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Mathematics and Metaphilosophy

- *Mathematics and Metaphilosophy* attempts to square two simple facts, and to sketch their broader ramifications for our theory of the world. The facts are:

    1. Mathematical truths are not up to us.

    2. We have substantial mathematical knowledge.

- Why believe (1)? Many will think that it needs no defense. It is obvious that we do not get to decide whether there are infinitely-many prime numbers, for example! However, in a philosophical context, an argument may be called for. So, here it is: *the physical and logical facts are not up to us, and physics and logic are up to their ears in mathematics*.

- Why believe (2)? Again, most will regard this as beyond reproach. What is more certain than that that the square root of 2 is irrational? But, again, if an argument is requested, then it is as follows: *skepticism about mathematics engenders skepticism quite generally* in light of the aforementioned indispensability of mathematics to physics and logic.

- How should we explain our substantial mathematical knowledge, given that the mathematical truths are not up to us? Note that the question is not how to explain *our coming to have the mathematical beliefs that we have*. That question presumably admits of an evolutionary psychological answer. Nor is the question how to *dialectically justify* our claim to mathematical knowledge. The task of explaining our knowledge of some kind, *assuming* that we have it, is very different from the task of convincing someone skeptical that we have such knowledge that we do have it. The second task, unlike the first, is notoriously intractable even in the perceptual case – as Descartes illustrated.

Overview of Arguments

- *Mathematics and Metaphilosophy* begins with the common suggestion that we know mathematical truths via proof. How do we know Thurston's Geometrization Conjecture? Because it was proved! Unfortunately, this answer is superficial. Even bracketing worries about logical knowledge (a matter to which I return at the end), mathematical proofs at most establish that *if* the axioms are true, *then* so too is the theorem proved.

Every claim admits of a proof in this sense! Gather together claims from which the one in question follows and call them 'axioms'. What matters is the standing of the axioms themselves – canonically, in the mathematical case, the axioms of *ZFC* set theory, perhaps plus large cardinals. How do we know *those*? It is often said that they are self–evident. But a cursory glance at the literature undercuts this hypothesis. There remain intractable disagreements over the axioms of Choice, Foundation, Replacement, Powerset, and Infinity. While some of these disagreements turn on outstanding conjectures (such as that the axioms imply a contradiction), a great many of them do not. They are recalcitrant in the way that stereotypically philosophical disagreements are.

- Perhaps axioms are 'analytic' – true in virtue of the meanings of the terms? In light of Quine's criticisms, most philosophers are careful to distinguish *epistemic* from *metaphysical* versions of this proposal. The metaphysical version says that the meaning of 'is a member of' somehow *makes it the case* that the set-theoretic axioms are true. This is hard to even understand. How could a meaning make a fact? The epistemic version says that competence with the concept of membership involves a disposition to believe the axioms of set theory, so those of us with that concept are at least defeasibly entitled in believing them. But even this is suspect. If we were worried that some sets lack Choice functions, for example, then, under the assumption that it is epistemically analytic that all sets have them, we should just worry *that our theory of set is not satisfied*. This is just what we concluded in connection with Frege's logical concept.

- Given the indispensability of mathematics to science, it might be thought that the axioms of mathematics could be known empirically. But this is doubtful. First, if this were the case, then it would be mysterious why scientists are happy to use any mathematics that is convenient for their purposes, while the postulation of new fields is scrutinized. Second, rigorous formalizations of scientific theories make use of only a relatively small portion of mathematics. So, even if some of the axioms of mathematics were empirically known in this way, a large – maybe *huge* – part of established mathematics would turn out not to be. Feferman conjectures that only (first-order) Peano Arithmetic is needed for science!

- Is there any other explanation of our mathematical knowledge, given that the truths are not up to us? There is one. Early on, Russell, like Frege, hoped to 'prove' mathematics in a logical as well as epistemic sense of that term. But he soon discovered the paradox that bears his name, and that any adequate foundation would require speculative hypotheses like the axioms of Infinity, Reducibility, and Choice. Russell concluded,

    - 'We tend to believe the premises [the axioms of mathematics] because we can see that their consequences are true, instead of believing the consequences because we know the premises….But the inferring of premises from consequences is the

> essence of induction; thus the method in investigating the principles of mathematics is really an inductive method, and is substantially the same as the method of discovering general laws in any other science [1973/1907, 273–4].'

- Russell's mature proposal is that the axioms of mathematics are analogous to laws of science. They are postulated to best systematize the data to be explained. The difference between the cases is the nature of the data: they are *intuitions*, or plausibility judgments, in the mathematical case, and *observations* in the scientific one. But, in both cases, the order of justification generally opposes the order of logical implication. Although we *deduce* theorems from axioms, we are justified in believing the axioms because we are justified in believing the theorems that they imply, rather than the other way around.

- Unfortunately, there is an important difference between mathematics and science that Russell does not address: *there is much less uniformity of intuition than observation*.

    - Consider a paradigmatic disagreement over a scientific theory, the theory of dark matter. Those who reject dark matter, like Milgrom, and propose amendments to Newtonian gravity, do so in order to account for *the same data*. They do not generally disagree over *it*. But disagreement in the foundations of mathematics seems *characteristically* to bottom out in conflicting intuitions. For example, while prominent set theorists maintain that Gödel's *Axiom of Constructibility* resolves questions in the wrong way, Devlin holds that it 'tends to decide problems in the 'correct' direction', and Jensen follows him in this. Similarly, while most accept the set-theoretic assumptions on which calculus depends, Weyl is explicit that 'in any wording [the Least Upper Bound Axiom] is false'. And so on.

- To be clear: *some* scientific disagreements bottom out in conflicting observations too. Observation is theory-laden, and perceptions can be impaired. The point is that such cases are exceptional. Unlike disagreements in the foundations of mathematics, scientific disagreements do not seem to be *primarily attributable* to disagreements over the data to be systematized – or what Jensen calls 'deeply rooted differences in mathematical tastes'.

- This difference between mathematics and science seriously aggravates the problem of explaining our mathematical knowledge, given that the mathematical truths are not up to us. It means that there is a surprising analogy between mathematics and philosophy. Not only are mathematical disagreements intractable; they bottom out in conflicting intuitions as philosophical disagreements tend to. If intuitions did not vary, then mathematical disagreement would be like ordinary scientific disagreement. They would reduce to disagreements about how to best systematize the agreed upon data. They would not constitute reason to *doubt the data itself*. But the existence of 'cognitive peers' with opposite intuitions is precisely a reason to doubt our intuitions – as the existence of cognitive peers with opposite observations would be a reason to doubt our observations. If my intuitions support the measurability of all sets, and yours support the non-measurability of some, then one of us has misleading intuitions. We can't both be right!

Mathematical Pluralism

- Or so it seems. I have been assuming that there is 'one true mathematical reality' just as physicists assume that there is one true physical reality – even if that reality incorporates a multiverse. There is a privileged metatheory, like some version of set theory, which constitutes the final court of appeals for mathematical questions. This metatheory engenders, not just an interpretation of 'is a member of', but also an interpretation of the powerset operation, higher-order quantification, and so on. Of course, our metatheory may house a rich plurality of mathematical structures – just as physical reality may encompass many universes. For example, all the geometries – Euclidean, hyperbolic, elliptic, variably curved – should find realizations there. But the question of what our metatheoretic resources are in the first place is supposed to admit of a once-and-for-all answer. That is what disagreements over the foundations of mathematics are all about.

- What if we gave up on the assumption of a unique mathematical reality – a uniquely right metatheory? Then intuitions relevant to the *Axiom of Choice*, for example, may vary in the banal sense that intuitions relevant to the Parallel Postulate do (understood as a claim of pure mathematics). Different intuitions may reflect different set theories, no one of which is preferred by 'Reality Itself' (however superior it may be for *our* purposes). The fact that our intuitions of Euclidean space diverge from our intuitions of hyperbolic space just shows that they are about different subjects – each of which is as real as the other one. Something similar may be true of our intuitions about 'the' set-theoretic universe.

- Let us call *mathematical pluralism* the view that mathematical reality is so rich, and the semantics of mathematical discourse is so cooperative, that any set theory (or potential metatheory) that we might have easily adopted truly describes some part of that reality.[1] Then pluralism affords an explanation of the fact that we have substantial mathematical knowledge that is consistent with the fact that mathematical truths are not up to us. It does so by giving up on the *objectivity* of the mathematical truths, but not on *realism* about mathematics. Different set theories can all be right. One is right of sets$_1$, another of sets$_2$, and so forth. Pluralism thus divorces two ideas that have long been associated: the idea that there is an independent reality and the idea that in a disagreement, only one

[1] See Balaguer, 'A Platonist Epistemology', for a closely related view in the mathematical case.

of us can be right. Pluralism says that mathematics is perfectly real, but it is not objective.

- Note that mathematical pluralism cannot be a *formal* theory. Any alleged formalization of pluralism would assume a metatheory of its own. This would wrongly take itself to be maximal. Mathematical pluralism is an inherently informal philosophy of mathematics.

Broader Relevance

- If mathematical knowledge is mysterious in the way that philosophical knowledge is, and pluralism demystifies the former, then could it demystify the latter? The final chapter of *Mathematics and Metaphilosophy* explores pluralism about logic, modality, and evaluative inquiry and the implications of a general pluralism for science and practical reason. The resulting system combines a naive realism with a *pragmatist* methodology. Divergent theories of what follows from what, how the world could have been different, and what we ought to do, are all independently true – albeit of subtly different subjects. So, truth factors out. It is as if the claims in question did not admit of truth at all. The only non-semantic questions (i.e., questions that are not just about what we mean by words) in logical, modal, evaluative and mathematical debates are *practical*. Instead of asking whether everything follows from a contradiction, we ask whether to use a paraconsistent logic. Instead of asking whether you could have had different parents, we ask whether to use a modal theory that incorporates that Necessity of Origins. And so on. Intractable questions from apparently a priori domains get traded for practical ones.

- Let me illustrate the revisionary implications of this position. Consider the intersection of a family of sets of points in spacetime. Does it exist? This depends on whether our background set theory includes the Powerset axiom. If it is *ZFC* set theory, then it does. But if it is, say, Kripke–Platek set theory, then it does not. Mathematical pluralism says that each such set theory truly describes some part of mathematical reality. So, if pluralism is correct, then basic questions about spacetime structure are not objective.

- The logical and modal cases are similar. Let T be a regimented physical theory, such as the Standard Model of particle physics (assuming that it can be regimented!). Does T imply O, where O is some observation statement that has been refuted? This depends on whether T is classically consistent. If it is not, then T implies everything, including O. So, one way that it could fail to be objective whether T has been refuted is that it fails to be objective whether T is classically consistent. But if logical, as opposed to mathematical, pluralism is true, then there is also another way. It fails to be objective *what logic is correct*. So, if O is classically, but not intuitionistically, implied by T, then the question of whether *T* has been refuted by ~*O* would again be like the question of whether the Parallel Postulate is true – understood as a question of pure mathematics.

- Finally, consider a quantum spin system, *S*, with eigenstates |↑> and |↓>. Could *S* have been in the *indeterminate* state of being *neither* |↑> *nor* |↓> *nor* a superposition of the two? What about the *contradictory* state |↑> *and* |↓>? As a matter of metaphysical, a fortiori, physical, possibility, the answer to both of these questions is, of course, 'no'. However, if the kind of possibility at issue is paracomplete, allowing for the possibility of indeterminacies, then the state, *neither* |↑> *nor* |↓> *nor* a superposition of the two, will be in *S*'s state space. If it is paraconsistent, then the state *both* |↑> *and* |↓> will be in that space. And if First-Degree Entailment governs the states of the system, then *neither* |↑> *nor* |↓>, nor a superposition of the two, as well as *both* |↑> *and* |↓> will all be possible states of the system. In general, if there is no objective fact as to how the world could have been, then there is no objective fact as to the state space of a physical system.

- For an evaluative example, consider the question of whether we ought to kill the one to save the five (in some situation). As in the set-theoretic case, we $\text{ought}_{\text{utilitarian}}$, $\text{ought}_{\text{deontological}}$ not, and so on. The only non-verbal question is *whether to consult* a utilitarian, deontological, or theory of ought. But, unlike the other cases, there is a connection between *this* practical question and a question of fact from the target domain. Evaluative facts, unlike mathematical, logical, or modal ones, are supposed to tell us what to do. The problem is that any self-respecting evaluative fact will tell us to consult *it*! We $\text{ought}_{\text{utilitarian}}$ consult a utilitarian theory, $\text{ought}_{\text{deontological}}$ consult a deontological theory (and so on for other evaluative theories). The question of *which theory to consult* remains open. This cannot be the question of which theory we *ought* to consult on pain of triviality. So, the facts, *even the evaluative ones* (if there are any), fail to settle what to do. This idea radicalizes Hume's dictum that one cannot derive an 'ought' from an 'is'. One cannot even derive *what to do* from what we *ought* to do – in any sense of 'ought'!

- When the dust settles, pluralism mimics Carnap's radical positivism, despite its antithetical metaphysics. The question of which mathematical, modal, or logical axioms are true *is* misconceived, as Carnap alleged. And the pressing questions in the vicinity are nonfactual practical questions of what to do. What concept of set to use? What concept of possibility and consequence to employ? Indeed, what concept of *ought* to follow – where this is *not* the (circular) question of what concept of ought we *ought* to follow. Theoretical questions dissolve into practical ones, questions of expedience. As Carnap puts it, 'the conflict between the divergent points of view … disappears … [B]efore us lies the boundless ocean of unlimited possibilities' [1937/2001, XV].