Strong Formulations for Convex Functions over Non-Convex Domains

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Introduction

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- F is constrained in a nasty way, e.g.

$$x_1 x_2 \;-\; 3\,x_3 \sin(x_4) + 2\cos(x_5) \;=\; 4$$

Or,

Convex constraint:

$$Q(x) \leq q, \quad \text{and} \quad x \in F,$$

- $Q(x)$ convex, especially: convex quadratic
- F nonconvex

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Examples: lattice-free sets, geometry

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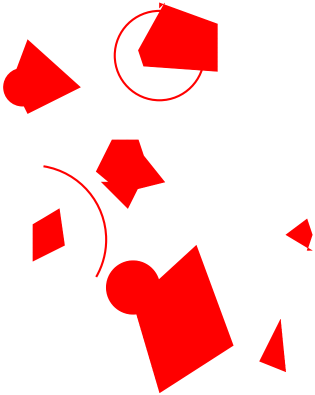
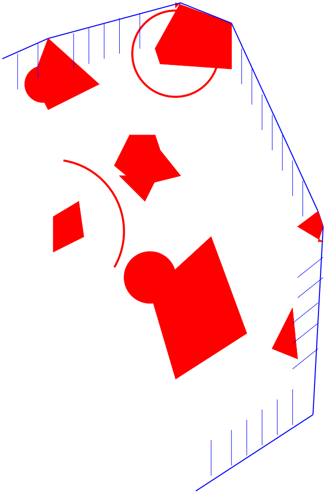
3. Add to the formulation an inequality $\mathbf{az \, + \, \alpha^T x \;\geq\; \alpha_0}$ valid for

$$\{\, (x, z) \; : \; x \in \mathbb{R}^n - S, \;\; z \geq Q(x) \,\}$$

but violated by (x^*, z^*).




= feasible region



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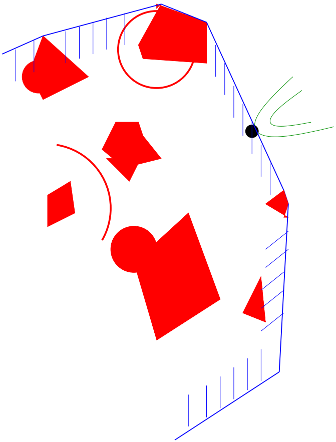

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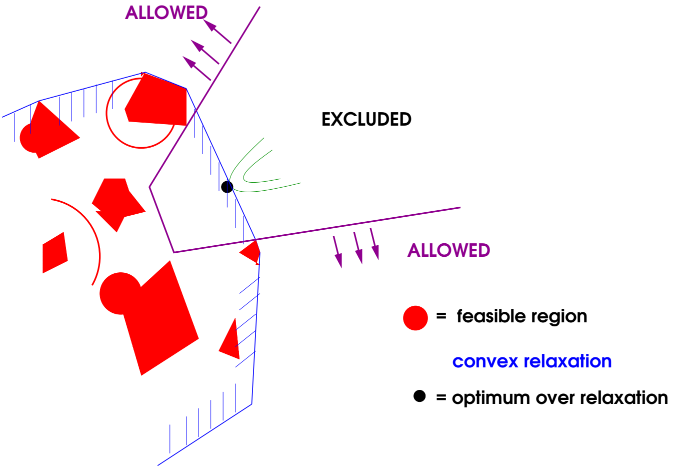
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- Can we incorporate the improvement into the formulation by adding *linear* inequalities?

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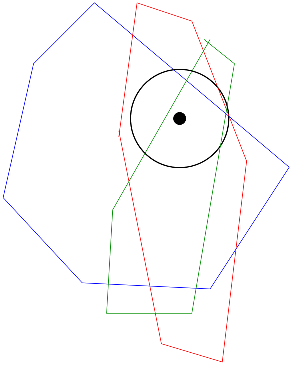
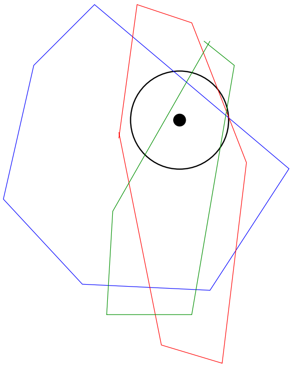
(application: X-ray lythography; see Ahmadia (2010))

(yes, it is NP-hard)

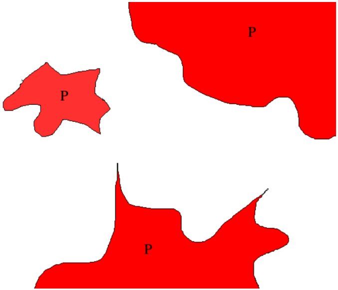




- Typical values for d (dimension): less than 20; usually even smaller
- Typical values for K (number of polyhedra): possibly hundreds, but often less than 50
- Very hard problem

First problem setting

- Let $Q(x)$ be a **strongly convex** function on $\mathbb{R}^d$,
- Let $P \subset \mathbb{R}^d$ be closed and such that $\mathbb{R}^d - P \neq \emptyset$.



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$$\left\{ (x, q) \in \mathbb{R}^{d+1} \, : \, Q(x) \leq q, \;\; x \in \mathbb{R}^d - \mathrm{int}(P) \right\}.$$

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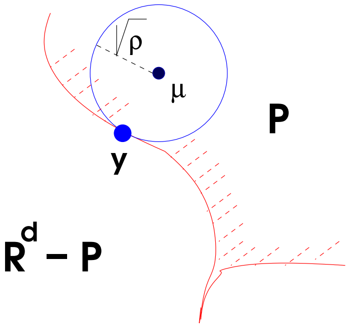
How can we use the structure of P to strengthen the inequality?

Definition:

Given $y \in \partial P$, say P is **locally flat** at y if
$\exists\ \mathcal{B}(\mu, \sqrt{\rho}) \subseteq P$ with $\|\mu - y\|^2 = \rho$ and $\rho > 0$.



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- Inequality is tight at $(y, Q(y))$, and cuts-off points $(x, Q(x))$ and $x \in \mathrm{int}(P)$.
- Largest possible α: "lifted first-order inequality".

$$\left\{(x,q) \in \mathbb{R}^{d+1} \,:\, Q(x) \,\le\, q, \quad x \in \mathbb{R}^d - \mathrm{int}(P)\right\}$$

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Theorem.

Any linear inequality valid for S is dominated by a lifted first-order inequality. More precisely,

$$conv\left\{(x,q) \in \mathbb{R}^{d+1} \,:\, Q(x) \le q, \quad x \in \mathbb{R}^d - \mathrm{int}(P)\right\} =$$

$$\{(x,q) \in \mathbb{R}^{d+1} \,:\, \text{LFO} \;+\; \text{FO}\,\}$$

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How do we make this computationally practicable?

First problem setting

- Let $Q(x)$ is a **positive definite** quadratic on $\mathbb{R}^d$,
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change in coordinates $\longrightarrow$

$$S \doteq \left\{(x, q) \in \mathbb{R}^{d+1} \,:\, \sum_{j=1}^{d} x_j^2 \le q, \quad x \in \mathbb{R}^d - \mathrm{int}(P)\right\}.$$

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Here $P = \{x \in \mathbb{R}^d \;:\; a_i^T x \le b_i,\ 1 \le i \le m\}$

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- **Theorem:** Let $(\hat{x}, \hat{q}) \in \mathbb{R}^{d+1}$ with $\hat{v} \in \mathrm{int}(P)$.

  We can compute a lifted first-order inequality maximally violated by $(\hat{x}, \hat{q})$, by solving m linearly constrained convex quadratic programs on $O(d)$ variables.

When does a point

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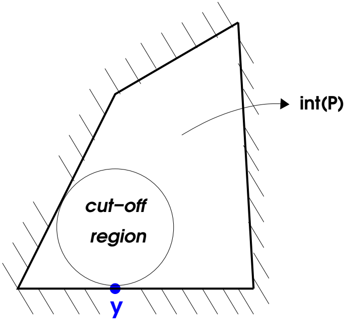
This describes the interior of a **ball**, which must be contained in $\mathrm{int}(P)$

Geometrical characterization



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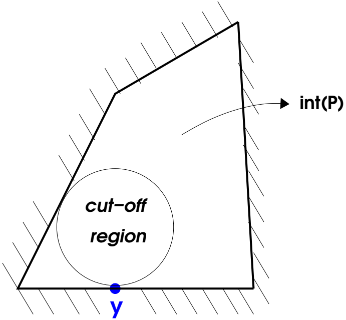

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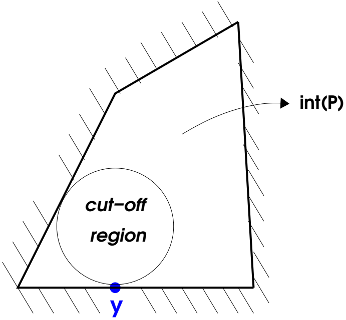

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Second setting: separating across a quadratic set

For $\mathbf{A} \succ \mathbf{0}$, polynomially separable linear inequality description for:

$$\left\{ (x, q) \in \mathbb{R}^{d+1} \, : \, \sum_{j=1}^{d} x_j^2 \, \leq \, q, \;\; x^T A x \, - \, 2c^T x \, + \, b \, \geq \, 0 \right\}$$

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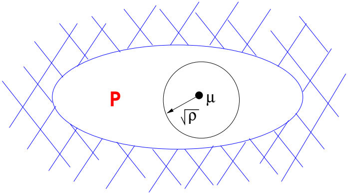

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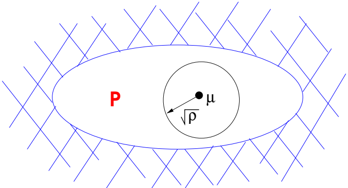

Separation problem: given$(\hat{x}, \hat{q}) \in \mathbb{R}^{d+1}$, find $\mathbf{B}(\mu, \sqrt{\rho}) \subseteq \mathbf{P}$ to max $\rho - (\hat{\mathbf{q}} - 2\mu^{\mathbf{T}}\hat{\mathbf{x}} + \mu^{\mathbf{T}}\mu)$

Second setting: separating across a quadratic set

For $\mathbf{A} \succ \mathbf{0}$, polynomially separable linear inequality description for:

$$\left\{(x,q) \in \mathbb{R}^{d+1} \,:\, \sum_{j=1}^{d} x_j^2 \le q, \;\; x^T A x - 2c^T x + b \ge 0\right\}$$

$$P = \{x \in \mathbb{R}^d \,:\, x^T A x - 2c^T x + b \le 0\}$$

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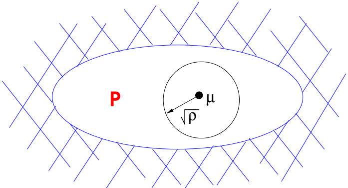

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Separation problem:

$$\min_{\mu,\rho} \quad \|\mu\|^2 - \rho - 2\bar{x}^T\mu$$

$$\text{Subject to:} \quad \{x : \|x - \mu\|^2 \leq \rho\} \subseteq P$$

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$$\min_{\mu,\rho} \quad \|\mu\|^2 - \rho - 2\bar{x}^T \mu$$

$$\text{Subject to:} \quad \{x : \|x - \mu\|^2 \le \rho\} \subseteq P$$

Theorem:
Optimal choices for μ and ρ are given by:

$$\hat{\mu} = \hat{\theta} b + (I - \hat{\theta} A)\bar{x}$$

and

$$\hat{\rho} = \|\hat{\mu}\|^2 - 2\bar{x}^T \hat{\mu} + \|\bar{x}\|^2 - \hat{\theta}(\bar{x}^T A \bar{x} - 2b^T \bar{x} + c).$$

Here, $\hat{\theta} \; = \; \frac{1}{\lambda_{\max} A}$.

Separating accross general quadratics

$$\Pi \doteq \{(x, w, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \; : \; z \ge x^T Q x + q^T x, \quad w \le x^T A x\}$$

$(A \succ 0, \; Q \succ 0)$.

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$(A \succ 0, \; Q \succ 0)$.

Linear transformation $\rightarrow$ Π is the set of points $\in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ s.t.

$$z \geq \|x\|^2 + q^T x, \quad w \leq x^T \Lambda x \quad (\Lambda > 0).$$

Write $P \doteq \{(x, w) \in \mathbb{R}^d \times \mathbb{R} \, : \, x^T \Lambda x - w \leq 0\}$, and for $\mu \in \mathbb{R}^d$, $\nu \in \mathbb{R}$,

$$M(\mu, \nu) \; \doteq \; \{(x, w) \; \in \; \mathbb{R}^d \times \mathbb{R} \; : \; \lambda_{max} \|x - \mu\|^2 + (\nu - w) \; \leq \; 0\}.$$

Then

$x \in \mathbb{R}^d - \mathrm{int}(P)$ iff $x \in \mathbb{R}^d - \mathrm{int}(M(\mu, \nu))$, for all μ, ν with $M(\mu, \nu) \subseteq P$.

- $\Pi = \{ (x, w, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \,:\, z \geq \|x\|^2 + q^T x, \ w \leq x^T \Lambda x \}$,
- $P = \{(x, w) \in \mathbb{R}^d \times \mathbb{R} \,:\, x^T \Lambda x \leq w\}$,
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Geometric characterization:

$x \in \mathbb{R}^d - \mathrm{int}(P)$ iff $x \in \mathbb{R}^d - \mathrm{int}(M(\mu,\nu)), \; \forall \mu,\nu$ with $M(\mu,\nu) \subseteq P$.

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Geometric characterization:

$x \in \mathbb{R}^d - \mathrm{int}(P)$ iff $x \in \mathbb{R}^d - \mathrm{int}(M(\mu,\nu))$, $\forall \mu, \nu$ with $M(\mu,\nu) \subseteq P$.

So, valid inequality for any μ, ν with $M(\mu,\nu) \subseteq P$:

$$\lambda_{max}\|\mu\|^2 - \lambda_{max}(2\mu + q)^T x + (\nu - w) + \lambda_{max} z \geq 0$$

Separation problem, given $(\bar{x}, \bar{w}) \in \mathrm{int}(P)$

$$\bar{w} - \nu - \lambda_{max}\|\mu\|^2 + 2\lambda_{max}\bar{x}^T\mu + 2\lambda_{max} q^T \bar{x}$$

subject to: $\nu + \min_x \{ \lambda_{max} \|x - \mu\|^2 - x^T \Lambda x \} \geq 0.$

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Lemma: This problem can be explicitly solved in polynomial time.

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and transformed it into:

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Theorem. Eigenspace not necessary for poly-time separation (only max eigenvalue of A).

Example: $f(x) \doteq 2(x_1x_2 \, + \, x_1x_3 \, + x_2x_3)$ over $[0,1]^3$.

McCormick relaxation gives zero lower bound on $f(\bar{x})$, where

$$\bar{x} = (1/2, 1/2, 1/2)^T.$$

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$f(x) = U(x) - L(x)$, where

$$\begin{aligned} U(x) &\doteq (x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_3)^2, \\ L(x) &\doteq 2(x_1^2 + x_2^2 + x_3^2). \end{aligned}$$

Example: $f(x) \doteq 2(x_1x_2 + x_1x_3 + x_2x_3)$ over $[0,1]^3$.

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$\rightarrow$ $f(x) \rightarrow F(p) \doteq (p_1^2 + p_2^2 + p_3^2) \; - \; (2p_1^2 + 2p_2^2 + \frac{1}{2}p_3^2)$.

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$$U(x) \doteq (x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_3)^2,$$
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$\longrightarrow$ $x \rightarrow (0, 0, \frac{2}{\sqrt{3}}(x_1 + x_2 + x_3))^T$

$f(x) \doteq 2(x_1 x_2 \, + \, x_1 x_3 \, + x_2 x_3)$ over $[0,1]^3$.

McCormick relaxation gives zero lower bound on $f((1/2,1/2,1/2)^T)$.

$\rightarrow$ $f(x) \, \rightarrow F(p) \; \doteq \; (p_1^2 + p_2^2 + p_3^2) \; - \; (2p_1^2 + 2p_2^2 + \frac{1}{2} p_3^2)$.

$\rightarrow$ $x \, \rightarrow \, (0, 0, \frac{2}{\sqrt{3}}(x_1 + x_2 + x_3))^T$

$\rightarrow$ **Paraboloid cut:**

$$p_1^2 + p_2^2 + (p_3 - 2\alpha\sqrt{3})^2 + \epsilon \quad \geq \quad 2p_1^2 + 2p_2^2 + \frac{1}{2} p_3^2.$$

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or,

$$f(x) \quad \geq \quad 8\alpha(x_1 + x_2 + x_3) - 12\alpha^2 - \epsilon$$

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When valid?

$f(x) \doteq 2(x_1 x_2 + x_1 x_3 + x_2 x_3)$ over $[0,1]^3$.

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$\rightarrow$ **Paraboloid cut:**

$$f(x) \quad \geq \quad 8\alpha(x_1 + x_2 + x_3) - 12\alpha^2 - \epsilon$$

- $\alpha = \epsilon = 1/10$, valid when $x_1 + x_2 + x_3 \geq 3/2$.

$f(x) \doteq 2(x_1 x_2 + x_1 x_3 + x_2 x_3)$ over $[0,1]^3$.

$\longrightarrow$ **Paraboloid cut:**

$$f(x) \quad \geq \quad 8\alpha(x_1 + x_2 + x_3) - 12\alpha^2 - \epsilon$$

- $\alpha = \epsilon = 1/10$, valid when $x_1 + x_2 + x_3 \geq 3/2$.

  get: $f(x) \geq \frac{4}{5}(x_1 + x_2 + x_3) - \frac{11}{50}$.

$f(x) \doteq 2(x_1 x_2 \; + \; x_1 x_3 \; + x_2 x_3)$ over $[0,1]^3$.

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  get: $f(x) \geq \frac{4}{5}(x_1 + x_2 + x_3) \; - \; \frac{11}{50}$. At $(.5, .5, .5)^T$ get .98

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- $\alpha = 1/2, \epsilon = 2$, valid when $x_1 + x_2 + x_3 \leq 3/2$.

$f(x) \doteq 2(x_1 x_2 \, + \, x_1 x_3 \, + x_2 x_3)$ over $[0,1]^3$.

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- $\alpha = 1/2, \epsilon = 2$, valid when $x_1 + x_2 + x_3 \leq 3/2$.

  get: $f(x) \geq 4(x_1 + x_2 + x_3) \; - \; 5$.

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$f(x) \doteq 2(x_1x_2 + x_1x_3 + x_2x_3)$ over $[0,1]^3$.

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Disjunction:

either

$$(x, f) \in \left\{(x, f) \;:\; x \in [0,1]^3,\ \sum_j x_j \geq 3/2,\ f \geq \tfrac{4}{5}(x_1 + x_2 + x_3) - \tfrac{11}{50}\right\},$$

or

$$(x, f) \in \left\{(x, f) \;:\; x \in [0,1]^3,\ \sum_j x_j \leq 3/2,\ f \geq 4(x_1 + x_2 + x_3) - 5\right\}.$$

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Proves: $f(x) \; \geq \; 1.245714(x_1 + x_2 + x_3) \; - \; 1.5571435$

valid throughout,

$f(x) \doteq 2(x_1x_2 + x_1x_3 + x_2x_3)$ over $[0,1]^3$.

$f(x) \geq \frac{4}{5}(x_1 + x_2 + x_3) - \frac{11}{50}$ valid for $x_1 + x_2 + x_3 \geq 3/2$

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Disjunction:

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Proves: $f(x) \ \geq \ 1.245714(x_1 + x_2 + x_3) \ - \ 1.5571435$

valid throughout, at $(.5, .5., .5)^T$ get .3114275.

Ongoing work

- Complement of union of two polyhedra
- Complement of union of two ellipsoids
- Complement of union of ellipsoid + half-plane