Recent results on solving QCQPs, and related topics

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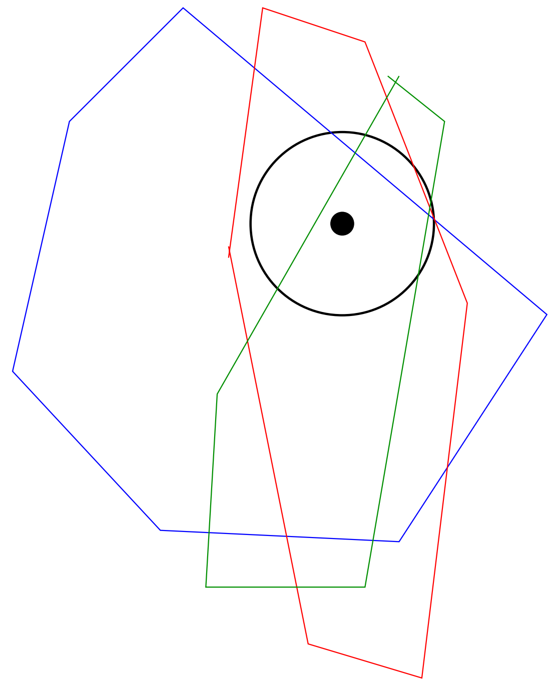
Three problems

1. The "SUV" problem

- given full-dimensional polyhedra $P^1, \ldots, P^K$ in $\mathbb{R}^d$,
- find a point closest to the origin *not* contained inside any of the P^h.

$$\begin{array}{rl} \min & \|x\|^2 \\ s.t. & x \in \mathbb{R}^d - \bigcup_{h=1}^{K} \operatorname{int}(P^h), \end{array}$$

(application: X-ray lythography)



- Typical values for d (dimension): less than 20; usually even smaller
- Typical values for K (number of polyhedra): possibly hundreds, but often less than 50
- Very hard problem

2. Cardinality constrained, convex quadratic programming.

$$\begin{aligned} \min \;\; & x^T Q x + c^T x \\ \text{s.t.} \;\; & Ax \le b \\ & x \ge 0, \quad \|x\|_0 \le k \end{aligned}$$

$\|x\|_0$ = number of nonzero entries in x.

- $Q \succeq 0$
- $x \in \mathbb{R}^n$ for n possibly large
- k relatively small, e.g. $k = 100$ for $n = 10000$
- VERY hard problem – just getting good bounds is tough

3. AC-OPF problem in rectangular coordinates

Given a power grid, determine voltages at every node so as to minimize a convex objective

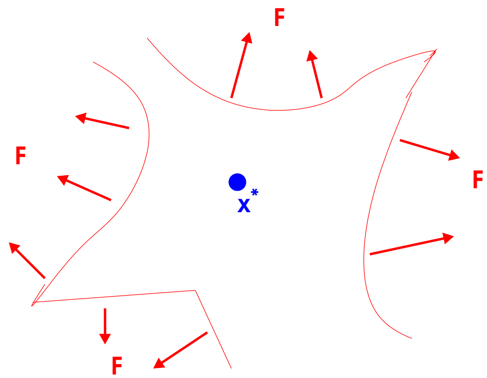
$$\begin{aligned} &\min \; v^T A v \\ &\text{s.t.} \quad L_k \le v^T F_k v \le U_k, \quad k = 1, \ldots, K \\ &\quad v \in \mathbb{R}^{2n}, \quad (n = \text{number of nodes}) \end{aligned}$$

- voltages are complex numbers; v is the vector of voltages in rectangular coordinates (real and imaginary parts)
- $A \succeq 0$
- n could be in the tens of thousands, or more
- the F_k are very sparse (neighborhood structure for every node)
- Problem HARD when grid under distress and $L_k \approx U_k$.

Why are these problems so hard

$$\text{Generic problem:} \quad \min \; Q(x), \quad s.t. \quad x \in F,$$

- $Q(x)$ (strongly) convex, especially: positive-definite quadratic
- F nonconvex

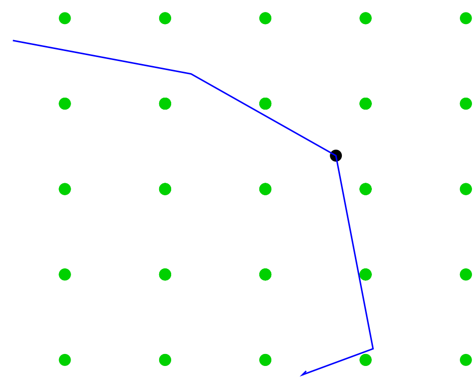


x^* solves $\min \left\{ Q(x), \quad : \quad x \in \hat{F} \right\}$ where $F \subset \hat{F}$ and $\hat{F}$ convex

$\longrightarrow$ straightforward relaxations are weak

Lattice-free cuts for **linear** integer programming

Generic problem: $\min \; c^T x, \quad s.t. \quad Ax \leq b, \;\; z \in Z^n$



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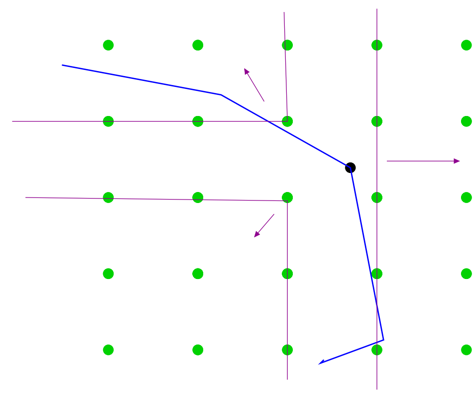

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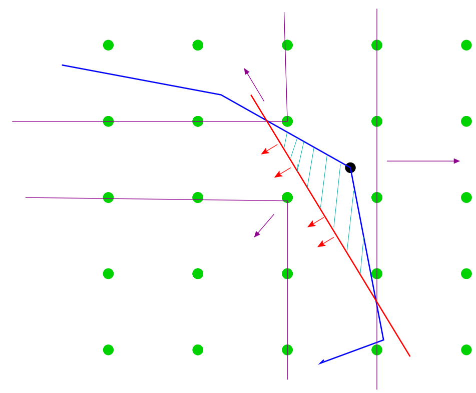

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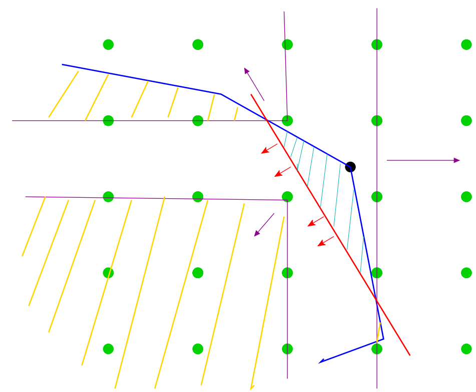

Special case: standard **disjunctions**

How to apply in a continuous, nonconvex setting?

Exclude-and-cut

$$\begin{aligned} & \min \ z \\ \text{s.t.} \quad & z \geq Q(x), \\ & x \in F \end{aligned}$$

0. $\hat{F}$: a **convex relaxation** of $\operatorname{conv}\{(x,z) \ : \ z \geq Q(x), \ x \in F\}$

1. Let $(x^*, z^*) \ = \ \operatorname{argmin}\{ \, z \ : (x,z) \in \hat{F}\}$

Exclude-and-cut

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2. Find an **open set** S s.t. $x^* \in S$ and $S \cap F = \emptyset$.
Examples: lattice-free sets, geometry

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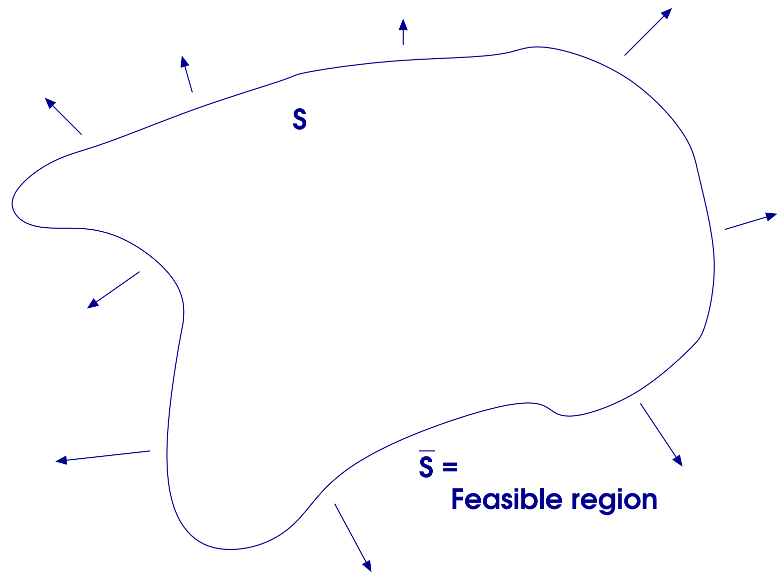
3. Add to the formulation an inequality $\boldsymbol{az + \alpha^T x \geq \alpha_0}$ valid for

$$\{\,(x,z) \, : \, x \in \overline{S}, \; z \geq Q(x)\,\}$$

but violated by (x^*, z^*).

Valid **linear** inequalities for $\{ (x, z) : x \in \overline{S}, \ z \geq Q(x) \}$.



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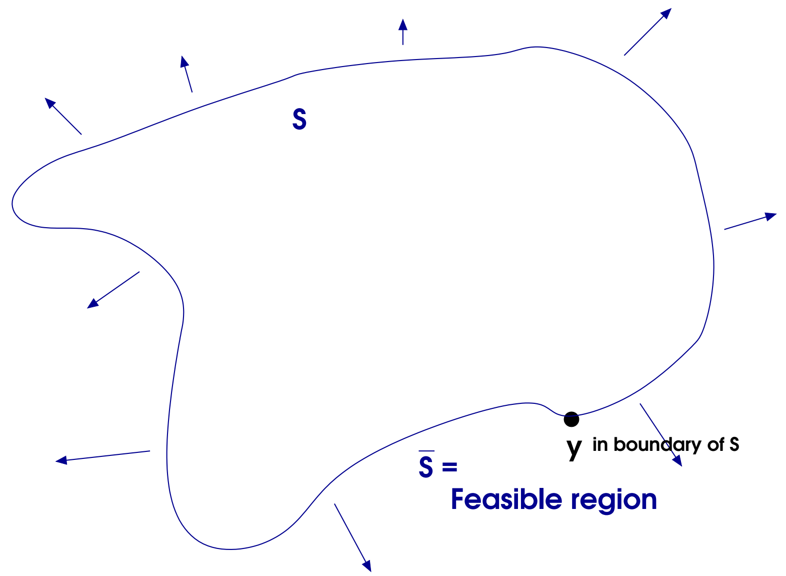

First order inequality:

$$z \;\; \geq \;\; [\nabla Q(y)]^T (x - y) + Q(y)$$

is valid EVERYWHERE – does not cut-off any points

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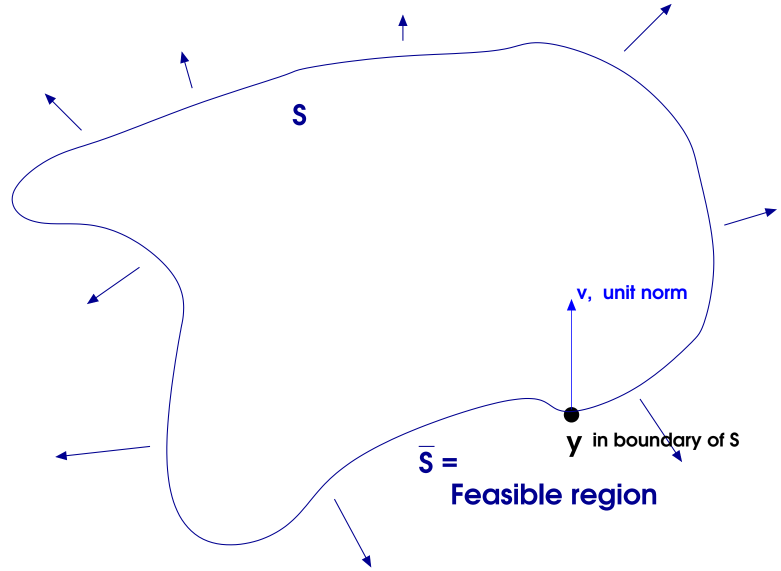

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$$z \ \geq \ \underbrace{[\nabla Q(y)]^T (x - y) + Q(y)}_{\text{first-order term} \approx Q(x)} + \underbrace{\alpha v^T (x - y)}_{\text{lifting}}$$

NOT valid EVERYWHERE: RHS $> Q(x)$ for $\alpha > 0$, $v^T(x - y) > 0$ and $x \approx y$.

– want $RHS \leq Q(x)$ in $\bar{S}$ ($\alpha = 0$ always OK)

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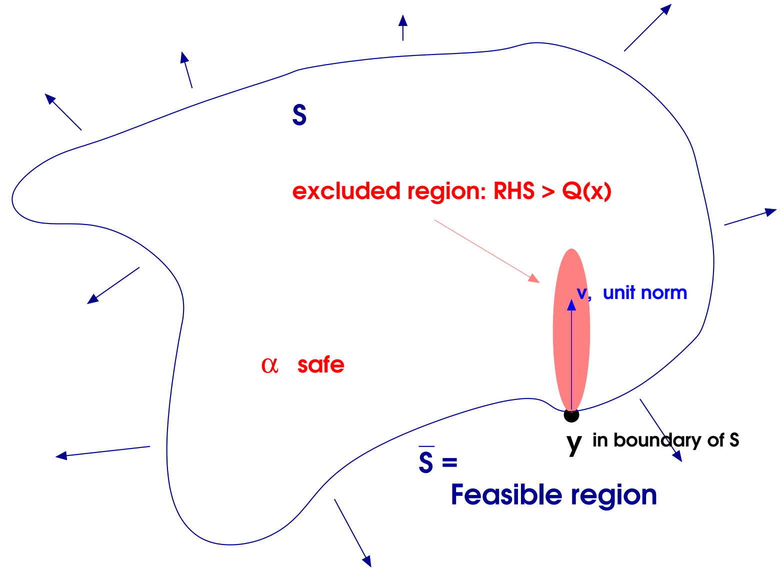

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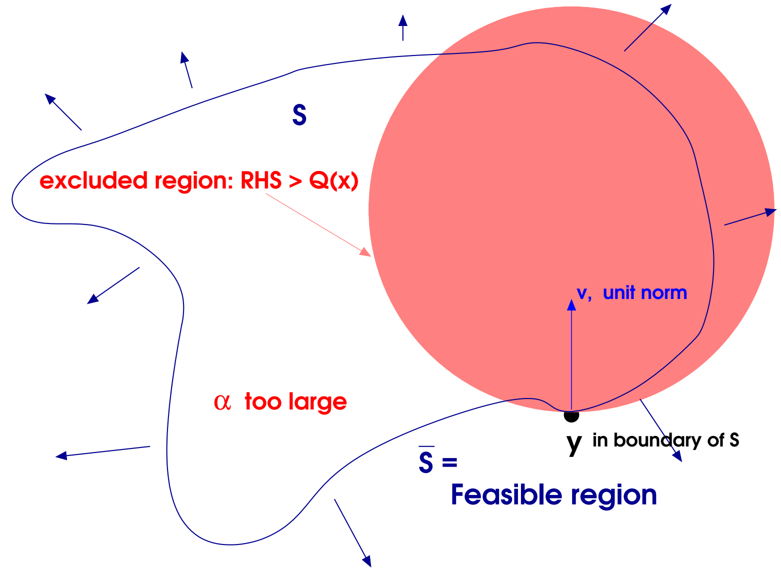

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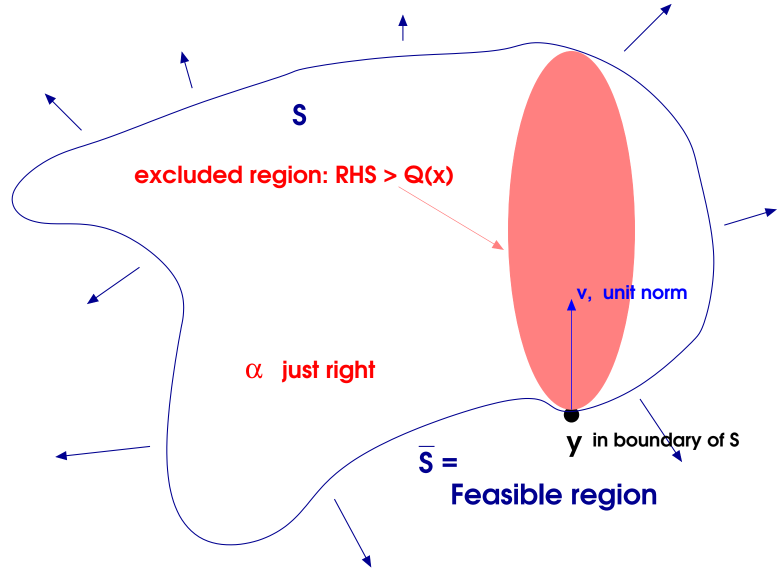

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Valid **linear** inequalities for $\mathcal{F} \doteq \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} \ : \ x \in \overline{S}, \ z \geq Q(x) \}$.

Given $y \in \partial S$, let

$$\alpha^* \doteq \ \boldsymbol{sup} \, \{ \, \boldsymbol{\alpha} \geq \mathbf{0} \ : \ Q(x) \ \geq \ [\nabla Q(y)]^T (x-y) + Q(y) + \alpha v^T (x-y) \ \}$$

valid for $\mathcal{F}$. Note: $\alpha^* = \alpha^*(v, y)$

Theorem. If Q is convex and differentiable, then $conv(\mathcal{F})$ is given by

$$Q(x) \ \geq \ [\nabla Q(y)]^T (x-y) + Q(y) \qquad \forall y$$
$$Q(x) \ \geq \ [\nabla Q(y)]^T (x-y) + Q(y) + \alpha^* v^T (x-y)$$
$$\forall v \text{ and } y \in \partial S.$$

(abridged)

Quadratics in action

Lifted first-order inequalities for $\mathcal{F} \; = \; \{ \, (x, z) \; : \; x \in \overline{S}, \;\; z \; \geq \; Q(x) \, \}$.

$$Q(x) \succ 0$$

Separation problem. Given $(x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}$, find a lifted-first order inequality maximally violated by (x^*, z^*) (if any)

Theorem: We can separate in polynomial time when:

- $\bar{S}$ (or S) is a union of polyhedra
- S is an ellipsoid or paraboloid (many cases)

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Key proof technique: S-Lemma

$$\begin{array}{rl} \min & Q_1(x) \\ \text{s.t.} & Q_2(x) \ \leq \ 0 \\ & \quad x \in \mathbb{R}^n \end{array}$$

($Q_i(x)$ **arbitrary** quadratics) is poly-time solvable

S-Lemma:

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Trust-region subproblem:

$$\begin{array}{rl} \min & Q_1(x) \\ \text{s.t.} & \|x\| \ \leq \ 1 \\ & x \in \mathbb{R}^n \end{array}$$

Extension

$$
\textbf{(TGEN):} \quad \begin{array}{lrcll}
\min & x^T A x + b^T x + c & & & \\
\text{s.t.} & \|x - x^k\|^2 & \le & f_k & k = 1, \ldots, L_k \\
 & \|x - y^k\|^2 & \ge & g_k & k = 1, \ldots, M_k \\
 & \|x - z^k\|^2 & = & h_k & k = 1, \ldots, E_k \\
 & a_i^T x & \le & b_i & i = 1, \ldots, m \\
 & x \in \mathbb{R}^n. & & &
\end{array}
$$

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\begin{array}{llrcll}
\textbf{(TGEN):} & \min & x^TAx + b^Tx + c & & & \\
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& & a_i^Tx & \leq & b_i & i = 1, \ldots, m \\
& & & x \in \mathbb{R}^n. & &
\end{array}
$$

- $P = \{x \, : \, a_i^Tx \ \leq \ b_i \quad i = 1, \ldots, m\}$
- $F^* =$ the number of **faces** of P that intersect $\bigcap_k \{x \, : \, \|x - x^k\| \leq f_k\}$.

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Theorem: For every **fixed** $L_k \geq 1, M_k \geq 0, E_k \geq 0$, problem **TGEN** can be solved in time polynomial in the problem size and F^*.

(SODA 2014)

Extends results by Ye, Ye-Zhang, Burer-Anstreicher, Burer-Yang

Even more general

Barvinok (STOC 1992):

For each fixed $\boldsymbol{p \geq 1}$, there is a polynomial-time algorithm for deciding feasibility of a system

$$\begin{aligned} x^T M_i\, x &= 0, \quad 1 \leq i \leq p, \\ \|x\| &= 1, \end{aligned}$$

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where the M_i are general matrices.

- **Non-constructive.** Algorithm says "yes" or "no."
- **Computational model?**

Theorem.

For each fixed $m \geq 1$ there is a polynomial-time algorithm that, given an optimization problem

$$\begin{aligned} \min \quad & f_0(x) \doteq x^T Q_0 x + c_0^T x \\ \text{s.t.} \quad & x^T Q_i x + c_i^T x + d_i \leq 0 \quad 1 \leq i \leq m, \end{aligned}$$

where $Q_1 \succ 0$, and $0 < \epsilon < 1$, either

(1) proves that the problem is infeasible,

or

(2) computes an ϵ-feasible vector $\hat{x}$ such that there exists no feasible $x \in \mathbb{R}^n$ with $f_0(x) < f(\hat{x}) - \epsilon$.

The complexity of the algorithm is polynomial in the number of bits in the data and in $\log \epsilon^{-1}$