Solving QCQPs

Daniel Bienstock, Columbia University

Quadratically constrained, quadratic programming:

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \quad 1 \leq i \leq m \\ & x \in \mathbb{R}^n \end{aligned}$$

Here,

$$f_i(x) = x^T M_i x + c_i^T x + d_i$$

is a general quadratic

Each M_i is $n \times n$, wlog symmetric

Folklore result: QCQP is NP-hard

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Let $w_1, w_2, \ldots, w_n$ be **integers**, and consider:

$$\begin{aligned} W^* \doteq \min \; & -\sum_i x_i^2 \\ \text{s.t.} \; & \sum_i w_i \, x_i = 0, \\ & -1 \leq x_i \leq 1, \quad 1 \leq i \leq n. \end{aligned}$$

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$\boldsymbol{W^* = -n}$, iff there exists a subset $J \subseteq \{1, \ldots, n\}$ with

$$\sum_{j \in J} w_j = \sum_{j \notin J} w_j$$

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$\rightarrow$

$$\begin{aligned} \min \ & c^T x \ - \ \boldsymbol{M} \sum_j x_j^2 \\ \text{s.t.} \ & Ax = b \\ & -1 \le x_j \le 1, \ \ 1 \le j \le n. \end{aligned}$$

(and many other similar transformations)

Even more general

Solving systems of polynomial equations:

Problem: given polynomials $p_i : \mathbb{R}^n \to \mathbb{R}$, for $1 \leq i \leq m$
find $x \in \mathbb{R}^n$ s.t. $p_i(x) = 0, \forall\, i$

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Example: find a root for $3v^6w - v^4 + 7 = 0$.

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Equivalent to the system on variables v, v_2, v_4, v_6, w, y and c:

$$
\begin{aligned}
c^2 &= 1 \\
v^2 - cv_2 &= 0 \\
v_2^2 - cv_4 &= 0 \\
v_2v_4 - cv_6 &= 0 \\
v_6w - cy &= 0 \\
3cy - cv_4 &= -7
\end{aligned}
$$

Smale's 17^{th} problem

Can a zero of n polynomial equations on n unknowns be found **approximately**, **on the average** in polynomial time, with a **uniform** algorithm?

(but we are cheating)

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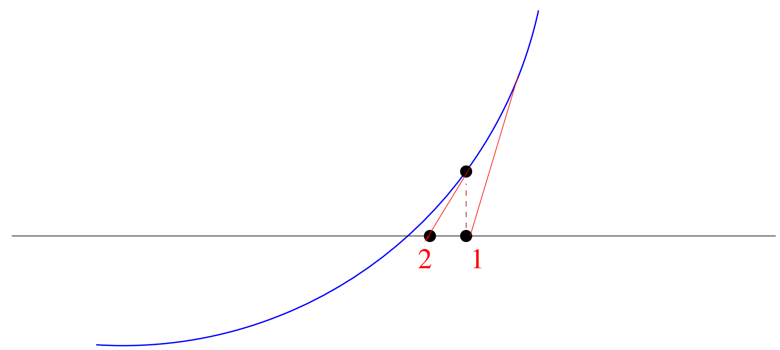
A: Newton-Raphson, of course!



$\rightarrow$ If we start near a solution, quadratic convergence

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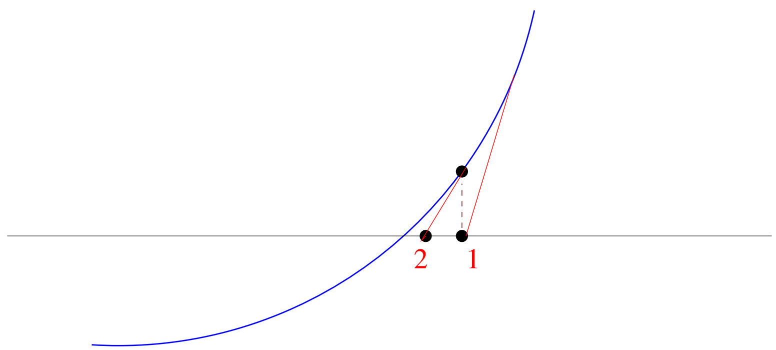

$\longrightarrow$ If we start near a solution, quadratic convergence

“Approximate” solution to a system of polynomials:

a point in the region of quadratic convergence (to a solution)

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“On the average” in polynomial time

A QCQP could be **quite** difficult!

e.g., a unique feasible solution, which additionally is an irrational vector

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- View a problem as a vector in an appropriate space
- Endow that space with an appropriate metric
  (Bombieri-Weyl Hermitian product)

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- View a problem as a vector in an appropriate space
- Endow that space with an appropriate metric (Bombieri-Weyl Hermitian product)
- In that space, uniformly sample a ball (of appropriate radius) around a given problem

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but a “nearby” problem instance could be much easier

- View a problem as a vector in an appropriate space
- Endow that space with an appropriate metric (Bombieri-Weyl Hermitian product)
- In that space, consider the set of problems given by a ball (of appropriate radius) around a given problem
- We want the algorithm to run in polynomial time, on average, in that ball

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Blum, Shub, Smale (89), Blum, Cucker, Shub, Smale (98)

First version: A non-uniform algorithm specifies the existence of an algorithm *for each input size.*

As such, we cannot write a "program" that implements the algorithm.

It is more a proof of existence of an algorithm for each input size.

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Second version: A uniform algorithm

- allows operations over real numbers
- at unit cost per operation
- with infinite precision

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- with infinite precision
- **Not!** the usual bit-model of computation

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- Bürgisser, Cucker (2012) – a deterministic $O(n^{\log \log n})$ (uniform) algorithm for computing approximate zeros
- **Techniques:** Homotopy (path-following method solving a sequence of problems), Newton's method

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So what can be done over the reals?

Take any $\{-1,1\}$-linear program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in \{-1,1\}^n. \end{aligned}$$

$\rightarrow$

$$\begin{aligned} \min \quad & c^T x \; - \; \boldsymbol{M} \sum_j x_j^2 \\ \text{s.t.} \quad & Ax = b \\ & -1 \le x_j \le 1, \;\; 1 \le j \le n. \end{aligned}$$

- Fixed number of linear constraints?
- Fixed number of quadratic constraints?
- Non-convex quadratic constraints?

The S-Lemma

Let $\boldsymbol{f}, \boldsymbol{g} : \mathbb{R}^{\boldsymbol{n}} \rightarrow \mathbb{R}$ be quadratic polynomials.

Suppose there exists $\bar{\boldsymbol{x}} \in \mathbb{R}^{\boldsymbol{n}}$ such that $\boldsymbol{g}(\bar{\boldsymbol{x}}) > \mathbf{0}$. Then

$$\boldsymbol{f}(\boldsymbol{x}) \geq \mathbf{0} \quad \text{whenever} \quad \boldsymbol{g}(\boldsymbol{x}) \geq \mathbf{0}$$

if and only if there exists $\boldsymbol{\gamma} \geq \mathbf{0}$ such that

$$\boldsymbol{f}(\boldsymbol{x}) \geq \boldsymbol{\gamma} \boldsymbol{g}(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}.$$

Yakubovich (1971), also much earlier, related work

Corollary: Can solve

$$\mathbf{min}\{\boldsymbol{f}(\boldsymbol{x}) : \boldsymbol{g}(\boldsymbol{x}) \geq \mathbf{0}\}$$

in polynomial time (using semidefinite programming)

Note: duality may not hold if there is more than one quadratic constraint

Special case: the trust-region subproblem

$$\min\{f(x)\ :\ g(x) \leq 0\}$$

can be solved in polynomial time, where f, g quadratics, g strictly convex

Scale, rotate, translate:

$$\min\{f(x)\ :\ \|x\| \leq 1\}$$

can be solved in poly time $\rightarrow$ $\log \epsilon^{-1}$

Y. Ye (1992) $\rightarrow$ $\log \log \epsilon^{-1}$

How about *extensions* of the trust-region subproblem?

Sturm-Zhang (2003)

Where $f(x)$ is a quadratic,

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad \|x\| &\leq 1 \\ a^T x &\leq b \qquad (\textbf{one} \text{ linear side constraint}) \end{aligned}$$

can be solved in polynomial time, as can

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad \|x\| &\leq 1 \\ \|x - x^0\| &\leq r_0 \qquad (\textbf{one} \text{ additional convex ball constraint}) \end{aligned}$$

Ye-Zhang (2003)

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad \|x\| &\leq 1 \\ a_i^T x &\leq b_i \quad i = 1, 2 \\ (a_1^T x - b_1)(a_2^T x - b_2) &= 0 \end{aligned}$$

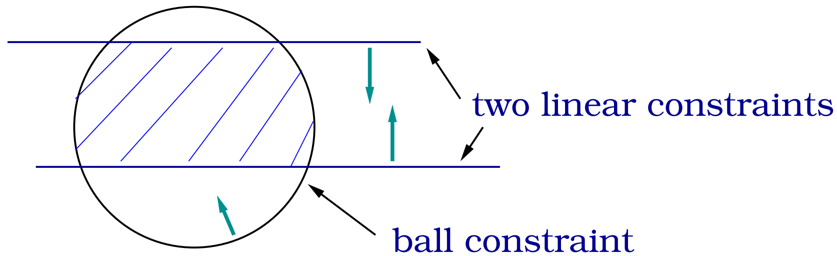
(two linear side constraints, but at least one binding)

Anstreicher-Burer (2012)

In polynomial time, one can solve a problem of the form

$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & \|x\| \le 1 \\ & a_i^T x \le b_i \quad i = 1,2 \end{aligned}$$

provided the two linear constraints are parallel:



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$$\begin{array}{rrcll} \min & x^TQx + c^Tx & & & \\ \text{s.t.} & \|x\| & \leq & 1 & \\ & a_i^Tx & \leq & b_i & i = 1, 2 \end{array}$$

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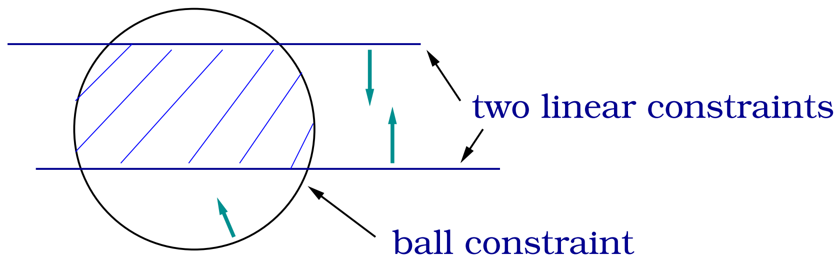

$$\rightarrow \min \{ \, x^TQx + c^Tx \, : \, l \leq x_1 \leq u, \; \|x\| \leq 1 \, \}$$

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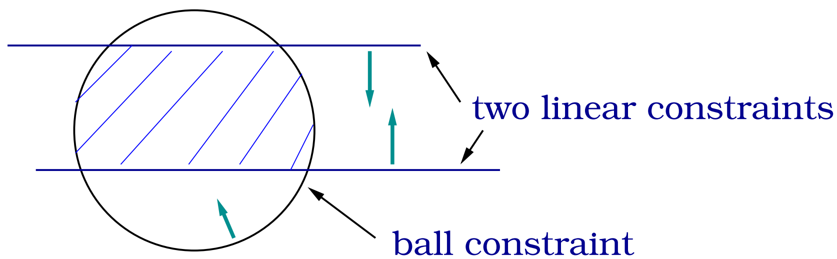

$$\rightarrow \min \{ x^T Q x + c^T x \, : \, l \le x_1 \le u, \; \|x\| \le 1 \}$$

restate as:

$$\begin{aligned} \min \quad & \sum_{i,j} q_{ij} X_{ij} \; + \; c^T x \\ \text{s.t.} \quad & X_{11} + lu \le (l+u) x_1 \\ & \|X_{.1} - l x\| \le x_1 - l \\ & \|u x - X_{.1}\| \le u - x_1 \\ & \sum_j X_{jj} \le 1 \quad , \quad X \succeq x x^T \end{aligned}$$

Lemma: This problem has an optimal solution with $X = xx^T$. Also: Ye-Zhang

Burer-Yang (2012)

In polynomial time, one can solve a problem of the form

$$
\begin{aligned}
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\end{aligned}
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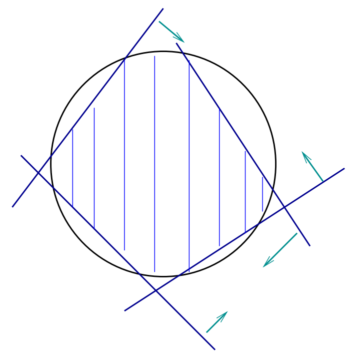
if no two linear inequalities are simultaneously binding in the feasible region



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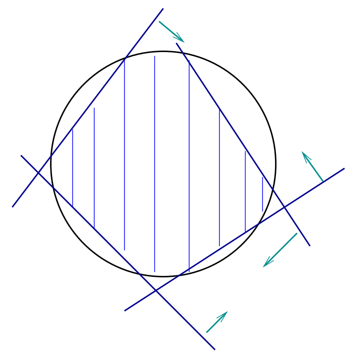

Lemma: the following problem has an optimal solution with $X = xx^T$.

$$\begin{aligned} \min \quad & \sum_{i,j} q_{ij} X_{ij} \; + \; c^T x \\ \text{s.t.} \quad X_{11} + lu & \le (l+u)x_1 \\ \|b_i x - X a_i\| & \le b_i - a_i^T x \qquad i \le m \\ b_i b_j - b_j a_i^T x - b_i a_j^T x + a_i^T X a_j & \le 0 \qquad i < j \le m \\ \sum_j X_{jj} \le 1 \quad , & \quad X \succeq xx^T \end{aligned}$$

This talk (B. and Alex Michalka, SODA 2014)

$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & \|x - \mu_h\| \le r_h, \quad h \in S, \\ & \|x - \mu_h\| \ge r_h, \quad h \in K, \\ & x \in P \doteq \{\, x \in \mathbb{R}^n \,:\, Ax \le b \,\} \end{aligned}$$

Theorem.
For each fixed $|S|$, $|K|$ can be solved in polynomial time if either

(1) $|S| \ge 1$ and polynomially large number of faces of P intersect

$$\bigcap_{h \in S} \{x \in \mathbb{R}^n \,:\, \|x - \mu_h\| \le r_h\},$$

or

(2) $|S| = 0$ and the number of rows of A is bounded.

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Anstreicher-Burer: Case (1) with 3 faces of P meeting the feasible region.
Burer-Yang: Case (1) with $m + 1$ faces of P meeting the feasible region.

More precise statement for case (1)

$$
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& x \in P \doteq \{\, x \in \mathbb{R}^n \,:\, Ax \leq b \,\}
\end{aligned}
$$

Theorem.
For each fixed $|S| \geq 1$, $|K|$ there is an algorithm that solves the problem, to tolerance $0 < \epsilon < 1$ in time

(a) Polynomial in the number of bits in the data and $\log \epsilon^{-1}$

(b) Linear in the number of faces of P that intersect

$$\bigcap_{h \in S} \{x \in \mathbb{R}^n \,:\, \|x - \mu_h\| \leq r_h\}.$$

Not hard **Lemma**

Given a collection of balls $B_h \subset \mathbb{R}^n$ $(h \in S)$

and a polyhedron

$$P \;=\; \{x \in \mathbb{R}^n \;:\; Ax \leq b\},$$

there is an algorithm that lists the faces of P that intersect $\bigcap_{h \in S} B_h$

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Proof sketch. Use e.g. breadth-first search on the faces of P, starting with P itself.

Basic step:

- Pick a row $a_i^T x \leq b_i$ of $Ax \leq b$.
- Impose $a_i^T x = b_i$.
- Test for feasibility. If feasible, found a new face.

Basic Idea

$$\min\{x^TQx+c^Tx \,:\, \|x-\mu_h\| \le r_h,\ h \in S, \quad \|x-\mu_h\| \ge r_h,\ h \in K, \quad Ax \le b\}$$

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Let $\boldsymbol{x^*}$ be optimal. Trivial: there exist (possibly empty) subsets

$\boldsymbol{S^=}$ of $\boldsymbol{S}$, $\boldsymbol{K^=}$ of $\boldsymbol{K}$, and $\boldsymbol{I^=}$ of the rows of $\boldsymbol{Ax \le b}$, such that

$$\|x^* - \mu_h\| = r_h \quad \forall\, h \in S^= \cup K^=, \quad a_i^T x^* = b_i \quad \forall\, i \in I^=$$

$$\|x^* - \mu_h\| < r_h \quad \forall\, h \in S - S^=, \quad \|x^* - \mu_h\| > r_h \quad \forall\, h \in K - K^=$$

$$a_i^T x^* < b_i \quad \forall\, i \notin I^=.$$

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$\boldsymbol{(S^=, K^=, I^=)}$: an **optimal** triple.

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Algorithm will **guess** $\boldsymbol{(S^=,\ K^=,\ I^=)}$ (actually, **compute** $\boldsymbol{I^=}$).

Basic Idea

$$\min\{x^TQx + c^Tx \;:\; \|x-\mu_h\| \le r_h,\ h \in S, \quad \|x-\mu_h\| \ge r_h,\ h \in K, \quad Ax \le b\}$$

Let $\boldsymbol{x^*}$ be optimal. Trivial: there exist (possibly empty) subsets

$\boldsymbol{S^=}$ of $\boldsymbol{S}$, $\boldsymbol{K^=}$ of $\boldsymbol{K}$, and $\boldsymbol{I^=}$ of the rows of $\boldsymbol{Ax \le b}$, such that

$$\|x^* - \mu_h\| = r_h \quad \forall\, h \in S^= \cup K^=, \quad a_i^T x^* = b_i \quad \forall\, i \in I^=$$

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$\boldsymbol{(S^=,\ K^=,\ I^=)}$: an **optimal** triple. x^*: **tight** for $(S^=,\ K^=,\ I^=)$

Algorithm will **guess** $\boldsymbol{(S^=,\ K^=,\ I^=)}$ (actually, **compute** $\boldsymbol{I^=}$).

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Geometry, 1

Notation. Given a ball $B = \{x \in \mathbb{R}^n \,:\, \|x - \hat{\mu}_i\| \leq \hat{r}\}$,

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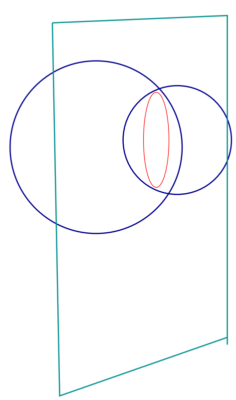
Lemma. Let $B_i = \{x \in \mathbb{R}^n : \|x - \mu_i\| \leq r_i\}$, $i = 1, 2$, be **distinct** and **intersecting**.

There exists an $(n-1)$-dim hyperplane $\boldsymbol{H}$, a point $\boldsymbol{v} \in \boldsymbol{H}$, and $\boldsymbol{r} \geq \mathbf{0}$ such that

$$\boldsymbol{\partial B_1 \cap \partial B_2 = \{x \in H : \|x - v\| = r\}}$$

and

$$\boldsymbol{\partial B_i \cap H = \{x \in H : \|x - v\| = r\}, \quad i = 1, 2}$$



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Corollary Given balls B_i, $i \in I$, not all equal, with

$$\bigcap_{i \in I} B_i \neq \emptyset,$$

there exists an $(n-t)$-dim hyperplane $\boldsymbol{H}$ ($\boldsymbol{t} \geq \mathbf{1}$), $\boldsymbol{v} \in \boldsymbol{H}$ and $\boldsymbol{r} \geq \mathbf{0}$ s.t.

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we

(1) Restrict to a lower dimensional space

(2) Obtain a single, binding, ball constraint

The original problem:

$$
\begin{aligned}
\min \quad & x^T Q x + c^T x \\
s.t. \quad \|x - \mu_h\| &\leq r_h, \quad h \in S, \\
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Given a guess, this becomes (ignoring the non-binding constraints):

$$
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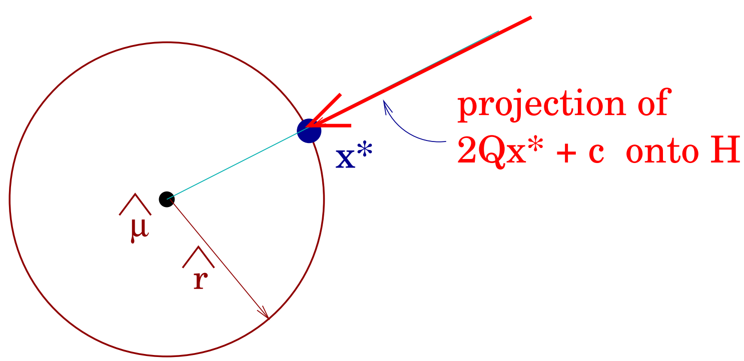
Almost correct: first-order condition restricted to H



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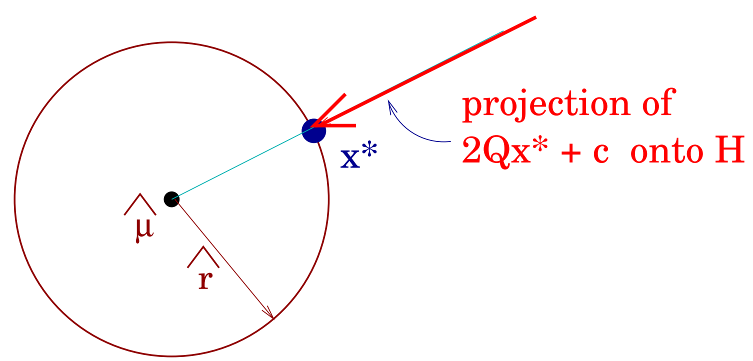

Better: Use projected quadratic representation

Theorem (abridged).

Given a triple $\boldsymbol{(\hat{S}, \hat{K}, \hat{I})}$ there is polynomially computable list of points $\boldsymbol{x^j,\ (j \in J)}$ tight for the triple, such that if $(\hat{S}, \hat{K}, \hat{I})$ is optimal, then either

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Generalization: CDT (Celis-Dennis-Tapia) problem

$$\begin{aligned} \min \quad & x^T Q_0 x + c_0^T x \\ \text{s.t.} \quad & x^T Q_1 x + c_1^T x + d_1 \;\leq\; 0 \\ & x^T Q_2 x + c_2^T x + d_2 \;\leq\; 0 \end{aligned}$$

where $\boldsymbol{Q_1 \succ 0}$, $\boldsymbol{Q_2 \succ 0}$

Even more general

Barvinok (STOC 1992):

For each fixed $\boldsymbol{p \geq 1}$, there is a polynomial-time algorithm for deciding feasibility of a system

$$\begin{aligned} x^T M_i\, x &= 0, \quad 1 \leq i \leq p, \\ \|x\| &= 1, \quad x \in \mathbb{R}^n \end{aligned}$$

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- **Non-constructive.** Algorithm says "yes" or "no."
- **Computational model?** Uniform algorithm? "Real-RAM"?

A (better?) alternative: ϵ-feasibility

For each fixed $\boldsymbol{p \geq 1}$, given a system

$$
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x^T M_i x &= 0, \quad 1 \leq i \leq p, \\
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and given $\boldsymbol{0 < \epsilon < 1}$, either

- **Prove** that the system is **infeasible**, or

- **Output** $\hat{\boldsymbol{x}} \in \mathbb{R}^{\boldsymbol{n}}$ with

$$
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Two issues: Constructiveness, and ϵ-feasibility

Modification to Barvinok's result

Assume that for each fixed $\boldsymbol{p \geq 1}$, there is an algorithm that given a system

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Assuming such an algorithm exists ...

Theorem (2014).

Assume that an algorithm for $\boldsymbol{\epsilon}$-feasibility as indicated above exists.

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Assume that an algorithm for $\boldsymbol{\epsilon}$-feasibility as indicated above exists.

For each fixed $m \geq 1$ there is a polynomial-time algorithm that, given an optimization problem

$$\begin{array}{lrcl} \textbf{min} & \boldsymbol{f_0(x)} & \doteq & \boldsymbol{x^T Q_0 x + c_0^T x} \\ \text{s.t.} & \boldsymbol{x^T Q_i x + c_i^T x + d_i} & \boldsymbol{\leq} & \boldsymbol{0 \quad 1 \leq i \leq m,} \end{array}$$

where $\boldsymbol{Q_1 \succ 0}$, and $\boldsymbol{0 < \epsilon < 1}$, either

(1) proves that the problem is infeasible,

or

(2) computes an $\boldsymbol{\epsilon}$-feasible vector $\boldsymbol{\hat{x}}$ such that there exists no feasible $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ with $\boldsymbol{f_0(x) < f(\hat{x}) - \epsilon}$.

The complexity of the algorithm is polynomial in the number of bits in the data and in $\boldsymbol{\log \epsilon^{-1}}$