The Role of Mathematics

- The applicability of mathematics to empirical science has been dubbed a 'miracle' by some scientific realists. But it is not obvious what the miracle is supposed to be. Logic is also applicable to empirical science. But few scientific realists have called that a miracle.
- Harty Field has developed a concrete version of the worry. There are two prongs, both of which are supposed to depend on the abstract nature of the subject matter of mathematics.
- *Note*: Taken at face-value, 'there are infinitely-many prime numbers' can only be true if there are numbers. Moreover, given that it would have been true no matter what we happened to do or think, the existence of numbers is counterfactually independent of us. Finally, since such entities would apparently lack spacetime locations, mass-energy, and so on, Field targets **platonism**, the view that mathematical entities are abstract objects.
- *Question*: What is it to take a sentence at face-value? Does Russell's analysis of proper names count as a face-value reading? Does Lewis's analysis of modal operators? What about a view on which the semantics mirrors the syntax but first-order variables are taken to range over sets in the iterative hierarchy and all properties are taken be set-theoretic?

Epistemology and Metaphysics

- There are two problems that motivate Field's **nominalism**, i.e., anti-platonism.
  - **Epistemological Problem**: How could we know any theories that make essential reference to numbers, tensors, fiber bundles, and so on, realistically construed?
    - *Field*: "[O]ur belief in a theory should be undermined if the theory requires that it would be a huge coincidence if what we believed about its subject matter were correct. But mathematical theories, taken at face value, postulat mathematical objects that are mind-independent and bear no causal or spatiotemporal relations to us, or any other kinds of relations to us that would explain why our beliefs about them tend to be correct; it seems hard to give any account of our beliefs about these...objects that doesn't make the correctness of the beliefs a...coincidence. [2005, 77]

- **Metaphysical Problem**: How could reference to (apparently) a-causal and non-spatiotemporal objects help to explain any goings on in the natural world?
  - *Field*: "*[E]ven on the assumption that mathematical entities exist*, there is a *prima facie* oddity in thinking that they enter crucially into explanations of what is going on in the non-platonic realm of matter….[T]he role of mathematical entities, in our explanations of the physical world, is very different from the role of physical entities in the same explanations...[because f]or the most part, the role of physical entities...is causal: they are assumed to be causal agents with a causal role in producing the phenomena to be explained [1989, 18–19, original italics]."

- The Epistemological Problem is commonly supposed to have been addressed by Quine. Our belief in numbers and electrons, for example, can both be empirically justified by the indispensable role they play in the best overall explanation of our sensory experiences. On this interpretation, Field is arguing, in part, that Quine's empiricist epistemology of mathematics fails because mathematics is not after all indispensable to empirical science.
  - *Colyvan*: "[L]et's take a...charitable reading of the….[Benacerraf] challenge, according to which the challenge is to explain the reliability of our systems of beliefs...Once the challenge is put this way, we see that Quine has already answered it: we justify our system of beliefs by testing it against bodies of empirical evidence" [2007, 111, emphasis in original].
- *Problem*: In order to explain the reliability of our mathematical beliefs it does not suffice to explain their justification. Gödel offered an explanation of the justification of our mathematical beliefs in terms of a phenomenology of intuition. The problem with his view was that it did not explain the reliability of our mathematical beliefs. It does not explain why being the content of an intuition would be a reliable symptom of being true. Exactly the same is true of Quine's epistemology. This says that whatever explains the justification of our beliefs in electrons is what explains our beliefs in numbers. But this does not show that what explains the reliability of our beliefs in the former explains the reliability of our beliefs in the latter! The latter are supposed to be abstract objects.
- Upshot: Even under the assumption that mathematics is indispensable to empirical science, it remains obscure how we could have substantial mathematical knowledge.

- The Explanatory Problem is harder to pin down. Field distinguishes between intrinsic and extrinsic explanations, and protests that mathematical explanations are extrinsic.
    - *Field*: "The role [the gravitational constant] plays [in the explanation of the moon's orbit] is as an entity extrinsic to the process to be explained, an entity related to the process to be explained only by a...rather arbitrarily chosen function ....Surely then it would be illuminating if we could show that a purely intrinsic explanation of the process was possible, an explanation that did not invoke functions to extrinsic and causally irrelevant entities….*[U]nderlying every good extrinsic explanation there is an intrinsic explanation*. Note that [this principle] is not a nominalistic principle: it could...be accepted by a platonist [1980, 44-5]."
- *Problem*: If intrinsicness is causal, and causal relevance is counterfactual, then mathematical explanations would not seem to be extrinsic in most physical explanations! Had the mathematical facts been different, the physical facts would have been different too. Had arithmetic been inconsistent, a machine checking for this would have said so!
- Perhaps, then, the problem with mathematical explanations is that they are non-local. If physical facts do counterfactually depend on mathematical ones, this would involve objects "operat[ing] upon and affect[ing] other matter without mutual contact (Newton)."
- *Problem*: On this reading, the de Broglie-Bohm or GRW theories count as extrinsic, since they are non-local. But arguably Bell's lesson is that any realist view must be non-local!
- Chen takes intrinsicality to be a matter of non-arbitrariness, distinguishing two levels.
    - A theory is intrinsic$_1$ if every basic quantity it postulates is invariant under gauge choices. This is achieved by factoring out what is common to equivalent theories.
            - *Example*: Distance in Euclidean are invariant under choices of orthogonal axes. To get an intrinsic$_1$ theory of space just take the class of coordinate systems related by Euclidean transformations.
        - *Problem*: How do we check whether theories differ merely by gauge?
    - A theory is intrinsic$_2$ if it is A theory is intrinsic$_1$ and it does not refer to gauge-dependent quantities. Such a theory explains the gauge freedom.
        - *Example*: Chen's reformulation of (part of) nonrelativistic quantum mechanics says that quantum wave functions are invariant under overall phase transformations and explains this *via* 'periodic difference structure'.

            - The intrinsic$_2$ relations postulated include *amplitude-sum*, *amplitude-geq*, *phase-congruence*, and *phase-clockwise-betweenness* relating mereological fusions of *N points in space*.
            - *Note*: Chen assumes Field's nominalization of Newtonian spacetime, which itself assumes substantivalism about spacetime.
            - *Note*: Chen follows Sider in rejecting "quotienting by hand". But a semantics-first approach to theoretical equivalence might allow it.
                - *Sider*: "[On this view] every good model has artifacts. It's ok [because] we can say which features of the model are genuinely representational and which are artifacts."
        - *Problem*: Chen's theory (like Balaguer's earlier nominalization of quantum mechanics) is not remotely unique in being intrinsic$_2$. Arguably, the arbitrariness that we were trying to avoid has merely been relocated.
            - *Example*: In response to my query along these lines, Chen himself offered a nice illustration. "Instead of invoking a two-place relation Amplitude-greater-than-or-equal-to, whose bearers are pairs of N-regions, we can invoke a 2N-place relation that obey the same axioms but whose bearers are points in Newtonian space-time" (where N is the number of particles in the universe).
        - *Note*: This points to either a deep underdetermination or to indeterminacy. If the latter, then perhaps we should regard all intrinsic$_2$ explanations explaining gauge invariance as really saying the same indeterminate thing.
    - *Problem*: A theory can be intrinsic$_2$ without being nominalist, as Field points out. So, a theory could be intrinsic$_2$, but fail to address the **Epistemological Problem**.

Field's Nominalist Vision

- Absent a better interpretation of 'intrinsic', let us focus on the project of formulating surrogates to our physical theories that simply avoid reference to mathematical entities.
- *Field*: We must show "that one can...reaxiomatize scientific theories so that there is no reference to or quantification over mathematical entities in the axiomatization (and one can do this in such a way that the resulting axiomatization is...simple and attractive)."

- The parenthetical caveat is needed because it is trivial to avoid reference to mathematical entities if we are allowed to use certain tricks. For example, if we help ourselves to the operator, *it is mathematically necessary that P, [M]*, and take this as a logical primitive, as Putnam did, then we could believe every sentence we previously did without believing in numbers. (This is a slight oversimplification because mixed statements might create problems.) Alternatively, if we assume that space has sufficient structure, we could simply find a model of mathematics in the physical world. (*Example*: Take '1' to refer to the left half of my desk, '2' to the refer to the left half of the left half, '3' to refer to the left half of the left half of the left half, and so forth to get a model of the Peano Axioms!)
- *Note*: Prior nominalists, like Quine & Goodman, were finitists. But Field definitely is not! He assumes the equivalent of the powerset of the continuum -- a very big infinity!
- Finally, bracketing assumptions about modality and physics, Craig's Theorem ensures that, for any first-order theory including mathematical language, there exists a recursively axiomatized non-mathematical theory with the same non-mathematical consequences.
- What about less crude, but still easy, strategies, such as the following counterfactual one?
    - *Williamson*: "The nominalist [may reason] in effect about how things would be if the mathematical theory were to obtain and concrete reality were just as it actually is. Thus the conclusion corresponds to this counterfactual

      (15) (M & A) []→ C

      Here M is the mathematical theory [realistically construed], A says that concrete reality is just as it actually is, and C says something purely about concrete reality. Thus, the truth of the counterfactual seems to guarantee the truth of its consequent, even though its antecedent is false (by [instrumental fictionalist] lights), because the relevant counterfactual worlds are the same as the actual world with respect to concrete reality, which C is purely about [2017]."
- The standard objection to this strategy, as Williamson notes, is that "the structure of the hierarchy of pure sets [and that of any mathematical object] seems to be a metaphysically non-contingent matter….[Nominalists] who implement their strategy with counterfactuals and regard the rival metaphysical theory as a useful but impossible fiction have therefore been compelled to deny orthodoxy about counterpossibles. (for instance, Dorr 2008)."

- However, this objection seems confused. Williamson notes that the standard account of counterfactuals is plausible only so long as "necessity" is taken to mean "the maximal objective" notion of necessity [2016, 460], where an an objective notion of necessity "is what the modal words express when they are not used in any epistemic or deontic sense...[Strohminger and Yli-Vakkuri 2017, 825]." But, then, (first-order) logical possibility (without or without the Necessity of Identity and Distinctness) counts as objective! So, counter-mathematicals are non-vacuous, even according to orthodoxy.
- A more serious problem with the counterfactual strategy is that it may be no better off than scientific instrumentalism, since "we should expect that the observed phenomena would be very different on the hypothesis that there are no such things [as electrons] [Leng, 202]". If mathematical entities are causally relevant, the same is true of them.
- Even supposing that we know what we mean by a "simple and attractive" nominalistic theory, however, there are two further questionable assumptions at play in Field's project.
    - (1) That there is a useful abstract/concrete distinction.
    - (2) That the data to be explained by our scientific theories is itself concrete.
- Regarding (1), while Field is skeptical of mathematical points, he freely postulates spacetime points. But just as the former are unobservable and have no spacetime location, so are the latter (Resnick). Field's position is that the latter are nevertheless causally relevant. But we already saw that by an ordinary criterion of causal relevance, in terms of counterfactual dependence, mathematics is apparently causally relevant too!
- *Note*: Because Field's spacetime encodes the structure of mathematical space, if we assume Choice for regions of it, then the Banach-Tarski paradox holds for *real* space!
- Regarding (2), consider the data that "the number of electrons in the box is indeterminate, but the state is $1/\sqrt{2}$(two electrons in the box) + $1/\sqrt{2}$(three electrons in the box)" (Putnam 2012, 196). What could it mean to accept this data only as they concern concrete reality? Whether a superposition like this can be factored into concrete and abstract components depends on what the right interpretation of quantum mechanics is. However, this is precisely what we might be trying to discover by appeal to such data!
- *Note*: A similar problem may plague statistical data. What is the concrete content of that?

Field's Strategy

- Field's main idea is that mathematical science affords a false but convenient shorthand for more fundamental theories that do not quantify over mathematical entities. It is uncontroversial that mathematics can facilitate reasoning about non-mathematical facts.
  - *Example*: Consider the inference from "I have two apples." "Jenn has three more." to the conclusion "So, we have five apples.", which refers to numbers. Now compare it to the surrogate inference in terms of quantifiers and identity!
    - $(\exists x)(\exists y)[Ax \,\&\, Ay \,\&\, Hix \,\&\, Hiy \,\&\, x \neq y \,\&\, (\forall z)[(Az \,\&\, Hiz) \rightarrow z=x \vee z=y]]$
    - $(\exists x)(\exists y)(\exists z)[Ax \,\&\, Ay \,\&\, Az \,\&\, Hjx \,\&\, Hjy \,\&\, Hjz \,\&\, x \neq y \,\&\, x \neq z \,\&\, y \neq z) \,\&\, (\forall q)[(Aq \,\&\, Hjq) \rightarrow q=x \vee q=y \vee q=z]]$
    - $(\exists x)(\exists y)(\exists z)(\exists q)(\exists r)[Ax \,\&\, Ay \,\&\, Az \,\&\, Aq \,\&\, Ar \,\&\, x \neq y \,\&\, x \neq z \,\&\, y \neq z \,\&\, x \neq q \,\&\, x \neq r \,\&\, y \neq q \,\&\, y \neq r \,\&\, z \neq q \,\&\, z \neq r \,\&\, q \neq r \,\&\, (Hix \vee Hjx) \,\&\, (Hiy \vee Hjy) \,\&\, (Hiz \vee Hjz) \,\&\, (Hiq \vee Hjq) \,\&\, (Hir \vee Hjr)]$
- However, Field adds that math cannot lead us astray about non-mathematical facts. His reasoning is parallel to Hilbert's reasoning in connection with infinitary mathematics.
  - Conservativeness: If $N_1, N_2\ldots$ are nominalistic premises (niether referring to nor quantifying over mathematical entities), M is a mathematical theory, and C is a consequence of $N_1, N_2\ldots + M$, then C is a consequence of $N_1, N_2\ldots$ on their own.
  - *Note*: If mathematics is conservative, then it must also be consistent (but not true). If it were inconsistent, then it would imply everything, nominalist and otherwise.
  - *Problem*: It matters what notion of consequence is invoked. But if it is a first-order notion of consequence, then Conservativeness fails, by Godel's First Incompleteness Theorem. If it is a second-order semantic notion, then it is not about what we can derive. It is a claim about what's true in all (full) models. (Moreover, the representation theorems that Field proves fail in this latter case.)
- Finally, Field formulates a nomalist surrogate of Newtonian Gravitation that appeals only to 'intrinsic' relations like at-least-as-massive-as. He then proves representation and uniqueness theorems. These theorems say, respectively, that if a nonmathematical structure satisfies certain constraints, then there is a homomorphism from it to a mathematical structure, and all such homomorphisms are "equivalent" -- e.g., similarity transformations of each other. These then legitimize the use of mathematical functions.

- *Problem*: The proofs of these theorems use standard mathematics!
- *Response (Field)*: It is enough that they convince a platonist of their conclusions.

The Problem of Metalogic

- We have been speaking as though if Field could successfully nominalize empirical science, then he would be done. But successful nominalization requires the proof of metatheorems, like conservativeness. More generally, everyone needs to be able to talk about what follows from what and what does not follow from what (e.g., a contradiction). But, as ordinarily understood, this amounts to talk of proofs (understood as sequences of symbol types) or models (understood as sets), both of which would be abstract entities.
- *Note*: It is tempting to think that sentences are more 'epistemically innocent' than numbers. But the symbols out of which sentences are made cannot literally be anything like the concrete items that we use to represent them. A concrete sign has shape and extension. For instance, the token, '0', is oval in shape. But the type '0' cannot literally be oval in shape, because types have no spatiotemporal properties at all. The notion of a sentence also brings to mind misleading geometrical intuitions. A sentence is a sequence of symbols from the alphabet, e.g., 001001. The first '0' is not to the left of the first '1'!
- Field is aware of this problem, and develops a nominalistic metalogic to support his view. Field argues that consistency is a theoretical primitive. "The claim that consistency should be regarded as a primitive notion does involve the claim that we can't clarify its meaning by giving a definition of it in more basic terms. Similarly, logical notions like 'and', 'not', and 'there is' are primitive…[W]e learn them by learning to use them in accordance with certain rules; and we clarify their meaning by unearthing the rules that govern them. The same holds for consistency and implication, I claim: there are "procedural rules" governing the use of these terms, and...these rules...give the terms the meaning they have, not...definitions, whether in terms of models or of proofs [MM, 5]."
    - *Response*: No model (or proof-)theoretic reductionist should hold that our knowledge of what is consistent depends on our knowledge of what models there are. By that reasoning, a Lewisian is not a reductionist about metaphysical possibility! We postulate worlds by appeal to prior judgments about what is

metaphysically possible. What matters is whether the model-theoretic (or proof-theoretic) reductions afford a theoretical account of consistency.

  - *Real Problem*: Is the avoidance of Field's ideology really worth the ontology?

- Field requires (1) a device for infinite conjunction and (2) two sentential operators.
  - (1) Field introduces a substitutional quantifier, #F, for conjunction, allowing us to assert, e.g., the infinitely-many axioms of ZF (some are given by schemas), rather than saying of them that they are true (which is to speak of mathematical entities).
    - *Example*: All instances of the Comprehension Axiom for set theory would get expressed (roughly) as #Φ∃z∀y(y∈z←→Φ), where Φ is a formula.
  - (2) The operators that Field introduces are [] and <>, which are read 'it is logically necessary that' and 'it is logically possible that', respectively. These are meant to be dual in the standard sense, so that []P←→~<>~P and <>P←→~[]~P.
- On Field's view, if AX_T is the conjunction of the axioms of a ordinary mathematical theory, T, then what we really know is: <>AX_T and [](AX_T→P), for 'proved' theorems, P.
- *Problem*: If metalogic only concerns a primitive notion of logical possibility, not proofs or models, then why is reasoning about models and proofs so useful in the discipline?
  - For instance, we infer from AX_T has a model that <>AX_T, and infer from there is a derivation of (P & ~P) from AX_T that ~<>AX_T. How can Field explain this?
- Field makes an argument inspired by Kriesel. Kriesel's argument proceeds:

1. If [](P□Q), then every model of P is a model of Q.
2. If there is a derivation from P to Q, then [](P□Q).
3. Completeness Theorem: If every model of P is a model of Q, then there is a derivation from P to Q.
4. *Conclusion*: Logical necessity is coextensive with truth in all models and derivability.

- *Problem*: According to Field, we don't know if 1-3 are (non-vacuously) satisfied!
- *Response*: Rather than knowing 1-3, Field suggests that we know (roughly) the following.

i. If [](P→Q), then [](ZFC→every model of P is a model of Q).
ii. If [](ZFC→There is a derivation from P to Q), then [](P→Q).
iii. [](ZFC→[every model of P is a model of Q]→[There is a derivation from P to Q])

  - *Upshot*: Given knowledge that <>ZFC, we may conclude that:
    - o [](P→Q) ←→ [](ZFC→ every model of P is a model of Q)

←→ [](ZFC→There is a derivation from P to Q).

- *Fundamental Problem*: Once we give up on intrinsicness, we are left with the Epistemological Problem as grounds for nominalism. But why would it be easier to know, e.g., <>(ZFC) than Con(ZFC)? In both cases, if there are objective facts about consistency, finiteness, proof, and so forth, we require arithmetic objectivity or its modal surrogate.
- *Answer 2 (Putnam):* The latter concerns abstract objects, while the former does not.
- *Response*: The epistemological problem has nothing to do with ontology. (If it did, then moral realists would face no such problem assuming nominalism about universals!) It has to do with mind-and-language independence and objectivity. We could even state it so as to avoid reference to truths. The problem is to explain instances of the schema: in general, if mathematicians believe P, for mathematical P, then P (where we use, and do not mention, P).
- *Note*: When P is not an arithmetic sentence (or its modal surrogate) we might reduce this problem to that of why mathematicians reliably believe consistent sentences. However, this will not work for arithmetic itself, which we need in order to state facts about consistency.
- *Answer 3*: The analogy with metaphysical possibility suggests that we know it by conceiving of a situation verifying P. Of course, it is not supposed to be conceivable in the standard sense that $\sim AX_{ZFC}$. But under a broader interpretation of "conceivable" this is defensible.
- *Response*: If alternative math is conceivable, then why is not alternative logic? This points to a problem with Field's program. The epistemology of logic does not seem *prima facie* to be much more tractable than the epistemology of mathematics. In particular, there is disagreement over basic principles that cannot be resolved *via* observation and experiment.
- *Response* (?): For each notion of conceivability, there is a (perhaps) primitive modal operator.
- *Answer 4*: "[K]nowledge of consistency of…is at least partly based on the idea that if the theories were inconsistent we would probably have discovered it by now [MKLK, 124]."
- *Response (Leng)*: "Unless we have reason to believe that the derivations we are able to produce so far are a suitably representative sample of all possible derivations, this kind of enumerative induction will provide only a very weak justification for our belief [105]."
- *Answer 5 and Problem (Leng)*: "[I]f the best explanation of the successful application of a piece of mathematics requires the mathematical theory that we apply to be consistent, then an application of inference to the best explanation would provide an inductive justification for our belief in the consistency of that theory. A problem with this kind of reasoning is that…if

we had reason to believe that any contradiction in our mathematical theory was only derivable in a derivation too long for humans to produce, then the best explanation of the applicability of that piece of mathematics might [not] require that it is consistent [106].”