# Zeno's Paradoxes of Space

*The Paradox of Plurality*

- We have discussed the three main paradoxes associated with Zeno, and will discuss the fourth, less influential, paradox -- The Stadium -- below. But G.E.L. Owen and others claim that the really fundamental problem lying behind all of them is the following:
- (5) Any finitely-extended line segment has basic parts. Its basic parts are either of finite size or zero size. No number of zero-size parts could compose a finite object. However, any finite part can itself be divided, and so cannot really be basic. Consequently, finitely-extended line segments -- and, more generally, finite objects -- are impossible.
- *Note*: It is certainly true that the infinite *series* $\Sigma(S_n)_{n\to\infty} = s_1 + s_2 + s_3\ldots = 0$, where $s_1 = 0$, $s_2 = 0 + 0$, $s_3 = 0 + 0 + 0$ and so forth, by the (standard) definition given last time!
  - *Salmon*: "Although [Zeno] talks about the possibility of subdividing the parts, he is not talking about the possibility of cutting up a physical object into separate physical parts that can be moved away from one another. He is not dealing with the physical hypothesis of the atomic constitution of matter….<u>[H]is arguments depend on the possibility of making conceptual or mathematical divisions</u> (52)."
  - *Question*: What *is* the "physical hypothesis of the atomic constitution of matter" and how does it differ from the question that Salmon is trying to ask? Salmon suggests that if we could *not* separate physical objects (as a matter of physical possibility?), then they *would* compose a *physical* atom. But quarks and gluons cannot be so separated, given the strong force ('confinement'). The whole point of the quark hypothesis is that *they*, not the things they make up, are basic!
- As before, it is natural to think that 19th century mathematics solves the problem. The points of a line, while infinite, are *uncountably* so -- i.e., not in bijective correspondence with the natural numbers. So their sum cannot be defined as an infinite series (whose terms are indexed by the natural numbers). Their treatment requires 'measure theory' -- and, indeed, in measure theory countable collections of points are assigned measure 0.
  - *Observation*: This implies that, if the number of basic particles in the universe is finite, and such particles are *point* particles, as they are commonly said to be

(since otherwise they would have to exhibit internal structure), then the *measure* of the set of all basic particles in the universe is precisely zero! But, depending on how we understand 'structure', might not an *extended* object be structureless?

- *Technical Note*: Assuming the Axiom of Choice, which is a standard axiom, some sets of points have *no* measure (not even 0). But Solovay proved that if there is an Inaccessible Cardinal, it is consistent with ZF that all sets of reals are measurable (and have other 'desirable' properties). Indeed, the alternative Axiom of Determinacy implies this.
- The arithmetic of measure is even further removed from finite arithmetic than that of infinite series. Every open interval is isomorphic to every other. Hence, the measure of an interval is not even fixed by its number of points *plus the order in which they occur*.
- Moreover, it is not hard to derive quite counterintuitive results, such as that Cantor's 'discontinuum' must have measure 0, since its complement of 'middle thirds' adds up to 1. However, of the two sets, *it* is uncountable, and the complement is countable!
- *Upshot*: The measure of a line segment is fixed by the *coordinates that are assigned to its endpoints*, not by the number of points it contains, or something about their structure. More carefully, they are fixed by the coordinates along with a *metric rule* for measuring.

...-----0(A)-----1(B)-----2(C)-----3(D)-----4(E)-----...
...-----0(A)-----1(B)-----4(C)-----9(D)-----16(E)-----...

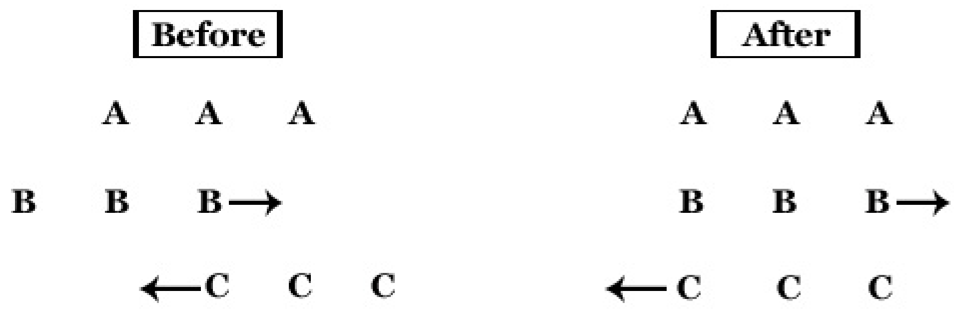
*Example*: The above coordinate systems are isomorphic, but give different answers to the question: *what is the interval between C & D?* When determined using the rule 'take the absolute value of the difference between the coordinates assigned to each' the first coordinate system gives the answer '1', while the second system gives the answer '5'.

- *Question*: Does it follow that space has no 'intrinsic metric' (as Grünbaum alleged)?

*The Stadium (Atomistic Interpretation)*

- It might be thought that the Paradox of Plurality could be solved by denying that any finite part can itself be divided. But even if this were granted, there would be problems.

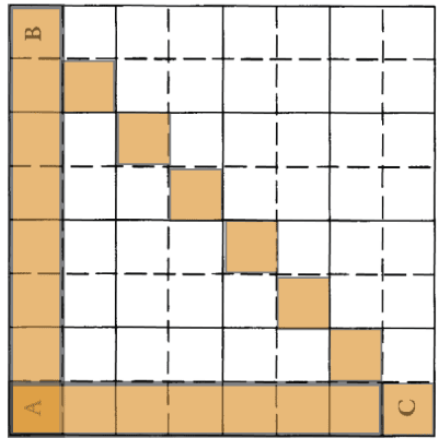
- First, if basic parts would all be of the *same* finite size, an infinity of them *would* seem to sum to an infinite size, as per the first three paradoxes discussed last time. Unlike the case of convergent series, the infinite series in question would not appropriately diminish.
- Second, even if the idea of quantized space and time makes sense, it seems to have the bizarre implication that objects can get past each other without ever *passing* each other.[1]




    - (4) Let each of the As, Bs, and Cs, occupy single atoms of space. Then, if the As do not move, but the Bs move at one unit of space per unit of time to the left, and the Cs move at one unit of space per unit of time to the right, then the middle B moves past the leftmost C without ever being *adjacent* to it (as shown above)!
- *Note*: Aristotle seems to have interpreted Zeno as arguing that the As and Cs disagree about *how fast the Bs move* (since C says they move at two units of space per unit of time, while A says that they move at one unit of space per unit of time). But, as Aristotle noted, this just shows that inertial motion is relative, which is now commonly accepted.
- Finally, even if we could learn to live with this implication (which, after all, seems more counterintuitive than paradoxical), there are problems with discrete space and time.
- *Problem 1*: It is not assured that discrete spacetime would approximate our spacetime.
    - *Example*: The Pythagorean Theorem clearly holds, for practical purposes, at the everyday scale. But Weyl pointed out that if we picture discrete spacetime as follows, it gives rise to manifest violations of the Pythagorean Theorem. *No matter how great or small the squares*, the hypotenuse equals the two other sides.[2]

---

[1] Image taken from https://www.iep.utm.edu/zeno-par/
[2] Imagine borrowed from https://i.pinimg.com/originals/45/f5/db/45f5db566af9576912ed74337b3fbed2.gif




- *Note*: We are certainly not logically forced to say that this *is* so. We could picture atoms as spheres. The point is that we cannot be assured that it is not, particularly because the 'shape' of a spacetime *atom* is presumably a meaningless concept.

- *Problem 2*: It's not obvious how discrete space squares with Relativity (to be discussed).
  - *Example*: Let h* be the minimum length relative to Joe, at rest. Then Katie, who is moving past Joe at nearly the speed of light, will see h* contract in the direction of her motion, as per Special Relativity. So, h* will be shorter relative to Katie. But if there is a minimum length, presumably every observer should agree to it!