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Russell's Regressive Method in Mathematics and Philosophy

Knowing the Axioms

- *Naive Account (Euclideanism)*: "Axioms are mathematical statements that are *self-evidently* true….If [philosophy] is like math, then the [philosophical] truths to which we appeal in our arguments must ultimately follow from [philosophical] axioms, from a manageable set of self-evident...truths" [Greene 2013, 184, italics in original].

- *Problem*: "The set-theoretical axioms that sustain modern mathematics are self-evident in differing degrees….[T]he most important of them, namely...the so-called axiom of infinity – has scarcely any claim to self-evidence at all" [Mayberry 2000, 10].

    - *Boolos*: "I am by no means convinced that any of the axioms of infinity, union, or power [set]…force themselves upon us or that all the axioms of replacement that we can comprehend do….[T]here is nothing *unclear* about the power set axiom… But it does not seem to me unreasonable to think that…it is not the case that for every set, there is a set of all its subsets..." [1999, 130-131, italics in original].

- *Better Answer (Russell)*: "We...believe the premises [i.e., the axioms of mathematics] because we can see that their consequences are true, instead of believing the consequences because we know the premises….But the inferring of premises from consequences is the essence of induction; thus the method in investigating the principles of mathematics is really an inductive method, and is substantially the same as the method of discovering general laws in any other science" [1973/1907, 273–274].

    - *Gödel*: "[Russell] compares the axioms of…mathematics with the laws of nature and logical evidence with sense perception, so that the axioms need not…be evident in themselves, but rather their justification lies (exactly as in physics) in the fact that they make it possible for these 'sense perceptions' to be deduced….I think that…this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future" [1990/1944, 121].

    - "[D]espite their remoteness from sense experience, we do have a perception also of the objects of set theory, as is seen from the fact that [at least some] axioms force themselves upon us as being true. I don't see why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them and...to believe that a question not decidable now has meaning and may be decided in the future" [1947, 483-4].

Benacerraf's Objection

- *Benacerraf*: "I find this picture both encouraging and troubling. What troubles me is that without an account of *how* the ['mathematical sense perceptions'] "force themselves upon us as being true," the analogy with sense perception and physical science is without much content. For what is missing is *precisely...*an account of the link between our cognitive faculties and the objects known. In physical science we have at least a start on such an account, and it is causal. We accept as knowledge only those beliefs which we can appropriately relate to our cognitive faculties….[T]here is a *superficial* analogy….[W]e "verify" axioms by deducing consequences from them concerning areas in which we seem to have more direct "perception" (clearer intuitions). But we are never told how we know even these, clearer, propositions" [1973, 674, italics in original].

- *Clarification (Field)*: "[W]e [should] grant...that there may be positive reasons for believing in [standard axioms]. These positive reasons might involve...initial plausibility ....But Benacerraf 's challenge ...is to...explain how our beliefs about these remote entities can so well reflect the facts about them….*[I]f it appears in principle impossible to explain this*, then that tends to *undermine* the belief in mathematical entities, *despite* whatever reason we might have for believing in them" [1989, 26, emphasis in original].

- *Interpretation*: Benaerraf's challenge partitions into two, which he does not distinguish:

  - (1) the challenge to explain the *justification* of our beliefs

  - (2) the challenge to explain their *reliability*

  - *Note*: There is another *non-epistemological* challenge raised by Benacerraf. This is to *explain the determinacy* of our mathematical beliefs. The epistemological challenges arise *only to the extent* that the determinacy challenge can be met.

- *Problem:* Gödel is offering a 'phenomenalist' (or 'dogmatist') answer to (1), in the spirit of Bengson [2015], Chudnoff [2013], Huemer [2005], and Pryor [2000]. *These theories, as such, do not even purport to answer the reliability challenge for perceptual realism*.

- *Upshot*: The fact that we can sketch a causal account of how our empirical, but not mathematical, beliefs 'track' the facts shows that we cannot answer the reliability challenge for mathematical realism *in the same way* that we answer it for empirical realism. It allows that we can answer the justificatory challenges in the same way.

Mathematics and Science

- Intuition is *not* just like observation, *even at the level of justification*. There is *disagreement over the data to be accounted for* in the mathematical case that has no analog in the empirical one.

- *Caveat*: There is not perfect agreement as to what the observational data is either. Our observational faculties can be impaired. Moreover, observational is theory-laden. However, disagreements that arise over empirical scientific hypotheses do not seem to be *primarily attributable* to disagreements over the empirical data to be accounted for.

- *Example*: Those who reject dark matter, and prefer instead to amend General Relativity, do so in order to account for the *same observational data*. They do not disagree over *it*.

  - *Milgrom*: "Dark matter is the only explanation...for the [observed]...mass discrepancies, if we cleave to the accepted laws of physics. But if we accept a departure from these...laws, we might do away with dark matter....I proposed a modification to Newton's second law that changed the relation between force and acceleration when the acceleration is low. This was the beginning of the idea called MOND, for Modified Newtonian Dynamics" [2002, 45].

- Contrast this characteristically scientific disagreement to disagreement over axioms:

  - *Replacement:* "Let me try to be as accurate, explicit, and forthright about my belief about the existence of k [= the least ordinal greater than all f(i), where f(0) = Aleph_0 and f(i + 1) = Aleph_f(i)] as I can…I…think it probably doesn't exist….I am also doubtful that anything could be provided that should be called a *reason* and that would settle the question" [Boolos 1999, 121, italics in original].

  - *Choice:* "Without any doubt the most problematic axiom of set theory is the axiom of choice....The current situation with AC is that the contestants have agreed to differ" [Forster 2016, 72].

  - *Foundation:* "ZF does not embody a philosophically coherent notion of set. There is a coherent constructivist position….There is also a coherent anti-constructivist position ….But ZF is an uneasy compromise between these two: it pays lip-service to constructivism without…meaning it…. Only a non-well-founded [set] theory can…be shown to modify the naive conception as much as, but no more than, is required…" [Rieger 2018, 17-18].

  - *New Axioms:* "Set theorists say that V = L has implausible consequences…. [They] claim to have a direct intuition which allows them to view these as so implausible that this provides "evidence" against V = L. However, mathematicians [like me] disclaim such direct intuition about complicated sets of

reals. Many say they have no direct intuition about all multivariate functions from N into N!" [Friedman FOM, 5.25.00][1]

- ○ ...

- *Caveat*: "Diagnosing a clash of intuitions…will typically involve attempting a careful hermeneutic reconstruction of the underlying dialectic, designed to reveal that the dispute rests ultimately with certain…premises that one side finds intuitive and the other does not. Any such reconstruction is bound to be controversial.…[W]hereas many philosophers agree that some questions boil down to...differences in intuition, there is considerable disagreement as to exactly which questions those are" [Mogensen 2016, 24].

Explaining Away the Difference

- *Argument 1*: Controversial mathematical intuitions are more like controversial observational ones. They are theoretical. Disagreement over whether there are more subsets than *L* allows, whether there is a Measurable Cardinal, or whether all projective sets are determined, say, is more like disagreement over the extent to which a biopsy harbors dysplasia -- an *observational* matter over which there is still disagreement.

- *Problem*: In the empirical case, we can contrast observational judgments of, e.g., the degree of dysplasia with "practically bedrock" such judgments about the overt appearance of the biopsy sample stated in neutral terms ("...not very circular"). But, in the mathematical case, there appears to be no such bedrock "observations" to begin with.

    - *Weyl*: "It will be recognized…that in any wording [the Least Upper Bound Axiom of the calculus] is false" (quoted in [Kilmister 1980, 157]).

    - *Nelson*: "The reason for mistrusting the induction principle [of arithmetic] is that it involves an impredicative concept of number.…A number is conceived to be an object satisfying every inductive formula" [1986, 1].

    - *Zeilberger*: "I am a platonist…[but] I deny even the...Peano axiom that every integer has a successor.…" [2004, 32-3].[2]

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[1] *Cf.*: "The axiom V=L, when added to ZFC, settles "nearly all" mathematical questions. Furthermore, it can be motivated by constructivist philosophy.…[L]ater developments on the structure of L, especially those due to Jensen, gave a wealth of powerful combinatorial principles that follow from the axiom V=L.….Given the effectiveness of the axiom V=L at settling mathematical questions, and the fact that it can be motivated by constructivist views that are still widely held today, why hasn't there been historically a stronger push to adopt it as a foundational axiom for mathematics" [Eskew 2019]?

[2] Nelson sometimes demurs from asserting that every natural number has a successor as well, at least inconnection with "actual" (or "genetic") numbers. See his [1986, 176] and [2013].

- - ■ *Friedman*: "I have seen some…go so far as to challenge the existence of 2^100…" [2002, 4].

- *Caveat*: I do not suggest that the notion of a practically bedrock proposition is precise!

- *Argument 2*: People with heretical intuitions are in the overwhelmingly *minority*, just like people with heretical observational judgments are in that minority. Moreover, *this sociological fact* affords (defeasible) evidence that the heretics are "hallucinating".

  - ○ *Koellner*: "[Projective Determinacy] has gained wide acceptance by the set-theorists...who know the details of the constructions and theorems involved in the case that has been made for PD" [2013, 21-22].

- *Problem 1*: The *relevant* group is presumably people who work on the disputed axioms and intimately related matters -- not "typical mathematicians". However, notoriously, specialist knowledge tends to turn "something so simple as not to seem worth stating" into "something so paradoxical that no one will believe it" [Russell 1918, 514]!

  - ○ *Forster*: "[F]or people who want to think of foundational issues as resolved… [standard axioms provide] an excuse for them not to think about [them] any longer. It's a bit like the role of the Church in Medieval Europe: it keeps a lid on things that really need lids" [2018, 15]!

  - ○ *Bell & Hellman*: "Contrary to the popular (mis)conception of mathematics as a cut-and-dried body of universally agreed upon truths...as soon as one examines the foundations of mathematics one encounters divergences of viewpoint...that can easily remind one of religious, schismatic controversy" [2006, 64].

- *Problem 2*: Even if poll numbers matter in the *empirical* cases, it is hard to see how they could matter in *armchair* cases like pure mathematics. There seems to be no reason to suppose that true set-theoretic intuitions would be popular *independent of their contents*.

  - *Martin*: "For individual mathematicians, acceptance of an axiom is probably often the result of nothing more than knowing that it is a standard axiom" [1998, 218].

  - *Cohen*: "[T]he attitudes that people profess towards the foundations seem to be greatly influenced by their training and their environment" [1971, 10].

- *Argument 3*: The justified intuitions are the ones whose contents are *indispensable to our empirical theories*, just like the justified observational judgments [Harman 1977, 208].

- *Problem 1*: If anything, those would support heretical intuitions!

  - ○ *Quine*: "I recognize indenumerable infinities only because they are forced on me by the simplest known systematizations of more welcome matters. Magnitudes in

excess of such demands, e.g., Beth$_\omega$ or inaccessible numbers, I look upon only as mathematical recreation and without ontological rights" [1986, 400].

- ○ *Feferman*: Weyl's system, W, exhausts what is indispensable for empirical science, and "W [itself can be] treated in an instrumental way, its entities outside the natural numbers are regarded as "theoretical" entities, and the justification for its use lies in whatever justification we give to the use of PA" [1992, 451].

- ○ *Field*: We can avoid "all appeal to mathematical entities in explanations when the chips are down: it must be possible, for instance, to develop theoretical physics without any appeal to mathematical entities" [1989, 6].

- *Problem 2*: The suggestion that indispensability is a guide to truth in the first place itself appears to turn on intuitions whose contents are not so indispensable [Bealer 1992]!

Mathematics and Philosophy

- *Upshot*: Even at the level of justification, mathematical knowledge is *not* just like scientific knowledge. Rather than resembling empirical science, mathematics (and metalogic, Goodman [63–64]) seems more closely to resemble paradigmatic *philosophy*.

- ○ *Lewis*: "Our "intuitions" are simply opinions: our philosophical theories are the same. Some are commonsensical, some are sophisticated; some are particular; some general; some are more firmly held, some less….[A] reasonable goal for a philosopher is to bring them into equilibrium. Our common task is to find out what equilibria there are that can withstand examination, but it remains for each of us to come to rest at one or another of them…" [1983: x-xi].

  - ■ *Cartwright*: "[T]he grounds on which ratings of attributes as essential or accidental are to be made [are obscure]....[O]ne is simply to reflect on the question of whether the object in question could or could not have had the attribute in question….But the criteria to which one appeals in such reflection are sufficiently obscure to leave me, at least, with an embarrassingly large number of undecided cases…." [1968, 626].

  - ■ ...

- Although this objection undercuts the Russell-Gödel analogy with science, it remains open what to make of it. Knowledge of it might be said to *undermine* our mathematical beliefs, either because *disagreement in intuition* constitutes a defeater [Mogensen 2016], or because such disagreement is evidence that the *reliability* challenge cannot be met.

- However, down that path lies *empirical* skepticism, in light of the indispensability of armchair theories, like mathematics, metalogic, and modality to our scientific theories.

- I favor a *pluralist* approach to armchair subjects (Clarke-Doane [2020]). According to it, while armchair questions answer to an independent reality, armchair -- unlike empirical -- reality is so rich as to witness any answers we might give. Hence, conflicts of intuition may be merely apparent, and our reliability can be explained in a non-causal way. Armchair reality is so inclusive that "one's cognitive faculties can't miss [Beall, 157]."

- Pluralism is a kind of transposition of Carnap's conventionalism to the key of realism. Although their metaphysics are totally different, their methodological upshot is the same.

  - *Carnap*: "[T]he conflict between the divergent points of view...disappears… [B]efore us lies the boundless ocean of unlimited possibilities [1937/2001, XV]."

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