{F(I_1, I_2)}$ **:** a monotonely increasing function of I_1, I_2

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(c) Maximize:

$$\sum_{i=1}^{n} F(I_1, I_{2,i}) p_i$$

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} F(l_1, l_{2,i}) p_i \\
\text{s.t.} \quad & P(\mathbf{H}(\boldsymbol{\tau}) > k) \le \epsilon
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Recall:

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\mathbf{H}(\boldsymbol{\tau}) \quad = \quad \int_0^{\tau} e^{-\nu(\tau - s)} \alpha I^2(s) ds + \mathbf{R}(1 - e^{-\nu\tau}) + C e^{-\nu\tau},
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\mathbf{H}(\boldsymbol{\tau}) &= \int_0^{\tau} e^{-\nu(\tau - s)} \alpha I^2(s) ds + \mathbf{R}(1 - e^{-\nu\tau}) + Ce^{-\nu\tau}, \\
&= v_1 \ I_1^2 \ + \ v_2 \ I_{2,i}^2 \ + \ r_i(1 - e^{-\nu\tau}) \ + \ Ce^{-\nu\tau} \qquad \textbf{in state i}
\end{aligned}
$$

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Recall:

$$\begin{aligned} \mathbf{H}(\tau) &= \int_0^{\tau} e^{-\nu(\tau - s)} \alpha I^2(s) ds + \mathbf{R}(1 - e^{-\nu\tau}) + C e^{-\nu\tau}, \\ &= v_1 \ I_1^2 \ + \ v_2 \ I_{2,i}^2 \ + \ r_i(1 - e^{-\nu\tau}) \ + \ C e^{-\nu\tau} \qquad \textbf{in state i} \end{aligned}$$

So chance constraint is of the form:

$$\sum_{i=1}^{n} \mathbb{I}\{v_1 \ I_1^2 \ + \ v_2 \ I_{2,i}^2 > u \ - \ r_i(1 - e^{-\nu\tau}) \ - \ C e^{-\nu\tau}\} p_i \ \leq \ \epsilon.$$

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\mathbf{H}(\boldsymbol{\tau}) &= \int_0^{\tau} e^{-\nu(\tau - s)} \alpha I^2(s) ds + \mathbf{R}(1 - e^{-\nu\tau}) + Ce^{-\nu\tau}, \\
&= v_1 \ I_1^2 \ + \ v_2 \ I_{2,i}^2 \ + \ r_i(1 - e^{-\nu\tau}) \ + \ Ce^{-\nu\tau} \qquad \textbf{in state i}
\end{aligned}
$$

So chance constraint s of the form:

$$
\sum_{i=1}^{n} \mathbb{I}\{\underbrace{v_1 \ I_1^2}_{z_1} \ + \ \underbrace{v_2 \ I_{2,i}^2}_{z_2(i)} > \underbrace{u \ - \ r_i(1 - e^{-\nu\tau}) \ - \ Ce^{-\nu\tau}}_{w_i}\} p_i \ \le \ \epsilon.
$$

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \tilde{F}(z_1, z_2(i))\ p_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} \mathbb{I}\{z_1 + z_2(i) > w_i\} p_i \ \le \ \epsilon \\
& z_1 + z_2(i) \ \le \ u_i \quad (w_i < u_i) \\
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Lemma: At optimality,

$$z_1 \ + \ z_2(i) \ = \ w_i \text{ or } u_i, \quad \text{all } i$$

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Lemma: At optimality,

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$\longrightarrow$ Use **binary** variable

$$y_i = \begin{cases} 0 & \text{when} \quad z_1 + z_2(i) = w_i \\ 1 & \text{when} \quad z_1 + z_2(i) = u_i \end{cases}$$

Continuous knapsack problem

$$\max \sum_{i=1}^{n} \tilde{F}(z_1, w_i - z_1) p_i (1 - y_i) + \tilde{F}(z_1, u_i - z_1) p_i y_i$$

$$\text{s.t.} \sum_{i=1}^{n} u_i p_i y_i \ \leq \ \epsilon$$

$$0 \ \leq \ z_1 \ \leq \ \bar{k}$$

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$$\max_{z_1 \in [0, \bar{k}]} M(z_1)$$

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Practicable!

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Practicable! Grid over z_1

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Practicable! Grid over z_1 + knapsack for given z_1

A useless result

Theorem. (B. and Mc Closky 2010)
Consider a $0-1$ knapsack problem on N variables

$$\max \sum_{j=1}^{N} p_j x_j, \qquad \text{s.t.} \sum_{j=1}^{n} a_j x_j \leq b,$$
$$x_j = 0 \text{ or } 1, \text{ all } j.$$

For each fixed tolerance $0 < \delta < 1$ there is a linear program LP with the following properties

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For each fixed tolerance $0 < \delta < 1$ there is a linear program LP with the following properties

- The number of variables and constraints in LP is $O(N^2)$.
- The x_j are among the variables of LP (a "lifted" formulation)
- The solution of LP, together with a simple rounding for the x_j variables yields a (binary) solution for the knapsack that is guaranteed to be within δ of the optimum.

$$
\begin{array}{ll}
\max & \sum_{i=1}^{n} F(I_1, I_{2,i}) p_i \\
\text{s.t.} & P(\mathbf{H}(\boldsymbol{\tau}) > k) \leq \epsilon \\
& \mathbf{H}(\boldsymbol{\tau}) \ \leq \ u \quad (k < u < \text{ critical temp}) \\
& I_1 \ \leq \ L(\tau/2, k), \quad \text{etc.}
\end{array}
$$

$$
\begin{aligned}
\mathbf{H}(\boldsymbol{\tau}) \quad &= \quad v_1 \ I_1^2 \ + \ v_2 \ I_{2,i}^2 \ + \ r_i(1 - e^{-\nu\tau}) \ + \ Ce^{-\nu\tau} \quad \textcolor{red}{\textbf{in state i}} \\
v_1 \ &= \ \int_0^{\tau/2} e^{-\nu(\tau - s)} \alpha \ ds, \quad v_2 \ = \ \int_{\tau/2}^{\tau} e^{-\nu(\tau - s)} \alpha \ ds
\end{aligned}
$$

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} F(I_1, I_{2,i}) p_i \\ \text{s.t.} \quad & P(\mathbf{H}(\boldsymbol{\tau}) > k) \leq \epsilon \\ & \mathbf{H}(\boldsymbol{\tau}) \ \leq \ u \quad (k < u < \text{ critical temp}) \\ & I_1 \ \leq \ L(\tau/2, k), \quad \text{etc.} \end{aligned}$$

$$\mathbf{H}(\boldsymbol{\tau}) \ = \ v_1 \ I_1^2 \ + \ v_2 \ I_{2,i}^2 \ + \ r_i(1 - e^{-\nu\tau}) \ + \ Ce^{-\nu\tau} \quad \textbf{in state i}$$

$$v_1 \ = \ \int_0^{\tau/2} e^{-\nu(\tau - s)} \alpha \ ds, \quad v_2 \ = \ \int_{\tau/2}^{\tau} e^{-\nu(\tau - s)} \alpha \ ds$$

What if $\mathbf{F(I_1, I_{2,i}) \ = \ v_1 \ I_1^2 \ + \ v_2 \ I_{2,i}^2}$?

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} F(I_1, I_{2,i}) p_i \\
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\end{aligned}$$

What if $\mathbf{F(I_1, I_{2,i}) \ = \ v_1 \ I_1^2 \ + \ v_2 \ I_{2,i}^2}$?
Lemma: (again) At optimality, $v_1 \ I_1^2 + v_2(i) \ I_{2,i}^2 \ = \ w_i$ or u_i, all i

A straight knapsack problem

$$\max \sum_{i=1}^{n} \tilde{F}(w_i)p_i(1-y_i) + \tilde{F}(u_i)p_i y_i$$

$$\text{s.t.} \sum_{i=1}^{n} u_i p_i y_i \ \leq \ \epsilon$$

$$y_i \ = \ 0 \text{ or } 1, \ \text{all } i.$$