Universal investment in markets with transaction costs[*]

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Abstract

In this paper we formulate and solve the problem of universal growth optimal investment in two-asset discrete time markets with proportional transaction costs. We do not make any distributional assumptions on the market return sequences and construct a policy with growth rate at least as large as any interval policy. Since interval policies are ϵ-optimal for independent identically distributed (IID) markets (Iyengar, 2002), it follows that our policy when used in an IID market is able to "learn" the optimal interval policy and achieve growth optimality, i.e. it is a universal growth optimal policy for IID markets.

1 Introduction

The objective of growth optimal investment is to maximize the long-run interest rate. Growth optimal investment in independent identically distributed (IID) discrete time horse race markets with no transaction costs was introduced in Kelly (1956). Kelly showed that log-optimum investment, where the investor maximizes the conditional expected logarithm of the one step return, maximizes the growth rate of the cumulative wealth. Breiman (1961) extended the growth optimal framework to investment in IID discrete time markets with general asset returns. In IID markets the log-optimum investment policy is a constant rebalance policy, i.e. in every market period the current portfolio is rebalanced to a single constant portfolio, namely the log-optimum portfolio. Algoet and Cover (1988) showed that conditionally log-optimum investment is growth optimal for stationary ergodic markets with general asset returns. Subsequently, Cover (1991) formulated the problem of investment in markets where one does not make any distributional assumptions on the sequence of asset returns and constructed a sequential policy that does as well in growth rate as any other constant rebalanced policy, even those chosen in hindsight. Such an investment policy was termed a *universal* investment policy in the sense that it dominates all the policies that could possibly be optimal for an IID market. The universal result was extended to an individual sequence result in Ordentlich and Cover (1998)(see also Cover and Ordentlich, 1996).

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All of this earlier work assumes that the markets do not have transaction costs. Unfortunately, the policies designed for markets without costs do not perform well in a market with costs. In fact, in a continuous time market such policies lead to immediate bankruptcy (Davis and Norman, 1990). Although the situation in discrete time markets is not as severe (Kalai and Blum, 1997), one can do substantially better by incorporating the costs into the model. Growth optimal investment with costs was introduced by Taksar et al. (1988) in the context of geometric Brownian markets with one risky asset and cash. In this paper we will assume that the transaction costs are proportional to the transaction amount. Several related works, notably Davis and Norman (1990), Shreve and Soner (1994), and Akian et al. (1996), study utility maximization in two-asset markets with costs. Iyengar (1997) and Akian et al. (2001) extended the growth optimal investment framework to continuous time markets with several risky assets. Iyengar and Cover (2000) investigated the effect of costs in the horse race setting and established that the growth optimal policy rebalances to a portfolio that maximizes a combination of the one-step return and transaction cost. This result was extended to IID discrete time markets with several assets in Iyengar (2002). In all of this work on markets with costs it is assumed that the distribution of the asset returns in known to the investor.

In this paper we consider two-asset markets where one does not make any distributional assumptions about the market returns. For such a market we construct an investment policy with growth rate at least as large as any interval policy (see Section 2), even those chosen in hindsight. Since the results in Iyengar (2002) imply that the class of interval policies (see Section 2) is ϵ-optimal for two-asset IID markets, it follows that our policy has a growth rate arbitrarily close to optimal for any fixed IID market, i.e. our policy is a universal growth optimal policy for IID markets. A crucial step in the proof of this result is to establish that the portfolio process corresponding to an interval policy $\boldsymbol{\alpha}$ is a continuous function $\boldsymbol{\alpha}$. Continuity is guaranteed for constant rebalance policies; therefore, previous universal results (Cover, 1991; Cover and Ordentlich, 1996) implicitly assume continuity and cannot be directly used in our context. Once continuity is established, the universal result follows from simple modifications of results in the literature.

Our contributions can be better understood by constrasting our results with those in Iyengar (2002). Iyengar (2002) studies growth optimal investment in IID markets with several (i.e. greater than 2) assets whereas this paper is concerned with two-asset markets where one does not make any distributional assumptions about the market returns. The central result of this paper is the construction of a policy that does as well as any interval policy. By itself, this result does not imply any optimality results for two-asset IID markets, i.e. growth optimal investment in two-asset IID markets is not a special case of the problem considered here. However, combining our result with the result that interval policies are ϵ-optimal (Iyengar, 2002) leads to the conclusion that the policy constructed in this paper is a universal growth optimal policy for IID markets.

The outline of this paper is as follows. In Section 2 we introduce our model for a two-asset discrete time market with proportional transaction costs and briefly review the optimality results for stochastic markets established in Iyengar (2002). In Section 3 we construct the universal investment policy and discuss its properties; and Section 4 concludes with some remarks, connection to general markets and open issues.

2 Market model

We consider a two-asset market which opens for trading at discrete instants in time. The price dynamics of the assets is described by the sequence of non-negative price relative vector $\{\mathbf{x}_n = (x_n(1), x_n(2))^T : n \geq 1\}$, where $x_n(i)$, $i = 1, 2$, is the ratio of the closing to the opening price of asset i over the period n. The observed sequence of price relatives over first n market periods is denoted by $\mathbf{x}_1^n = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. We do not make any distributional assumptions about the sequence $\mathbf{x}_1^n$. However, we assume the following regularity condition.

Assumption 1 *There exist* $0 < \underline{r} < \overline{r}$ *such that* $\underline{r} \leq \frac{x_k(1)}{x_k(2)} \leq \overline{r}$, *for all* $k \geq 1$.

This regularity condition is almost always satisfied by real markets and is closely related to the notion of diversity introduced by Fernholz (2000).

The market levies proportional transaction costs at a rate λ_i on asset i, i.e., the sale of one dollar of asset i nets only $(1 - \lambda_i)$ dollars in cash. Similarly, one dollar of asset i costs $(1 + \lambda_i)$ dollars. We assume that cash mediates all transactions. Therefore, selling one dollar of asset i and investing the proceeds in asset j involves first selling one dollar of asset i to receive $(1 - \lambda_i)$ dollars in cash; then using this capital to buy $\frac{(1-\lambda_i)}{(1+\lambda_j)}$ dollars of asset j. Although we have assumed symmetric transaction costs, all the results carry over to the case of asymmetric costs.

The investor starts with an initial wealth of S_1 invested in a portfolio b_1, where the portfolio $0 \leq b \leq 1$ represents the proportion of the total wealth invested in asset 1 (note that we do not allow any short selling). In the the beginning of the n-th market period, the cumulative wealth is denoted by S_n and the portfolio by b_n, i.e. the dollar amount invested in asset 1 (resp. asset 2) is $S_n b_n$ (resp. $S_n(1 - b_n)$). In every investment period, the investor has the option to trade from b_n to a new portfolio $z_n \in [0, 1]$. However, this trade results in a loss of wealth because of the transaction costs in the market. Let $w(b_n, z_n)$ denote the net realized wealth when one dollar at portfolio b_n is traded to portfolio z_n. By equating the transaction costs $\lambda_1 |wz_n - b_n| + \lambda_2 |w(1 - z_n) - (1 - b_n)|$ to the loss in wealth $1 - w$, it follows that $w(b_n, z_n)$ is the unique positive solution of the nonlinear equality

$$\lambda_1 |wz_n - b_n| + \lambda_2 |w(1 - z_n) - (1 - b_n)| = 1 - w. \tag{1}$$

Since the transaction costs are proportional, wealth S_n invested in portfolio b_n would net $w(b_n, z_n)S_n$ at portfolio z_n.

Subsequently the market reveals the price relative vector $\mathbf{x}_n$. Since one dollar of asset i is worth $x_n(i)$ dollars, one dollar invested in portfolio z_n yields the wealth $W(z_n)$ given by

$$W(z_n) = z_n x_n(1) + (1 - z_n) x_n(2) \tag{2}$$

and the new portfolio b_{n+1} is given by

$$b_{n+1} = \frac{z_n x_n(1)}{z_n x_n(1) + (1 - z_n) x_n(2)} = \frac{z_n r_n}{z_n(r_n - 1) + 1}, \tag{3}$$

where $r_n = \frac{x_n(1)}{x_n(2)}$.

This process repeats itself in every market period. Thus, the cumulative wealth S_n is given by

$$\begin{aligned} S_n &= S_{n-1} w(b_n, z_n) W(z_n), \\ &= S_1 \prod_{k=1}^{n-1} w(b_k, z_k) W(z_k), \end{aligned} \tag{4}$$

and is invested in the portfolio b_n. In this work, we are interested in maximizing the growth rate of the wealth of the investor, i.e. very loosely speaking we want to maximize

$$g = \lim_{n\to\infty} \left\{ \frac{1}{n} \log S_n \right\} = \lim_{n\to\infty} \left\{ \frac{1}{n} \log S_1 + \frac{1}{n} \sum_{k=1}^{n} \big[\log w(b_k, z_k) + \log W(z_k) \big] \right\}.$$

This optimization problem is not yet well defined because we have not specified the constraints on the decisions of the investor. These restrictions will be motivated by the structure of the growth optimal policies in stochastic markets.

Consider the stochastic version of the problem where the sequence $\{\mathbf{x}_n : n \geq 1\}$ satisfies Assumption 1 and is, in addition, IID according to a known distribution. Suppose the investor in this stochastic market is restricted to policies that are non-anticipating and self-financing, i.e. all transaction costs and future investments are financed by the wealth generated by the policy without any fresh wealth. Denote the set of feasible policies by Π. Then the growth optimal investment problem in IID markets is as follows: characterize the optimal growth rate function $g(b)$ given by

$$g(b) = \sup_{\pi\in\Pi} \left\{ \liminf_{n\to\infty} \frac{1}{n} \mathbf{E} \log S_n^{\pi}(b) \right\}, \tag{5}$$

where $S_n^{\pi}(b)$ is the total wealth generated by policy π up to time n starting from the initial portfolio $b_1 = b$. And if the supremum in (5) is achieved, then characterize a policy π_{opt} that achieves $g(b)$, i.e. a policy that is growth optimal.

A policy π is called an *interval policy* if it is stationary, Markov, i.e. the portfolio z_k is only a function of the current portfolio b_k, and has the following structure:

$$\pi(b_k) = \begin{cases} \alpha_1, & b_k < \alpha_1, \\ b_k, & b_k \in [\alpha_1, \alpha_2], \\ \alpha_2, & b_k \geq \alpha_2, \end{cases} \tag{6}$$

for some $[\alpha_1, \alpha_2] \subseteq [0, 1]$. Thus, an interval policy exercises control only if the portfolio leaves the no-trade interval $[\alpha_1, \alpha_2]$ and in that case maps the portfolio to the closest end-point.

Growth optimal investment in general (i.e. number of assets ≥ 2) discrete-time IID markets with proportional costs was studied in Iyengar (2002). The summary of the results as applicable to two-asset markets is as follows.

Theorem 1 (Corollary 3 and Theorem 6 in Iyengar (2002)) *The following results hold for the growth optimal investment problem in two-asset discrete-time IID markets with proportional transaction costs:*

(a) The growth rate function $g(b)$ is a constant g independent of the initial portfolio b.

(b) For every $\epsilon > 0$ there exists an ϵ-optimal interval policy π, i.e. the growth rate g_π of policy π is at least $g - \epsilon$.

(c) An interval policy π achieves its growth rate g_π almost surely, i.e.

$$g_\pi = \lim_{n\to\infty} \frac{1}{n}\mathbf{E}\log S_n^\pi = \lim_{n\to\infty}\frac{1}{n}\log S_n^\pi, \quad a.s.$$

We now return to markets where one does not make any distributional assumptions on the market price relatives. The motivation for studying such markets is that they are a good model for stochastic markets where the distribution is unknown, or maybe non-stationary. Since Theorem 1 asserts that an interval policy will be arbitrarily close to optimal (we, however, do not know which particular one), we restrict the investor to interval policies. Thus, the set of admissible policies is

$$\mathcal{I} = \big\{\boldsymbol{\alpha} = [\alpha_1, \alpha_2] : 0 \leq \alpha_1 < \alpha_2 \leq 1\big\}. \tag{7}$$

Since the growth rate g is independent of the initial wealth and initial portfolio, we assume that the investor begins with initial wealth $S_1 = 1$ and initial portfolio $b = \frac{1}{2}$.

We do not restrict the investors to use non-anticipating policies. In fact, we allow the investors to choose the investment policy with hindsight, i.e. the interval policy up until the n-th market outcome can be chosen after observing the market sequence $\mathbf{x}_1^n$. The main result of this paper is that there exists a non-anticipating policy that performs as well as any interval policy, to first order in the exponent, even those chosen with hindsight, i.e. the policy is a univeral growth optimal policy for IID markets. This universal policy, although non-anticipating and self-financing, is not stationary.

3 Universal investment in two asset markets

Let $\boldsymbol{\alpha} \in \mathcal{I}$ denote the interval policy with the no-trade interval $[\alpha_1, \alpha_2] \subseteq [0,1]$, $\big\{b_n^\alpha : n \geq 1\big\}$ denote the sequence of market opening portfolios corresponding to policy $\boldsymbol{\alpha}$, $\big\{z_n^\alpha : n \geq 1\big\}$ denote the corrected portfolios dictated by $\boldsymbol{\alpha}$, and $\big\{S_n^\alpha : n \geq 1\big\}$ denote the sequence of cumulative wealths generated by $\boldsymbol{\alpha}$.

Let $\boldsymbol{\alpha}_n^*$ denote the optimal interval policy in hindsight, i.e. the interval $\boldsymbol{\alpha}_n^* = [\alpha_{n,1}^*, \alpha_{n,2}^*]$ is chosen after observing the sequence $\mathbf{x}_1^n$ and maximizes the wealth S_n defined in (4). Denote the optimal wealth by S_n^*. Note that the subscript n refers to the fact that the policy $\boldsymbol{\alpha}_n^*$ is chosen after observing $\mathbf{x}_1^n$; the policy itself is a stationary interval policy.

Next we define our candidate universal policy $\widehat{\pi}$. The policy $\widehat{\pi}$ invests $(d\boldsymbol{\alpha}/\int_{\mathcal{I}} d\boldsymbol{\alpha})$-fraction of the initial dollar according to the policy $\boldsymbol{\alpha}$ and manages the pools separately. At time n, a dollar invested in policy $\boldsymbol{\alpha}$ is worth S_n^α and is invested in portfolio b_n^α. Since the initial wealth invested in the policy $\boldsymbol{\alpha}$ was $(d\boldsymbol{\alpha}/\int_{\mathcal{I}} d\boldsymbol{\alpha})$, we have that the wealth $\widehat{S}_n$ generated by the policy $\widehat{\pi}$ is given by

$$\widehat{S}_n = \frac{\int_{\mathcal{I}} S_n^{\alpha} d\boldsymbol{\alpha}}{\int_{\mathcal{I}} d\boldsymbol{\alpha}}. \tag{8}$$

Furthermore, at time n, the policy $\boldsymbol{\alpha}$ dictates a move from the portfolio b_n^{α} to the portfolio z_n^{α}, resulting in a wealth $w(b_n^{\alpha}, z_n^{\alpha})S_n^{\alpha}$ at z_n^{α}. The portfolio $\widehat{z}_n$ corresponding to the universal policy $\widehat{\pi}$ is obtained by weighting the portfolio of each of the individual policies $\boldsymbol{\alpha}$ by the corresponding wealth $w(b_n^{\alpha}, z_n^{\alpha})S_n^{\alpha}$, i.e.

$$\widehat{b}_n = \frac{\int_{\mathcal{I}} w(b_n^{\alpha}, z_n^{\alpha}) S_n^{\alpha} z_n^{\alpha} d\boldsymbol{\alpha}}{\int_{\mathcal{I}} w(b_n^{\alpha}, z_n^{\alpha}) S_n^{\alpha} d\boldsymbol{\alpha}}. \tag{9}$$

This policy $\widehat{\pi}$ is the counterpart for the market with transaction costs to the universal policy defined in Cover (1991). The policy $\widehat{\pi}$ is clearly non-anticipating and self-financing but not stationary. It is easy to see that a policy that manages the pools on *paper*, thereby saving the transaction costs resulting from offsetting trades, would result in a higher growth rate.

Although the policy $\widehat{\pi}$ is not practical, its performance can be approximated by practical policies as follows. The wealth $\widehat{S}_n$ can be written as $\widehat{S}_n = \mathbf{E}[S_n^{\alpha}]$ where the expectation is with respect to a uniform distribution over $\mathcal{I}$. Let $\boldsymbol{\alpha}_j$, $j = 1, \ldots, N$, be IID samples from the uniform distribution on $\mathcal{I}$. Suppose we invest $\frac{1}{N}$ dollars in each of these N policies and manage them separately. Then the time n wealth $\frac{1}{N}\sum_{j=1}^{N} S_n^{\alpha_j} \approx \widehat{S}_n$, provided N is reasonably large. Kalai and Blum (1997) give explicit bounds for N as a function of the desired accuracy and confidence.

The main result of this paper is that, under some mild regularity conditions, the investment policy $\widehat{\pi}$ performs as well as any fixed interval policy $\boldsymbol{\alpha}$, including the optimal policy $\boldsymbol{\alpha}_n^*$, at least to first order in the exponent. We need some preliminary results in order to prove that $\widehat{\pi}$ is a universal investment policy.

Lemma 1 *Let $\boldsymbol{\theta} = [\theta_1, \theta_2]$ and $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]$ be any two interval policies. Then*

$$b_{n+1}^{\alpha} - b_{n+1}^{\theta} = (z_n^{\alpha} - z_n^{\theta}) \cdot \frac{r_n}{(z_n^{\theta}(r_n - 1) + 1)(z_n^{\alpha}(r_n - 1) + 1)}, \tag{10}$$

where $r_n = \frac{x_n(1)}{x_n(2)}$. Suppose policies $\boldsymbol{\theta}$ and $\boldsymbol{\alpha}$ do not exercise any control in market periods $j = k, \ldots, n$, i.e. $z_j^{\theta} = b_j^{\theta}$ and $z_j^{\alpha} = b_j^{\alpha}$, $j = k, \ldots, n$. Then iterating (10) we get

$$b_{n+1}^{\alpha} - b_{n+1}^{\theta} = (z_{k-1}^{\alpha} - z_{k-1}^{\theta}) \cdot \prod_{j=k-1}^{n} \frac{r_j}{(z_j^{\theta}(r_j - 1) + 1)(z_j^{\alpha}(r_j - 1) + 1)}. \tag{11}$$

Remark 1 *Since $W_n(z) = z_n x_n(1) + (1 - z)x_n(2) \geq 0$ for all $z \in [0, 1]$, the evolution relation (10) implies that the market does not alter the order of the portfolios, i.e. $z_n^{\alpha} \geq z_n^{\theta} \Rightarrow b_n^{\alpha} \geq b_n^{\theta}$.*

Proof: From (3) we have that

$$\begin{aligned} b_{n+1}^{\alpha} - b_{n+1}^{\theta} &= \frac{z_n^{\alpha} r_n}{z_n^{\alpha}(r_n - 1) + 1} - \frac{z_n^{\theta} r_n}{z_n^{\theta}(r_n - 1) + 1}, \\ &= (z_n^{\alpha} - z_n^{\theta}) \cdot \frac{r_n}{(z_n^{\theta}(r_n - 1) + 1)(z_n^{\alpha}(r_n - 1) + 1)}. \end{aligned} \tag{12}$$

■

Lemma 2 establishes that the portfolios $\{z_n^\alpha : n \geq 1\}$ are equicontinuous functions of $\boldsymbol{\alpha}$. This equicontinuity is crucial for proving the universal investment result. For constant rebalance policies $z_n = b^*$, $b^* \in [0,1]$, $n \geq 1$; therefore, all the previous universal results (see, e.g. Cover, 1991; Cover and Ordentlich, 1996) could implicitly assume equicontinuity of $\{z_n : n \geq 1\}$.

Lemma 2 *Fix a non-empty interval policy* $\boldsymbol{\alpha} = [\alpha_1, \alpha_2] \subset (0,1)$*, and a market sequence* $\mathbf{x}_1^\infty$ *satisfying Assumption 1. Then there exist* δ_0 *and* $m > 1$ *(functions of* $\boldsymbol{\alpha}$*,* $\overline{r}$ *and* $\underline{r}$*) such that for all interval policies* $\boldsymbol{\theta} = [\theta_1, \theta_2]$ *with* $|\theta_i - \alpha_i| \leq \delta \leq \delta_0$*,* $i = 1,2$*, we have*

$$|b_n^\theta - b_n^\alpha| \leq m\delta, \qquad |z_n^\theta - z_n^\alpha| \leq m\delta, \qquad \textit{for all } n \geq 1.$$

Remark 2 *The assumptions that interval* $\boldsymbol{\alpha} = [\alpha_1, \alpha_2] \neq \emptyset$ *and* $\boldsymbol{\alpha} = [\alpha_1, \alpha_2] \subset (0,1)$ *are both critical to the result. The proof does not work if either of these conditions are violated.*

Proof: We first establish that for any interval policy $\boldsymbol{\theta} = [\theta_1, \theta_2]$ with $|\theta_i - \alpha_i| \leq \delta$, $i = 1,2$,

$$|z_n^\theta - z_n^\alpha| \leq \max\{|b_n^\theta - b_n^\alpha|, \delta\}. \tag{13}$$

The bound (13) is established by considering the following cases:

(a) $b_n^\theta \in [\theta_1, \theta_2]$ and $b_n^\alpha \in [\alpha_1, \alpha_2]$: Since $z_n^\theta = b_n^\theta$ and $z_n^\alpha = b_n^\alpha$, it follows that $|z_n^\theta - z_n^\alpha| = |b_n^\theta - b_n^\alpha|$.

(b) $b_n^\theta \in [\theta_1, \theta_2]$ and $b_n^\alpha < \alpha_1$: In this case, we have

$$\begin{aligned}
|z_n^\theta - z_n^\alpha| &= |b_n^\theta - \alpha_1|, \\
&= \max\{b_n^\theta - \alpha_1, \alpha_1 - b_n^\theta\}, \\
&\leq \max\{b_n^\theta - b_n^\alpha, \alpha_1 - \theta_1\}, \qquad (14)\\
&\leq \max\{|b_n^\theta - b_n^\alpha|, \delta\},
\end{aligned}$$

where (14) follows from the fact that $b_n^\alpha < \alpha_1$ and $b_n^\theta \geq \theta_1$.

(c) $b_n^\theta \in [\theta_1, \theta_2]$ and $b_n^\alpha > \alpha_2$: Here,

$$\begin{aligned}
|z_n^\theta - z_n^\alpha| &= |b_n^\theta - \alpha_2|, \\
&= \max\{b_n^\theta - \alpha_2, \alpha_2 - b_n^\theta\}, \\
&\leq \max\{\theta_2 - \alpha_2, b_2^\alpha - b_n^\theta\}, \qquad (15)\\
&\leq \max\{\delta, |b_n^\theta - b_n^\alpha|\},
\end{aligned}$$

where (15) follows from the fact that $b_n^\alpha > \alpha_2$ and $b_n^\theta \leq \theta_2$.

All the remaining cases can be resolved in a manner similar to the cases (b) and (c) above. Thus, the proof of the lemma reduces to establishing the inequality $|b_n^\theta - b_n^\alpha| \leq m\delta$.

Let $b^{\alpha}_{\max}$ (resp. $b^{\alpha}_{\min}$) denote the maximum (resp. minimum) possible portfolio value when interval policy $\boldsymbol{\alpha}$ is employed. Then

$$b^{\alpha}_{\max} = \frac{\alpha_2 \overline{r}}{\alpha_2(\overline{r}-1)+1}, \qquad b^{\alpha}_{\min} = \frac{\alpha_1 \underline{r}}{\alpha_1(\underline{r}-1)+1}. \tag{16}$$

Let $\delta_1 = \min\left\{\frac{\alpha_1}{2}, \frac{1-\alpha_2}{2}\right\}$, $b_{\max} = \frac{(\alpha_2+\delta_1)\overline{r}}{(\alpha_2+\delta_1)(\overline{r}-1)+1}$, and $b_{\min} = \frac{(\alpha_1-\delta_1)\underline{r}}{(\alpha_1-\delta_1)(\underline{r}-1)+1}$. Then $b_{\max}$ (resp. $b_{\min}$) is the maximum (resp. minimum) possible portfolio value when an interval policy $\boldsymbol{\beta} = [\beta_1, \beta_2]$ with $|\beta_i - \alpha_i| \leq \delta_1$, $i = 1, 2$, is employed.

Define

$$m = \max\left\{ \frac{b^{\alpha}_{\max}(1-b_{\min})}{\alpha_1(1-(\alpha_2+\delta_1))}, \frac{(1-b^{\alpha}_{\min})(\alpha_2+\delta_1)}{(1-\alpha_2)(\alpha_1-\delta_1)} \right\}, \tag{17}$$

and

$$\delta_0 = \min\left\{ \delta_1, \frac{(\alpha_2-\alpha_1)}{2m} \right\}. \tag{18}$$

Fix a comparison policy $\boldsymbol{\theta}$ such that $\max_{\{i=1,2\}} |\theta_i - \alpha_i| \leq \delta \leq \delta_0$. Define $b^{\theta}_{\max}$ and $b^{\theta}_{\min}$ as in (16). The uniform bound on the differences $|b^{\theta}_n - b^{\alpha}_n|$ is established by induction on n. To this end, define $\mathcal{T} = \{t_k : k \geq 1\}$ as follows:

$$t_k = \begin{cases} \min\{n \geq 1 : b^{\alpha}_n \neq z^{\alpha}_n \text{ or } b^{\theta}_n \neq z^{\theta}_n\}, & k = 1, \\ \min\{n \geq t_{k-1}+1 : b^{\alpha}_n \neq z^{\alpha}_n \text{ or } b^{\theta}_n \neq z^{\theta}_n\}, & k \geq 2, \end{cases} \tag{19}$$

i.e. t_1 is the first time either policy makes a portfolio correction; and t_k is the first time after t_{k-1} that either policy makes a correction.

The starting portfolio $b^{\theta}_1 = b^{\alpha}_1 = \frac{1}{2}$, therefore $|b^{\alpha}_k - b^{\theta}_k| = 0$, for $k = 1, \ldots, t_1$. Since $\delta_0 \leq (\alpha_2 - \alpha_1)/2m$ and at time t_1 at least one of the policies makes a correction, only the following four cases are possible.

(a) $b^{\alpha}_{t_1} = b^{\theta}_{t_1} \leq \min\{\theta_1, \alpha_1\}$: $|z^{\theta}_{t_1} - z^{\alpha}_{t_1}| = |\theta_1 - \alpha_1| \leq \delta$.

(b) $b^{\alpha}_{t_1} = b^{\theta}_{t_1} \geq \max\{\theta_2, \alpha_2\}$: $|z^{\theta}_{t_1} - z^{\alpha}_{t_1}| = |\theta_2 - \alpha_2| \leq \delta$.

(c) $\min\{\theta_1, \alpha_1\} \leq b^{\alpha}_{t_1} = b^{\theta}_{t_1} \leq \max\{\theta_1, \alpha_1\}$: Without loss of generality, assume $\alpha_1 \leq \theta_1$. Then $|z^{\theta}_{t_1} - z^{\alpha}_{t_1}| = \theta_1 - b^{\alpha}_{t_1} \leq \theta_1 - \alpha_1 \leq \delta$.

(d) $\min\{\theta_2, \alpha_2\} \leq b^{\alpha}_{t_1} = b^{\theta}_{t_1} \leq \max\{\theta_2, \alpha_2\}$: Without loss of generality, assume $\alpha_2 \leq \theta_2$. Then $|z^{\theta}_{t_1} - z^{\alpha}_{t_1}| = b^{\theta}_{t_1} - \alpha_2 \leq \theta_2 - \alpha_2 \leq \delta$.

Thus, $|z^{\theta}_{t_1} - z^{\alpha}_{t_1}| \leq \delta$.

Let n_0 be the first instant when $\left|b^{\theta}_n - b^{\alpha}_n\right| > m\delta$. Let $n_0 \in [t_k, t_{k+1}-1)$. Assume without loss of generality that $b^{\theta}_n > b^{\alpha}_n$. (If this is not true then $(1-b^{\theta}_n) > (1-b^{\alpha}_n)$ and an identical analysis can be applied to the mirror image.) Neither policy makes any correction in periods $j = t_k+1, \ldots, t_{k+1}-1$, therefore (3) implies

$$\begin{aligned} 1 - b^{\theta}_{n_0} &= (1 - z^{\theta}_{t_k}) \cdot \textstyle\prod_{j=t_k}^{n_0-1} \left(\frac{1}{z^{\theta}_j(r_j-1)+1}\right), \\ b^{\alpha}_{n_0} &= z^{\alpha}_{t_k} \cdot \textstyle\prod_{j=t_k}^{n_0-1} \left(\frac{r_j}{z^{\alpha}_j(r_j-1)+1}\right), \end{aligned} \tag{20}$$

and (11) in Lemma 1 implies that

$$b_{n_0}^{\theta} - b_{n_0}^{\alpha} = (z_{t_k}^{\theta} - z_{t_k}^{\alpha}) \cdot \prod_{j=t_k}^{n_0-1} \left(\frac{r_j}{(z_j^{\theta}(r_j-1)+1)(z^{\alpha}(r_j-1)+1)} \right). \tag{21}$$

At time t_k, at least one of policies makes a correction, i.e. at least one of $z_{t_k}^{\alpha}$ and $z_{t_k}^{\theta}$ must be at one of the end-points of the corresponding no-trade interval. From (21) we have $z_{t_k}^{\theta} > z_{t_k}^{\alpha}$ and $\delta \leq \delta_0 \leq \frac{\alpha_2 - \alpha_1}{2m}$; therefore, only the following cases are possible:

(a) $z_{t_k}^{\theta} = \theta_1 \geq z_{t_k}^{\alpha}$. Since $z_{t_k}^{\alpha} \geq \alpha_1$,

$$\begin{aligned}
b_{n_0}^{\theta} - b_{n_0}^{\alpha} &= (\theta_1 - \alpha_1) \cdot \prod_{j=t_k}^{n_0-1} \left(\frac{1}{z_j^{\theta}(r_j-1)+1} \right) \prod_{j=t_k}^{n_0-1} \left(\frac{r_j}{z_j^{\alpha}(r_j-1)+1} \right), \\
&\leq \delta \left(\frac{1 - b_{n_0}^{\theta}}{1 - z_{t_k}^{\theta}} \right) \left(\frac{b_{n_0}^{\alpha}}{z_{t_k}^{\alpha}} \right), && (22) \\
&\leq \delta \left(\frac{1 - b_{\min}^{\theta}}{1 - \theta_2} \right) \left(\frac{b_{\max}^{\alpha}}{\alpha_1} \right), && (23) \\
&\leq m\delta, && (24)
\end{aligned}$$

where (22) follows from (20), (23) follows from the definition of $b_{\max}^{\alpha}$ and $b_{\min}^{\theta}$, and (24) follows from the definition of m in (17). Thus, we have a contradiction.

(b) $z_{t_k}^{\theta} \geq z_{t_k}^{\alpha} = \alpha_2$. Since $z_{t_k}^{\theta} \leq \theta_2$,

$$\begin{aligned}
b_{n_0}^{\theta} - b_{n_0}^{\alpha} &= (\theta_2 - \alpha_2) \cdot \prod_{j=t_k}^{n_0-1} \left(\frac{1}{z_j^{\theta}(r_j-1)+1} \right) \prod_{j=t_k}^{n_0-1} \left(\frac{r_j}{z_j^{\alpha}(r_j-1)+1} \right), \\
&\leq \delta \left(\frac{1 - b_{n_0}^{\theta}}{1 - z_{t_k}^{\theta}} \right) \left(\frac{b_{n_0}^{\alpha}}{z_{t_k}^{\alpha}} \right), \\
&\leq \delta \left(\frac{1 - b_{\min}^{\theta}}{1 - \theta_2} \right) \left(\frac{b_{\max}^{\alpha}}{\alpha_2} \right), \\
&\leq m\delta.
\end{aligned}$$

The arguments for the bounds are similar to those in case (a). Thus, another contradiction.

(c) Either ($z_{t_k}^{\theta} = \theta_2$ and $z_{t_k}^{\alpha} < \alpha_2$) or ($z_{t_k}^{\theta} > \theta_1$ and $z_{t_k}^{\alpha} = \alpha_1$): In either of these two cases a direct contradiction cannot be shown. However, we show that one can recourse back to t_{k-1} and argue that one would either recourse all the way back to t_1 or terminate in either cases (a) or (b). We will prove the result for the case ($z_{t_k}^{t} = \theta_2$ and $z_{t_k}^{\alpha} < \alpha_2$) – the other case can be proved in an identical fashion.

Since $z_{t_k}^{\theta} = \theta_2$ it must be that $b_{t_k}^{\theta} \geq z_{t_k}^{\theta}$ or equivalently $1 - b_{t_k}^{\theta} \leq 1 - z_{t_k}^{\theta}$. Also, $z_{t_k}^{\alpha} < \alpha_2$

implies that $b_{t_k}^{\alpha} = z_{t_k}^{\alpha}$. Therefore,

$$\begin{aligned} 1 - b_{n_0}^{\theta} &= (1 - z_{t_k}^{\theta}) \prod_{j=t_k}^{n_0-1} \left(\frac{1}{z_j^{\theta}(r_j - 1) + 1} \right), \\ &\geq (1 - b_{t_k}^{\theta}) \prod_{j=t_k}^{n_0-1} \left(\frac{1}{z_j^{\theta}(r_j - 1) + 1} \right), \\ &= (1 - z_{t_{k-1}}^{\theta}) \prod_{j=t_{k-1}}^{n_0-1} \left(\frac{1}{z_j^{\theta}(r_j - 1) + 1} \right); \end{aligned} \tag{25}$$

and similarly,

$$b_{n_0}^{\alpha} = z_{t_{k-1}}^{\alpha} \prod_{j=t_{k-1}}^{n_0-1} \left(\frac{1}{z_j^{\theta}(r_j - 1) + 1} \right). \tag{26}$$

Since $z_{t_k}^{\theta} - z_{t_k}^{\alpha} \leq b_{t_k}^{\theta} - b_{t_k}^{\alpha}$, Lemma 1 implies that

$$\begin{aligned} b_{n_0}^{\theta} - b_{n_0}^{\alpha} &= (z_{t_k}^{\theta} - z_{t_k}^{\alpha}) \prod_{j=t_k}^{n_0-1} \left(\frac{r_j}{(z_j^{\theta}(r_j - 1) + 1)(z_j^{\alpha}(r_j - 1) + 1)} \right), \\ &\leq (b_{t_k}^{\theta} - b_{t_k}^{\alpha}) \prod_{j=t_k}^{n_0-1} \left(\frac{r_j}{(z_j^{\theta}(r_j - 1) + 1)(z_j^{\alpha}(r_j - 1) + 1)} \right), \\ &= (z_{t_{k-1}}^{\theta} - z_{t_{k-1}}^{\alpha}) \prod_{j=t_{k-1}}^{n_0-1} \left(\frac{r_j}{(z_i^{\theta}(r_j - 1) + 1)(z_j^{\alpha}(r_i - 1) + 1)} \right). \end{aligned} \tag{27}$$

The bound (27) implies $z_{t_{k-1}}^{\theta} \geq z_{t_{k-1}}^{\alpha}$. Thus, one is again faced with the same three cases at time t_{k-1}. At t_{k-1} if case (a) or (b) prevails, then we have that

$$\begin{aligned} b_{n_0}^{\theta} - b_{n_0}^{\alpha} &= \delta \prod_{j=t_{k-1}}^{n_0-1} \left(\frac{r_j}{(z_j^{\theta}(r_j - 1) + 1)} \right) \prod_{j=t_{k-1}}^{n_0-1} \left(\frac{r_j}{z_j^{\alpha}(r_j - 1) + 1} \right), \\ &\leq \delta \left(\frac{1 - b_{n_0}^{\theta}}{1 - z_{t_{k-1}}^{\theta}} \right) \left(\frac{b_{n_0}^{\alpha}}{z_{t_{k-1}}^{\alpha}} \right), \qquad (28) \\ &\leq \delta \left(\frac{1 - b_{\min}^{\theta}}{1 - \theta_2} \right) \left(\frac{b_{\max}^{\alpha}}{\alpha_2} \right), \\ &\leq m\delta, \end{aligned}$$

where (28) follows from the bounds (25) and (26); i.e. a contradiction. Otherwise we recourse to t_{k-2}; and all the way back to t_1. At t_1, we have shown that that $\left| z_{t_1}^{\theta} - z_{t_1}^{\alpha} \right| \leq \delta$. Thus, again a contradiction.

■

We are now in position to state and prove the main result of this paper. Recall that $\boldsymbol{\alpha}_n^* = [\alpha_{n1}^*, \alpha_{n2}^*]$ was the optimal stationary Markov interval policy chosen after observing $\mathbf{x}_1^n$. The subscript n refers to the observation horizon $\mathbf{x}_1^n$ and is not be taken to mean that the policy is a function of n.

Theorem 2 *Let* $\boldsymbol{\alpha}_n^* = [\alpha_{n1}^*, \alpha_{n2}^*]$ *be the optimal interval policy in hindsight and let* S_n^* *denote the wealth at time* n *associated with the policy* $\boldsymbol{\alpha}_n^*$*. Let* $\widehat{\pi}$ *be the universal policy and* $\widehat{S}_n$ *be the corresponding wealth. Suppose that* $\boldsymbol{\alpha}_n^* = [\alpha_{n,1}^*, \alpha_{n,2}^*]$ *is eventually non-empty, i.e.* $\liminf_{n\to\infty}(\alpha_{n,2}^* - \alpha_{n,1}^*) > 0$*, and contained in* $(0,1)$*, i.e.* $\limsup_{n\to\infty} \alpha_{n,2}^* < 1$ *and* $\liminf_{n\to\infty} \alpha_{n,2}^* > 0$*. Then for all market sequences* $\{\mathbf{x}_n : n \geq 1\}$*,*

$$\liminf_{n\to\infty} \frac{1}{n} \log\left(\frac{\widehat{S}_n}{S_n^*}\right) \geq 0. \tag{29}$$

Remark 3 *Since in the absence of costs it is optimal to rebalance to a fixed portfolio* $b \in (0,1)$ *(Breiman, 1961; Cover, 1991), one expects that for small transaction costs the optimal policy* $\boldsymbol{\alpha}_n^*$ *will not allow the portfolio to drift to* $b = 0$ *or* $b = 1$*, i.e* $\boldsymbol{\alpha}_n^* \subset (0,1)$*. Also, for non-zero costs we expect that a constant rebalance policy (see Cover, 1991) will not be optimal, i.e.* $\alpha_{n,2}^* - \alpha_{n,1}^* > 0$*. The proof does not hold if any of these assumptions is violated.*

Proof: Choose a subsequence $\{n_j : j \geq 1\}$ such that

$$\lim_{j\to\infty} \frac{1}{n_j} \log\left(\frac{\widehat{S}_{n_j}}{S_{n_j}^*}\right) = \liminf_{n\to\infty} \frac{1}{n} \log\left(\frac{\widehat{S}_n}{S_n^*}\right).$$

From the hypothesis of the theorem, we have that the policies $\{\boldsymbol{\alpha}_{n_j}^* : j \geq 1\}$ are eventually non-empty and contained in $(0,1)$. Therefore, there exists $\epsilon > 0$ and j_0 such that for $j \geq j_0$ we have

$$\alpha_{n_j,1}^* > \epsilon, \qquad \alpha_{n_j,2}^* < 1 - \epsilon, \qquad \alpha_{n_j,2}^* - \alpha_{n_j,1}^* > \epsilon. \tag{30}$$

Fix $j > j_0$ and denote the optimal interval policy $\boldsymbol{\alpha}_{n_j}^*$ by $\boldsymbol{\alpha}^*$. In the rest of this proof we will denote $b_k^{\alpha_{n_j}^*}$, $z_k^{\alpha_{n_j}^*}$, $S_k^{\alpha_{n_j}^*}$ by b_k^*, z_k^* and S_k^*, respectively.

Let m^j, δ_0^j be the constants that satisfy Lemma 2 for $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{n_j}^*$. Then for any interval policy $\boldsymbol{\theta} = [\theta_1, \theta_2]$, with $|\theta_i - \alpha_{n,i}^*| \leq \delta \leq \delta_0$, $i = 1,2$, we have that for all $n \geq 1$,

$$|b_n^\theta - b_n^*| \leq m^j \delta, \qquad |z_n^\theta - z_n^*| \leq m^j \delta. \tag{31}$$

From (30) and the definition of δ_0^j in (18), it follows that there exist δ_0 and m such that for all $j \geq j_0$,

$$\delta_0^j \geq \delta_0, \qquad m^j \leq m. \tag{32}$$

The wealth S_n^θ corresponding to any interval policy $\boldsymbol{\theta}$ can be expressed as follows:

$$\begin{aligned} S_{n_j}^\theta &= \prod_{k=1}^n w(b_k^\theta, z_k^\theta) W(z_k^\theta), \\ &\geq \prod_{k=1}^n \left[\left(w(b_k^\theta, b_k^*) w(b_k^*, z_k^*) w(z_k^*, z_k^\theta) \right) \cdot \left(W(z_k^*) \cdot \frac{W(z_k^\theta)}{W(z_k^*)} \right) \right], \qquad (33) \\ &= S_{n_j}^* \cdot \left[\prod_{k=1}^n w(b_k^\theta, b_k^*) w(z_k^*, z_k^\theta) \right] \cdot \left[\prod_{k=1}^n \frac{W(z_k^\theta)}{W(z_k^*)} \right], \end{aligned}$$

where (33) follows from the fact that cost of the trade $b_k^\theta \rightarrow z_k^\theta$ is no more than the cost of the trade $b_k^\theta \rightarrow b_k^* \rightarrow z_k^* \rightarrow z_k^\theta$.

From (1) it follows that for all $|b - z| \leq m\delta$,

$$w(b,z) \geq 1 - \Big(\frac{m^j \lambda_{\max}}{1-\lambda_{\max}}\Big)\delta \geq 1 - \Big(\frac{m\lambda_{\max}}{1-\lambda_{\max}}\Big)\delta, \tag{34}$$

where $\lambda_{\max} = \max\{\lambda_1, \lambda_2\}$ and the second bound follows from the fact that $m^j \leq m$, for all $j > j_0$.

Suppose $z_k^* < z_k^\theta \leq 1$. Then $z_k^\theta = \mu z_k^* + (1-\mu)$, where (30) and (32) imply that

$$\mu = \frac{1 - z_k^\theta}{1 - z_k^*} \geq \frac{1 - z_k^* - m\delta}{1 - z_k^*} = 1 - \Big(\frac{m^j}{1-\alpha_2^*}\Big)\delta \geq 1 - \Big(\frac{m^j}{\epsilon}\Big)\delta \geq 1 - \Big(\frac{m}{\epsilon}\Big)\delta.$$

Thus, we have that

$$\begin{aligned} W(z_k^\theta) &= \mu W(z_k^*) + (1-\mu)W(1), \\ &\geq \mu W(z_k^*), \qquad (35) \\ &\geq \Big(1 - \Big(\frac{m}{\epsilon}\Big)\delta\Big) W(z_k^*), \qquad (36) \end{aligned}$$

where (35) follows from the fact that $W(1) \geq 0$, and (36) follows from the lower bound on μ. Similarly, for $0 \leq z_k^\theta < z_k^*$, we have

$$W(z_k^\theta) \geq \Big(1 - \Big(\frac{m}{\alpha_1^*}\Big)\delta\Big) W(z_k^*) \geq \Big(1 - \Big(\frac{m}{\epsilon}\Big)\delta\Big).$$

Therefore,

$$\frac{W(z_k^\theta)}{W(z_k^*)} \geq 1 - (\frac{m}{\epsilon})\delta. \tag{37}$$

From the lower bounds (34) and (37) we have

$$\begin{aligned} \frac{S_{n_j}^\theta}{S_{n_j}^*} &\geq \Big(1 - \Big(\frac{m\lambda_{\max}}{1-\lambda_{\max}}\Big)\Big)^{2n_j} \Big(1 - (\frac{m}{\epsilon})\delta\Big)^{n_j} \\ &\geq (1-\delta)^{cn_j}, \qquad (38) \end{aligned}$$

where $c = \frac{2m\lambda_{\max}}{1-\lambda_{\max}} + \frac{m}{\epsilon}$ and (38) is a consequence of the fact that the concavity of the log-function implies the bound $\log(1 - c\delta) \geq c\log(1-\delta)$.

Define $\mathcal{G}_\delta$ as follows.

$$\mathcal{G}_\delta = \Big\{\boldsymbol{\theta} \in \mathcal{I} : |\theta_i - \alpha_i^*| \leq \delta\Big\},$$

where $\delta \leq \delta_0$. From (38) it follows that for all $\boldsymbol{\theta} \in \mathcal{G}$,

$$S_{n_j}^\theta \geq (1-\delta)^{cn_j} S_{n_j}^*.$$

Using the characterization in (8) and the bounds developed above, the wealth $\widehat{S}_n$ generated by the universal policy $\widehat{\pi}$ can be bounded by,

$$\begin{aligned} \widehat{S}_{n_j} &= \frac{\int_{\mathcal{I}} S^{\alpha}_{n_j}\, d\alpha}{\int_{\mathcal{I}} d\alpha}, \\ &\geq \frac{\int_{\mathcal{G}_\delta} S^{\alpha}_{n_j}\, d\alpha}{\int_{\mathcal{I}} d\alpha}, \\ &\geq \frac{\int_{\mathcal{G}_\delta} (1-\delta)^{cn_j} S^{*}_{n_j}\, d\alpha}{\int_{\mathcal{I}} d\alpha}, \\ &= (1-\delta)^{cn_j} S^{*}_{n_j} \cdot \frac{\int_{\mathcal{G}_\delta} d\alpha}{\int_{\mathcal{I}} d\alpha}, \\ &= (1-\delta)^{cn_j} S^{*}_{n_j} \delta^2. \end{aligned} \tag{39}$$

Choose $\delta = \frac{1}{n_j+1}$. Then (39) implies that for all $j \geq \max\left\{j_0, \frac{1}{\delta_0}\right\}$,

$$\frac{\widehat{S}_{n_j}}{S^{*}_{n_j}} \geq e^{-c}\Big(\frac{1}{n_j+1}\Big)^2.$$

Therefore,

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log\left(\frac{\widehat{S}_n}{S^{*}_n}\right) &= \lim_{j\to\infty} \frac{1}{n_j} \log\left(\frac{\widehat{S}_{n_j}}{S^{*}_{n_j}}\right) \\ &\geq \lim_{j\to\infty} \frac{1}{n_j}\big(-c - 2\log(n_j+1)\big) = 0. \end{aligned}$$

This completes the proof of the theorem. ■

The following corollary immediately follows from the previous result.

Corollary 1 *Let $\boldsymbol{\alpha} = [\alpha_1, \alpha_2] \subset (0,1)$, $\boldsymbol{\alpha} \neq \emptyset$, be a given interval policy. Then for all market sequences $\{\mathbf{x}_n : n \geq 1\}$ satisfying Assumption 1 and $n \geq \frac{1}{\delta_0}$,*

$$\frac{\widehat{S}_n}{S^{\alpha}_n} \geq \frac{e^{-c}}{(n+1)^2},$$

where $c = \frac{2m\lambda_{\max}}{1-\lambda_{\max}} + \max\left\{\frac{m}{\alpha_1}, \frac{m}{1-\alpha_2}\right\}$, and (m, δ_0) are any constants that satisfy Lemma 2.

Proof: The proof of this result is a minor modification of the proof of Theorem 2. ■

Suppose the market was stochastic with the price relative vectors IID according to a fixed but unknown distribution μ. It is well known that one can approximate the optimal growth rate to any degree of accuracy using interval policies (Iyengar, 2002); and since the policy $\widehat{\pi}$ dominates all interval policies to first-order, one would expect that the growth rate of this policy $\widehat{\pi}$ in an IID market should be arbitrarily close to optimal. The following Corollary establishes this result under the mild regularity condition that market price relatives satisfy Assumption 1.

Corollary 2 *Suppose the market sequence* $\{\mathbf{X}_n : n \geq 1\}$ *is IID according to a fixed, unknown distribution* μ *and satisfies Assumption 1. Let* d_μ *be the optimal achievable growth rate. Then for all* $\epsilon > 0$*,*

$$\liminf_{n\to\infty} \frac{1}{n} \log \widehat{S}_n \geq d_\mu - \epsilon, \quad a.s.$$

Proof: Fix $\epsilon > 0$. Let $\boldsymbol{\alpha}^\mu = [\alpha_1^\mu, \alpha_2^\mu]$ be an ϵ-optimal interval policy for the market and let $\{S_n^\mu : n \geq 1\}$ be the associated wealth sequence. From Corollary 3 and Theorem 6 in Iyengar (2002) we have that

$$\lim_{n\to\infty} \frac{1}{n} \log S_n^\mu = \lim_{n\to\infty} \mathbf{E}\Big[\frac{1}{n} \log S_n^\mu\Big] \geq d_\mu - \epsilon.$$

For all $0 < \lambda < 1$, we have that $\boldsymbol{\alpha}^\mu \subset (0,1)$ and $\boldsymbol{\alpha}^\mu \neq \emptyset$ (see Iyengar, 2002, for details). Thus, Corollary 1 implies that

$$\liminf_{n\to\infty} \frac{1}{n} \log \widehat{S}_n \geq \limsup_{n\to\infty} \frac{1}{n} \log S_n^\mu, \quad \text{a.s.}$$

Therefore,

$$\liminf_{n\to\infty} \mathbf{E}\Big[\frac{1}{n} \log \widehat{S}_n\Big] \geq d_\mu - \epsilon.$$

This establishes the result. ■

Corollary 2 shows that the policy $\widehat{\pi}$ learns the ϵ-optimal interval $\boldsymbol{\alpha}^\epsilon = [\alpha_1^\epsilon, \alpha_2^\epsilon]$; and is, thus, able to achieve a growth exponent arbitrarily close to optimal, i.e. it is a universal growth optimal policy. Thus, in return for an infinitesimal decrease in the growth rate exponent, one can design a policy that is robust to errors in estimating the distribution of the asset returns.

4 Conclusion

In this paper we formulated and solved the problem of identifying universal growth optimal strategies for discrete time two-asset markets with proportional transaction costs. We constructed an investment policy that weakly dominates all interval policies in growth rate. Since interval policies are ϵ-optimal for IID markets (Iyengar, 2002), it follows that our policy is a universal growth optimal policy. Crucial to the proof of the universal investment result is the special structure of interval policies – interval policies entail a natural continuity of the portfolio process that is captured in Lemma 2; and the "size" of the set of interval policies is not too large which is critical in the proof of the lower bound in Theorem 2.

The universal result proved in this paper is of limited value since it holds only for markets with two assets. The natural analog of interval policies for markets with more than two assets are policies that correct the portfolio to a closed connected no-trade set. Although this class contains ϵ-optimal policies for all $\epsilon > 0$ (Iyengar, 2002), it does not satisfy the two regularity properties required for establishing a universal investment result. If the investor could be restricted to convex no-trade sets without violating the ϵ-optimality property, a universal investment result would follow.

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