1   The Single Index Model (Review)

One possible model for the returns is

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i$$

where α_i, and β_i are constants, R_m is the return of a market index and ϵ_i is a random variable with mean 0 and variance τ_i^2.

If the α_i, β_i and τ_i^2 are estimated via regression analysis it turns out that $\text{Cov}(\epsilon_i, R_m) = 0$, so we can and do assume without loss of generality that this holds. The key assumption of the single index model is $\text{Cov}(\epsilon_i, \epsilon_j) = 0$ for all $i \neq j$. We shall see that this assumption is inconsistent, but this inconsistency does not preclude the practical usefulness of this model.

It is easy to see that

$$r = \alpha + \overline{R}_m \beta,$$
$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2,$$

when $j \neq i$ and

$$\sigma_i^2 = \sigma_{ii} = \beta_i^2 \sigma_m^2 + \tau_i^2$$

when $j = i$. If we let T denote the diagonal matrix with τ_i^2, $i = 1, \ldots, n$ in the diagonal, and let β denote the column vector of betas, then we can write

$$V = \sigma^2 \beta \beta' + T.$$

For this reason the single-index model is also known as the diagonal model.

The single index model requires estimating $3n + 2$ parameters compared with $n + n(n+1)/2$ for the full covariance model.

Let x_p be a portfolio, then

$$\overline{R}_p = r' x_p = \alpha_p + \beta_p \overline{R}_m$$

and

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + x_p' T x_p,$$

where $\alpha_p = \alpha' x_p$ and $\beta_p = \beta' x_p$.

The above equation decomposes the variance of a security or portfolio x_p into a market risk term $\beta_p^2 \sigma_m^2$ and unique risk $x_p' T x_p$. The market risk is often called systematic or undiversifiable risk. The unique risk is often called unsystematic risk or diversifiable risk. Indeed, if x_p is a well diversified portfolio then the unique risk $x_p' T x_p$ goes to zero as $n \to \infty$ if each component of x_p is bounded by k/n for some constant k and $\tau_i^2 \leq s^2$ for some constant s.

Let x_m denote the market index portfolio. We have

$$\overline{R}_m = \alpha_m + \beta_m \overline{R}_m.$$

The only values of α_m and β_m that hold for all values of $\overline{R}_m$ are $\alpha_m = 0$, $\beta_m = 1$. Then

$$R_m = R_m + \sum_{i=1}^{n} x_{mi} \epsilon_i$$

so

$$\sum_{i=1}^{n} x_{mi} \epsilon_i = 0.$$

But this last statement contradicts the assumption that the ϵ's are uncorrelated. Since the market portfolio is by definition well diversified, the inconsistency of the single index model is not fatal in practice.

1.1 Estimating α_i and β_i.

Notice that

$$
\begin{aligned}
\sigma_{im} &= \sum_{j=1}^{n} \sigma_{ij} x_{mj} \\
&= \beta_i \beta_m \sigma_m^2 + \tau_i^2 x_{mi} \\
&= \beta_i \sigma_m^2 + \tau_i^2 x_{mi} \\
&\simeq \beta_i \sigma_m^2.
\end{aligned}
$$

The last equation follows from $\beta_m = 1$ and the approximation from $\tau_i^2 x_{mi} \simeq 0$. Using $\sigma_{im} = \beta_i \sigma_m^2$ as an approximation we see that

$$
\beta_i = \frac{\sigma_{im}}{\sigma_m^2}. \tag{1}
$$

We can estimate β_i by estimating the right hand side of (1). If we have data R_{it} and R_{mt} over periods $t = 1, \ldots, T$ we can estimate β_i by

$$
\hat{\beta}_i = \frac{\sum_{t=1}^{T} (R_{it} - \overline{R}_{it})(R_{mt} - \overline{R}_{mt})}{\sum_{t=1}^{T} (R_{mt} - \overline{R}_{mt})^2},
$$

and α by

$$
\hat{\alpha}_i = \overline{R}_{it} - \hat{\beta}_i \overline{R}_{mt}
$$

2 Multi-Index Models

Factor models or index models assume that the return on a security is sensitive to the movements of various factor or indices. Multiple-factor models are potentially more useful than a single index model based on a market index because it appears that actual security returns are sensitive to more than movements in a market index.

We can think of index models as return-generating processes. That is, a statistical model that describes how the return of a security is produced. As a return generating process a factor model attempts to capture the major economic forces that systematically move the prices of all securities. Implicit is the assumption that the returns on two securities will be correlated only through common reactions to one or more factors specified in the model. Any aspect of a security's return unexplained by the factor model is assumed to be unique or specific to the security and therefore uncorrelated with the unique elements of returns on other securities.

As with the case of single index model, multi-index factor models can be used to:

1. Supply the information needed to calculate r and V.

2. It can be used to characterize a portfolio's sensitivity to movements in the factors and to decompose risk in several ways

3. Analyze current portfolio risk

4. Design future portfolios

5. Evaluate past performance

We will discuss three approaches to factor models: time-series, cross-sectional, and factor analytic.

2.1 Time-series approach

The time-series approach is perhaps the most intuitive to investors. The model begins with the assumption that he or she knows in advance the factors that influence security returns. The model builder collects information concerning the values of the factors and security returns from period to period. Using these data, the model builder can calculate the αs βs and τs. In this approach, accurate measurement of factor values is crucial and can be quite difficult in practice.

The standard form of the multi-index model can be written as follows:

$$R_i = \alpha_i + \sum_{k=1}^{K} \beta_{ik} I_k + \epsilon_i \tag{2}$$

where the α_is and the β_{ik}s are constants. Let τ_i^2 denote the variance of ϵ_i and let $\sigma_{I_k}^2$ denote the variance of index I_k. By construction we can and an do select $E[\epsilon_i] = 0$, $\mathrm{Cov}(I_k, I_l) = 0$ for $k \neq l$, and $\mathrm{Cov}(\epsilon_i, I_k) = 0$ for all i, k. In addition, we assume that $\mathrm{Cov}(\epsilon_i, \epsilon_j) = 0$.

Under this assumption:

$$\overline{R}_i = \alpha_i + \sum_{k=1}^{K} \beta_{ik} \overline{I}_k$$

$$\sigma_i^2 = \sum_{k=1}^{K} \beta_{ik}^2 \sigma_{I_k}^2 + \tau_i^2.$$

$$\sigma_{ij} = \sum_{k=1}^{m} \beta_{ik} \beta_{jk} \sigma_{I_k}^2.$$

Example of factors:

1. Gross domestic product (GDP)

2. Inflation

3. The level of interest rates

4. The level of oil prices

5. Economic sectors

Example: The following table presents a time series of GDP, Inflation, and the return of a specific security. Regressing the return of the security against GDP and inflation we obtain

$$\overline{R} = 5.80 + 2.17 I_1 - 0.68 I_2$$

where I_1 is GDP and I_2 is Inflation. A similar regression is needed for each security.

Year	GDP	Inflation	Return
1	5.7%	1.1.%	14.3%
2	6.4%	4.4%	19.2%
3	7.9%	4.4%	23.4%
4	7.0%	4.6%	15.6%
5	5.1%	6.1%	9.2%
6	2.9%	3.1%	13.0%

2.1.1 Industry Models

A natural extension to the single index model is to add industry factors resulting in the model

$$R_i = \alpha_i + \beta_{im} I_m + \sum_{k=1}^{K} \beta_{ik} I_k + \epsilon_i$$

where I_m is the market index and $I_k \ k = 1, \ldots, K$ are industry indices that are uncorrelated with the market and with each other. If we further assume that a firm i is industry k the equation simplifies to

$$R_i = \alpha_i + \beta_{im} I_m + \beta_{ik} I_k + \epsilon_i$$

resulting in expected return

$$\overline{R}_i = \alpha_i + \beta_{im} \overline{I}_m + \beta_{ik} \overline{I}_k.$$

The covariance between securities i and j can be written as

$$\beta_{im} \beta_{jm} \sigma_m^2 + \beta_{ik} \beta_{jk} \sigma_{I_k}^2$$

for firms in the same industry (k) and

$$\beta_{im} \beta_{jm} \sigma_m^2$$

for firms in different industries.

2.2 Cross-sectional approach

In the cross sectional approach the model builder starts with estimates of securities' sensitivities to certain factors. Then in a particular time period, the values of the factors are estimated based on securities' returns and their sensitivities to the factors. The process is repeated over multiple time periods, thereby providing an estimate of the factors' standard deviations and their correlations.

Imagine plotting the returns for a number of different stocks in a given time period and the values of certain attributes such as dividend-yield, size, momentum, etcetera. In contrast to the time series approach this plot is based on many stocks for one period of time. To quantify the relationship we postulate the model

$$R_i = \alpha + \sum_{k=1}^{K} \beta_{ik} I_k + \epsilon_i$$

across a large number of stocks over one period of time rather than across time for a single stock. For these models we know the exposures or sensitivities β_{ik} and the objective is to determine the actual value of the factors $I_k, \ k = 1, 2, \ldots, K$.

For example, in a two factor model with dividend-yield and size as factors, β_{i1} is the dividend yield of security i and β_{i2} is the size of security i. The objective is to find out how these factors affect returns. Regression will give us the values of α, I_1 and I_2 over the period. The value α represents the expected return on a typical stock with a dividend yield of zero and size zero. The slope I_1 represents the increase in expected return for each percent of dividend yield and I_2 represents the increase in expected return for each percent increase in size.[1]

For example, if we obtain

$$\overline{R}_i = 7 + .4b_{i1} - .3b_{i2}$$

from regression, then the zero factor is 7% meaning that a stock with zero dividend yield and zero size would have been expected to have a return of 7%. During this time period, higher-dividend yields and smaller sizes were both associated with larger returns.

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[1]The size attribute that is often computed by taking the logarithm of the total market value of the firms' outstanding equity measured in millions. This convention is based on the empirical observation that the impact of the size factor on a security with a large total market value is likely to be twice as great as that on a security with one-tenth the value.

The procedure is repeated over several periods to obtain the mean and variance of I_1 and I_2 and the covariance of I_1 and I_2. This can then be used to obtain estimates of the covariance between securities.

2.3   Factor Analytic Approaches

Here the model builder knows neither the factor values nor the securities' sensitivities to those factors. A statistical technique called factor analysis is used to extract the number of factors and securities' sensitivities based simply on a set of securities' past returns. Factor analysis takes the returns over many time periods on a sample of securities and attempts to identify one or more statistically significant factors that could have generated the covariances of returns observed within the sample. Unfortunately, factor analysis does not specify what economic variables the factors represent.

Not surprisingly, multi-index models are better at explaining historical covariances than the single index model. There is no reason to assume that a good factor model for one period will be a good one for the next period. Key factors change as in the effect of energy prices on security markets in the 1970s and more recently during the war in the Persian Gulf.

3   Capital Asset Pricing Model

The CAPM is a cornerstone of financial economics. It answers a puzzle that had intrigued many economists. What is the risk return relationship for specific assets and for portfolios of assets?

We know that when short sales and borrowing and lending at the risk free rate is allowed efficient portfolios have expected return given by

$$\overline{R} = R_f + S_q \sigma,$$

where S_q is the Sharpe ratio of the tangency portfolio x_q. Individual securities and inefficient portfolios will lie below this line. Form an earlier assignment we know that if x_q is the tangency portfolio and $\beta = \frac{V x_q}{\sigma_q^2}$ then for any portfolio x_p we have

$$q_p = x_p' q = x_p' \beta x_q' q = \beta_p q_q$$

which translates into

$$\overline{R}_p = R_f + \beta_p(\overline{R}_q - R_f),$$

and in particular to

$$\overline{R}_i = R_f + \beta_i(\overline{R}_q - R_f).$$

The CAPM goes further by concluding (under some assumptions to be stated shortly) that in equilibrium the tangent portfolio is the "market portfolio," resulting in

$$\overline{R}_i = R_f + \beta_i(\overline{R}_m - R_f)$$

where $\overline{R}_m$ is the expected return of the market portfolio.

3.1   Utility Theory

Economists have developed a theory for decision under risk. Under this theory every investors has a utility function relating its future wealth to the utility derived from the investor from this wealth. It doesn't matter how utility is measured. What is important is that the existence of a utility function allows the investor to rank different investment opportunities, and this is done by selecting the investment opportunity that maximizes expected utility. If U is the utility function of an investor than he or she will prefer wealth X to wealth Y if

$$EU(X) > EU(Y).$$

If $U(X) = aX + b$ with $a > 0$ we say that the investor is risk neutral because $EU(X) > EU(Y)$ if and only if $EX > EY$.

For all investors U is increasing. For most investors U is also concave. This means that all investors prefer more wealth, but for most investors there is a diminishing marginal utility of wealth. Investors with increasing concave utility functions are said to be risk-averse. In a way this means that these investors are willing to pay money to avoid risk. Indeed, by Jensen's inequality

$$EU(X) \leq U(EX) = EU(EX)$$

which means that a risk-averse investor will always prefer wealth EX to wealth X.

Example: Suppose $U(x) = \sqrt{x}$ and X is a random variable taking value 10,000 with probability .05 and value 40,000 with probability .95. Then

$$EU(X) = .05(100) + .95(200) = 195.$$

On the other hand, $EX = 500 + 38,000 = 38,500$, so

$$U(EX) = \sqrt{38,500} = 196.21.$$

3.1.1　Certainty Equivalence

What certain wealth W would make the investor indifferent between W and random wealth X? We need to find W so that the expected utility of W, i.e., $EU(W) = EU(EW) = U(EW) = U(W)$ is equal to the expected utility of X, i.e., $EU(X)$. The equation is

$$U(W) = EU(X)$$

and the solution is

$$W = U^{-1}(EU(X)).$$

In the above example, we need to solve for

$$U(W) = EU(X) = 195$$

or

$$\sqrt{W} = 195$$

or

$$W = 195^2 = 38,025.$$

This tells us that this investor is willing to pay $EX - W = 475$ to remove risk.

3.1.2　Indifference Curves

Suppose that utility maximization is consistent with mean variance analysis. Given a random variable X with mean μ and standard deviation σ we can find its certainty equivalent, which plots as $(0, W)$ in standard deviation expected return space. An indifference curve in this space is the set of all (σ, μ) pairs with certainty equivalent equal to W. This is an increasing convex function reflecting the fact the risk-averse investors require increasingly more compensation for taking more risk.

For example, if $U(x) = ax - bx^2$ over $x \in [0, a/2b]$ then

$$EU(X) = a\mu - b(\mu^2 + \sigma^2)$$

has certainty equivalence W such that

$$aW - bW^2 = a\mu - b(\mu^2 + \sigma^2).$$

Then,

$$\sigma^2 = \frac{a(\mu - W) - b(\mu^2 - W^2)}{b}.$$

which has the stated form.

3.1.3 Utility Theory and Mean Variance Analysis

Let X and Y be two random variables with the same expected return μ and assume that $\sigma_x^2 = \text{Var}[X] < \sigma_y^2 = \text{Var}[Y]$. Under what conditions would an investor with utility function U prefer X over Y?

Let X and Y be two random variables with the same variance σ^2 and assume that $\mu_x = E[X] > \mu_y = E[Y]$. Under what conditions would an investor with utility function U prefer X over Y?

It can be shown that the mean variance criterion is consistent with maximizing expected utility if either:

- The utility function is the increasing concave part of a quadratic function.

- The random variables are normally distributed.

In addition, mean variance analysis is consistent with the second order Taylor approximation of strictly risk-averse utility functions. Indeed, suppose that U is twice differentiable and $U'' < 0$. Then

$$U(X) \simeq U(\mu) + (X - \mu)U'(\mu) + \frac{1}{2}(X - \mu_X)^2 U''(\mu).$$

Taking expectations we find

$$EU(X) \simeq U(\mu) + \frac{1}{2}U''(\mu)\sigma^2$$

If X and Y have the same mean but difference variance we see that

$$EU(X) - EU(Y) \simeq \frac{1}{2}U''(\mu)[\sigma_x^2 - \sigma_y^2]$$

and the right hand side is positive if and only if $\sigma_x^2 < \sigma_y^2$.

A similar argument can be used to show that investors prefer the security with highest expected return among those with the same standard deviation.

3.2 Assumptions of Capital Asset Pricing Model

CAPM1 There are no transactions costs

CAPM2 Assets are infinitely divisible

CAPM3 Absence of personal income tax

CAPM4 Individuals cannot affect prices by buying and selling

CAPM5 Individuals make decisions in terms of means and variances

CAPM6 Short sales are allowed

CAPM7 Borrowing and lending at risk-free rate

CAPM8 All investors have same one-period horizon

CAPM9 All investors have same r and V

"The relevant question to ask about assumptions of a theory is not whether they are descriptively realistic, for they never are, but whether they are sufficiently good approximations for the purpose in hand. And this question can be answered only by seeing whether the theory works, which means whether it yields sufficiently accurate predictions." (Milton Friedman, Nobel Prize in Economics 1976)

3.3   Equilibrium Argument of CAPM

Under these assumptions all investors would determine the composition of the same tangency portfolio, and all investors will have portfolios involving combinations of the agreed-upon tangency portfolio and either risk free lending or borrowing. As all investors face the same efficient set, the only reason they will choose different portfolios is that they have different indifference curves. Note, however, that although the chosen portfolios will be different, each investor will choose the same combination of risky securities.

In equilibrium each security must have a nonzero proportion in the composition of the tangency portfolio. If a security has zero weight in x_q then nobody is investing in this security and its price must fall, thereby causing the expected returns of these security to rise until the resulting tangency portfolio has a nonzero proportion.

On the other hand, if each investor concludes that the tangency portfolio should include x_{qi} of security i but there are not enough shares outstanding to meet the demand, orders to buy this share will raise the price in search of sellers.

Ultimately, everything balances out. When all the price adjusting stops, the market will have been brought into equilibrium:

- Each investor will want to hold a certain positive amount of each security

- The current market price of each security will be at a level where the number of shares demanded equal the number of shares outstanding.

- The risk-free rate will be at a level where the total amount borrowed equals the total amount of money lent.

As a result, in equilibrium the proportions of the tangency portfolio will correspond to the proportion of what is known as the market portfolio. That is $x_m = x_q$.

3.4   Definition of Market Portfolio

Market Portfolio The market portfolio is a portfolio consisting of all securities where the proportion invested in each security corresponds to its relative market value. The relative market value of a security is simply equal to the aggregate market value of the security divided by the sum of the aggregate market value of all securities.

In theory, portfolio x_m consists not only of common stocks but also bonds, preferred stocks, real estate, etc.. In practice, some people restrict x_m to common stocks.

3.5   The Capital Market Line

Identifying x_m with x_q and S_m with S_q we see that efficient portfolios have expected return

$$\overline{R} = R_f + S_m \sigma.$$

This is known as the capital market line (CML). Individual securities and other inefficient portfolios will have expected returns that lie below the CML.

3.6   The Security Market Line

Identifying x_m with x_q we define beta as

$$\beta = \frac{V x_m}{x_m' V x_m} = \frac{V x_m}{\sigma_m^2}.$$

For any portfolio x_p let $\beta_p = \beta' x_p$. In particular security i, represented by portfolio e_i has

$$\beta_i = \beta' x_i = \frac{\sigma_{im}}{\sigma_m^2},$$

where we have used the fact that $\sigma_{im} = e_i'Vx_m$.

Now, from

$$q_p = \beta'x_pq_q = \beta_pq_q,$$

(recall assignment two), we have for any portfolio x_p

$$\overline{R}_p - R_f = \beta_p(\overline{R}_m - R_f).$$

In particular, for security i we have

$$\overline{R}_i = R_f + \beta_i(\overline{R}_m - R_f). \tag{3}$$

Equation (3) is known as the security market line (SML), and can be written in vector notation as

$$r = R_fe + (\overline{R}_m - R_f)\beta,$$

where $\beta = \frac{Vx_m}{\sigma_m^2}$. As described above, the SML works for portfolios as well as for individual securities.

CAPM suggests that

$$R_i = R_f + \beta_i(R_m - R_f) + \epsilon_i.$$

Then

$$\text{Cov}(R_i, R_m) = \beta_i\sigma_m^2 + \text{Cov}(\epsilon_i, R_m).$$

but the definition of β_i forces $\text{Cov}(\epsilon_i, R_m) = 0$.

What can we say about risk?

$$\sigma_i^2 = \beta_i^2\sigma_m^2 + \tau_i^2.$$

The first part $\beta_i^2\sigma_m^2$ is known as systematic risk. The second part τ_i^2 is known as unique risk. The market compensates you only for bearing systematic risk.

The main implications of the CAPM is that the market is efficient. An investor will then be in the efficient frontier if he or she invests in the market in combination with the risk-free rate. This is the theoretical underpinning of passive investment strategies that have done very well in the last decade.

The above derivation of the SML is completely formal. An informal (but perhaps more intuitive) derivation often presented to business school students goes as follows. Since

$$\sigma_m^2 = x_m'Vx_m = \sum_{i=1}^{n} x_{mi}\sigma_{mi},$$

$$\begin{aligned} \overline{R}_m &= R_f + \frac{S_m}{\sigma_m}\sigma_m^2 \\ &= R_f + \frac{S_m}{\sigma_m}\sum_{i=1}^{n} x_{mi}\sigma_{mi}, \end{aligned}$$

and

$$\overline{R}_m - R_f = \sum_{i=1}^{n} x_{mi}(\overline{R}_i - R_f).$$

Then

$$\sum_{i=1}^{n} x_{mi}(\overline{R}_i - R_f) = \sum_{i=1}^{n} x_{mi}\frac{S_m}{\sigma_m}\sigma_{mi},$$

which suggests that

$$\overline{R}_i - R_f = \frac{S_m}{\sigma_m}\sigma_{mi}$$

or equivalently

$$\overline{R}_i - R_f = \frac{\overline{R}_m - R_f}{\sigma_m^2}\sigma_{mi},$$

or

$$\overline{R}_i - R_f = \frac{\sigma_{mi}}{\sigma_m^2}(\overline{R}_m - R_f),$$

which is equivalent to

$$\overline{R}_i = R_f + \beta_i(\overline{R}_m - R_f).$$

As a final note, we relate Sharpe ratios of portfolios to their correlation with the market.

$$
\begin{aligned}
S_p &= \frac{\overline{R}_p - R_f}{\sigma_p} \\
&= \frac{(\overline{R}_m - R_f)\beta_p}{\sigma_p} \\
&= S_m\frac{\beta_p\sigma_m}{\sigma_p} \\
&= S_m\frac{\sigma_{pm}}{\sigma_m\sigma_p} \\
&= S_m\rho_{pm}.
\end{aligned}
$$

Which implies that

$$\rho_{pm} = \frac{S_p}{S_m}.$$

This implies that the higher the correlation of a portfolio with the market, the higher its return for a given level of risk.

3.7 CAPM as a Pricing Formula

Let V_0 be the current price of an asset or portfolio and let $E[V_1]$ be the expected value at the end of the horizon. The CAPM suggests that

$$\frac{E[V_1] - V_0}{V_0} = R_f + \beta(\overline{R}_M - R_f)$$

where β is the Beta of the asset. This implies that

$$V_0 = \frac{E[V_1]}{1 + R_f + \beta(\overline{R}_M - R_f)}.$$

This tells us that the fair price of the asset today is the expected value of the asset a period from now discounted at rate $R_f + \beta(\overline{R}_M - R_f)$.

Example: You have \$100 and invest 10% at the risk-free rate and 90% in the market. Suppose that $R_f = 7\%$ and $\overline{R}_m = 15\%$. The portfolio has expected value $E[V_1] = 10(1.07)+90(1.15) = 114.2$. Since the portfolio Beta is 0.9, the present value is

$$V_0 = \frac{E[V_1]}{1 + R_f + 0.9(R_m - R_f)} = \frac{114.2}{1.142} = 100.$$

There is another way we can use the CAPM as a pricing formula. Recall that

$$
\begin{aligned}
\beta &= \mathrm{Cov}(R, R_m)/\sigma_m^2 \\
&= \mathrm{Cov}(\frac{V_1 - V_0}{V_0}, R_m)/\sigma_m^2 \\
&= \frac{1}{V_0}\mathrm{Cov}(V_1, R_m)/\sigma^2.
\end{aligned}
$$

Consequently,

$$V_0 = \frac{E[V_1]}{1 + R_f + \frac{1}{V_0}\text{Cov}(V_1, R_m)(\overline{R}_M - R_f)/\sigma_m^2}.$$

Solving for V_0 we obtain

$$V_0 = \frac{E[V_1] - \text{Cov}(V_1, R_m)\frac{\overline{R}_m - R_f}{\sigma_m^2}}{1 + R_f}.$$

Here we are subtracting from $E[V_1]$ the cost of risk and discounting using the risk-free rate.

3.8 Comparison with Single Index Market Model

The Single Index Market Model is not an equilibrium model. The SIMM uses a broad market index, while CAPM uses the market portfolio. In practice, people use broad indices to approximate the market portfolio. Implications: The SIMM postulates

$$\overline{R}_i = \alpha_i + \beta_{iI}\overline{I}$$

while the CAPM postulates

$$\overline{R}_i = R_f + \beta_{im}(\overline{R}_m - R_f).$$

In practice, we may use I instead of the return of the market portfolio. In this case,

$$\overline{R}_i - R_f = \alpha_i - (1 - \beta_{iI})R_f + \beta_{iI}(\overline{I} - R_f)$$

which implies that

$$\alpha_i = (1 - \beta_{iI})R_f.$$

4 Style Analysis

The return of security i is modeled as

$$R_i = \sum_{k=1}^{K} b_{ik}I_k + \epsilon_i$$

where

- I_k is the value of factor k

- The factors represent returns of asset classes

- b_{ik} is the sensitivity of R_i to factor I_k

- Force $\sum_k b_{ik} = 1..$

We assume that $\text{Cov}(\epsilon_i, \epsilon_j) = 0$.

Under this model the return of asset i is represented as the return on a portfolio invested in K asset classes plus a residual component ϵ_i. The component of $\sum_{k=1}^{K} b_{ik}I_k$ of R_i is attributable to style, while the component ϵ_i is attributable to selection.

Desirable properties of asset classes:

- mutually exclusive

- collectively exhaustive

- have statistically different returns

An example of asset classes with these properties would be:

- Treasury bills (Salomon Brothers 90 day)

- Intermediate Government Bonds (Lehman Brothers I)

- Long Term Government Bonds (Lehman Brothers L T)

- Corporate Bonds (Lehman Brothers C)

- Mortgage Related Securities (Lehman Brothers MB)

- Large Cap Value Stocks (BARRA V)

- Large Cap Growth Stocks (BARRA G)

- Mid-Cap Stocks (Barra M)

- Small-Cap Stocks (Barra S)

- Non US Bonds

- European Stocks

- Japanese Stocks

- Emerging Market Stocks

In the list above, Barra V is comprised by half the stocks in the SP 500 with the highest Book-to-Price Ratio, while Barra G is comprised by half the stocks in the SP 500 with the lowest Book-to-Price Ratio. Barra M is the top 80% of non SP 500 in terms of market capitalization, while Barra S is the bottom 20% of non SP500 in terms of market cap.

The exposure of an investor to the different asset classes is a function of the amounts invested in various funds, the amount invested by these funds in different securities, and the exposure of securities to asset classes.

There are two ways we can approach finding the exposure:

- Internal. Detailed analysis of securities.

- External. Used realized returns to infer typical exposure of funds to asset classes.

We prefer the second approach.

Data: $R_{ih}, I_{1h}, \ldots I_{Kh}, h = 1, \ldots, H$. Select $b_{ik}, k = 1, \ldots, K$ to minimize

$$\sum_{h=1}^{H} (R_{ih} - \sum_{k=1}^{K} b_{ik} I_{kh})^2$$

subject to $b_{ik} \geq 0$ and $\sum_{k=1}^{K} b_{ik} = 1$.

Once the exposure of each security is found, we can find the exposure of a portfolio

$$R_p = \sum_{i=1}^{n} X_i R_i$$

so

$$\overline{R}_p = \sum_{k=1}^{K} b_{pk} \overline{I}_k + \epsilon_p$$

where

$$b_{pk} = \sum_{i=1}^{n} x_i b_{ik}$$

and

$$\epsilon_p = \sum_{i=1}^{n} x_i \epsilon_i$$

Performance measurement: A passive fund manager provides an investor with an investment style. Active fund manager provides both style and selection. To evaluate the selection component we look at ϵ_p. If we can reject the hypothesis that $\epsilon_p < 0$, we can be confident that the selection is adding value to our portfolio. Otherwise, we would be better off simply following the style component.