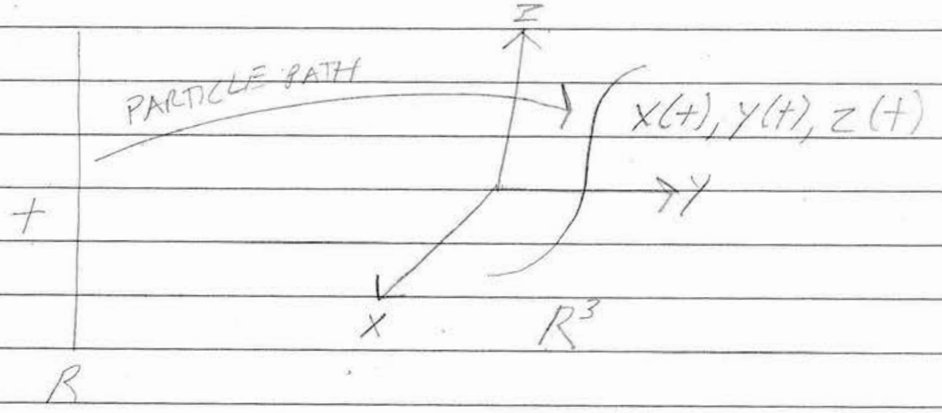
SPACETIME THEORIES

- *RAINE & HELLER, THE SCIENCE OF SPACE-TIME
- EPSTEIN, DIFFERENTIAL GEOMETRY
- GEROCH, GENERAL RELATIVITY FROM A TO B

· ARISTOTELIAN PHYSICS

KINEMATICS

· A PATH OF A PARTICLE IN SPACE IS A FUNCTION, $R \to R^3$:
$t \to (x(t), y(t), z(t))$, WRITTEN $t \to \underset{\sim}{x}(t)$.



MAKES SENSE USING USUAL ORDER ON R

· DEFINITION: A SET OF POINTS, S, IS OPEN WHEN (i) BETWEEN ANY TWO POINTS OF S THERE IS ANOTHER IN S, & (ii) ANY POINT OF S IS BETWEEN TWO OTHERS IN S.

TOPOLOGY

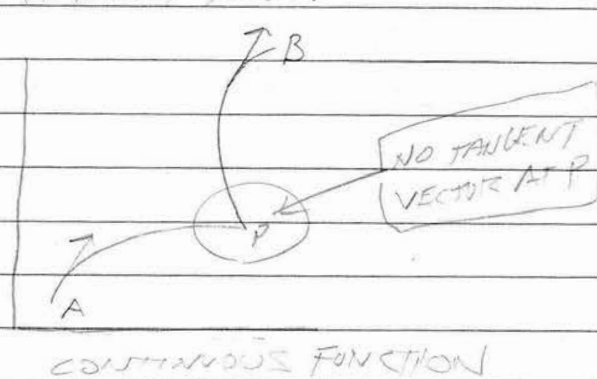
· BY SPECIFYING WHAT SUBSETS OF A COLLECTION OF SETS ARE OPEN WE ENDOW IT WITH A TOPOLOGY.

· DEFINITION: A PARTICLE PATH IN R^3, $x(t)$, IS CONTINUOUS WHEN $t \to x(t)$, $t \to y(t)$, AND $t \to z(t)$ ARE ALL CONTINUOUS FUNCTIONS $R \to R$.

· NOTE: JUST BECAUSE A PATH IS CONTINUOUS DOES NOT MEAN THAT IT HAS A TANGENT VECTOR AT EVERY POINT.

· HOWEVER, AS R^3 IS A DIFFERENTIABLE MANIFOLD WE CAN SAY WHICH FUNCTIONS ON R^3 ARE DIFFERENTIABLE, AND, HENCE, WHICH PATHS HAVE TANGENT VECTORS AT EVERY POINT.

...ERENTIAL MANIFOLD (OF ...ASS C^k): A SET, E, WITH ... MAXIMAL ATLAS (OF ...SS C^k) THE DIMENSION ... WHICH IS THAT OF THE ...CE, R^n, ON WHICH ... IS MODELED.



CONTINUOUS FUNCTION

ATLAS OF CHARTS

(A, E) IS A CHART

• UPSHOT: WHILE THE SPECIFICATION OF PARTICLE TRAJECTORIES DEPENDS ONLY ON CONTINUUM, THE INTRODUCTION OF TANGENT VECTORS APPEALS TO THE STRUCTURE OF SPACE AS A DIFFERENTIABLE MANIFOLD.

AFFINE STRUCTURE

• WE HAVE AN OPERATION OF PARALLEL TRANSPORT, SO WE CAN ALSO COMPARE VECTORS AT DIFFERENT POINTS.

METRIC

• WE WILL ASSUME A STANDARD CLOCK BY WHICH WE CAN MEASURE DISTANCES ACCORDING TO $d(y, Y) = |Y - y|$, AND A SIMILARLY EUCLIDEAN METRIC FOR SPACE:

$$[d(\underset{\sim}{x}, \underset{\sim}{X})]^2 = (x - X)^2 + (y - Y)^2 + (z - Z)^2$$

"EFFICIENT"

• NOTE: GIVEN A METRIC STRUCTURE WE CAN QUANTIFY VELOCITIES.

CANNOT BE USED TO PREDICT CIRCULAR MOTION

• ARISTOTLE'S LAW #1: EVERY OBJECT IN MOTION IS KEPT IN MOTION BY A CAUSE/FORCE. AN OBJECT SUBJECT TO NO CAUSE IS AT ABSOLUTE REST.

• ARISTOTLE'S LAW #2: THE FORCE APPLIED TO AN OBJECT IS PROPORTIONAL TO ITS MASS TIMES VELOCITY, i.e.

$$\underset{\sim}{F} = m\underset{\sim}{v} = m\left(\frac{d\underset{\sim}{x}}{dt}\right)$$

DYNAMICS

• NOTE: IF $F = 0$, $\underset{\sim}{v} = 0$, AS PER LAW #1. BUT LAW #1 IS NOT REDUNDANT. LAW #2 PRESUPPOSES IT. WE IDENTIFY ABSOLUTE REST AS THE STATE OF THE EARTH, AND VELOCITY RELATIVE TO IT – THUS GIVING PRINCIPLE #2 NON-CIRCULAR CONTENT.

• A "PRIVILEGED OBSERVER" WITH RESPECT TO WHICH

THE LAWS ARE SUPPOSED TO BE VALID IS AT REST AT THE CENTER OF THE EARTH.

- MACH'S PROBLEM: THE LAWS ARE STATED WITH RESPECT TO AN ABSOLUTE FRAME OF REST WHICH JUST HAPPENS TO CORRESPOND TO THE EARTH'S FRAME.

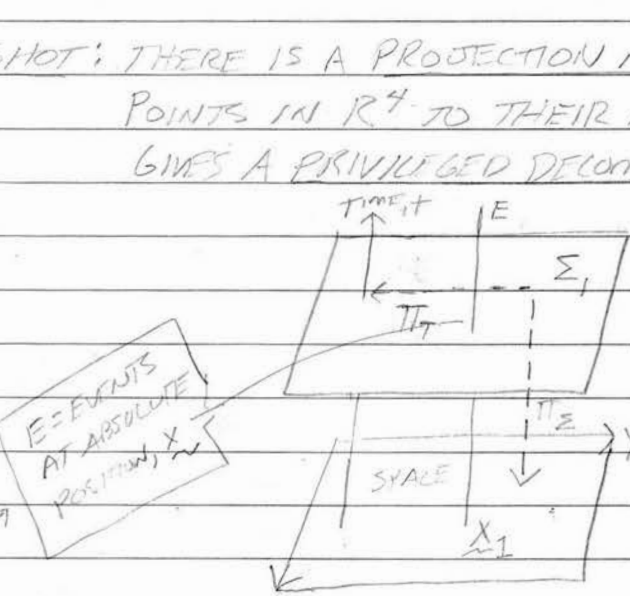
- IT WILL BE HELPFUL FOR PURPOSES OF COMPARISON TO FORMULATE ARISTOTLE'S THEORY AS AS A SPACE-TIME THEORY. WORLDLINES ARE NOW FUNCTIONS FROM A REAL PARAMETER, λ, TO EVENTS IN THE 4-D EUCLIDEAN MANIFOLD, $\mathbb{R}^4$.

BUT WE CAN SHIFT THE ORIGIN & UNITS ONLY

- NOTE: LAW #2 INVOLVES AN ABSOLUTE TIME SINCE

$$v = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} \neq \frac{dx}{d\tau} \text{ FOR } \tau = f(t) \text{ (AND } \frac{df}{dt} \neq 1\text{)}.$$

- UPSHOT: THERE IS A PROJECTION MAP, $\Pi_T : M \to T$ FROM POINTS IN $\mathbb{R}^4$ TO THEIR ABSOLUTE TIME IN $\mathbb{R}$. THIS GIVES A PRIVILEGED DECOMPOSITION OF THE MANIFOLD.



ABSOLUTE SIMULTANEITY SLICES

- $\Pi_T : M \to T$ (ABSOLUTE TIME)

$$\boxed{\Pi_T(m) = \Pi_T(t, x) = t}$$

- $\Pi_\Sigma : M \to \Sigma$ (ABSOLUTE SPACE)

$$\boxed{\Pi_\Sigma(m) = \Pi_\Sigma(t, x) = x}$$

- NOTE: ABSOLUTE TIME AND ABSOLUTE SPACE ARE INDEPENDENT COMMITMENTS, AS WE WILL SEE.

$\{\langle t, x\rangle : t \in T \wedge x \in \Sigma$

- THE FUNCTIONS Π_T AND Π_Σ MAKE M A DIRECT PRODUCT OF A TIME MANIFOLD, T, AND A SPACE MANIFOLD, Σ, i.e. $\boxed{M = T \times \Sigma}$

- COPERNICAN PHYSICS

  - COPERNICAN SPACETIME HAS THE SAME [illegible] STRUCTURE AS ARISTOTELIAN. WE SIMPLY CHANGE THE PRIVILEGED POSITION FROM THE EARTH TO THE SUN. ONLY IN THE HELIOCENTRIC FRAME (RATHER THAN GEOCENTRIC) DO THE LAWS TAKE SIMPLE FORM.

  - DEFINITION: A THEORY IS COVARIANT IF ITS LAWS TAKE THE SAME FORM IN ARBITRARY REFERENCE FRAMES.

  - NOTE: A THEORY CAN BE COVARIANT EVEN IF IT POSTULATES A PRIVILEGED REFERENCE FRAME.

  - COPERNICUS WAS SENSITIVE TO THE APPARENT RELATIVITY OF MOTION, BUT RESISTED GIVING UP ON ABSOLUTE SPACE AS THAT WOULD SEEM TO MAKE GEOCENTRISM-HELIOCENTRISM DEBATE MERELY VERBAL.

- NEWTONIAN PHYSICS

  - NEWTON'S PRINCIPIA IS OFTEN SAID TO MARK THE TRANSITION FROM 'EXPLAINING EVERYTHING WHILE CALCULATING NOTHING' TO 'CALCULATING EVERYTHING WHILE EXPLAINING NOTHING' (THOM 1975).

  - NEWTON'S FIRST LAW: EVERY OBJECT IN NON-UNIFORM MOTION IS KEPT IN NON-UNIFORM MOTION BY SOME FORCE.

  - NOTE: THIS DEFINES A PRIVILEGED CLASS OF 'INERTIAL' MOTIONS DYNAMICALLY, IN TERMS OF ABSENCE

OF FORCES. HOWEVER, THE SPACETIME NEWTON POSTULATED IS ACTUALLY APPROPRIATE TO ARISTOTLE'S LAWS, NOT HIS.

- NEWTON'S SECOND LAW: THE FORCE APPLIED TO AN OBJECT IS PROPORTIONAL TO IS MASS TIMES ACCELERATION:

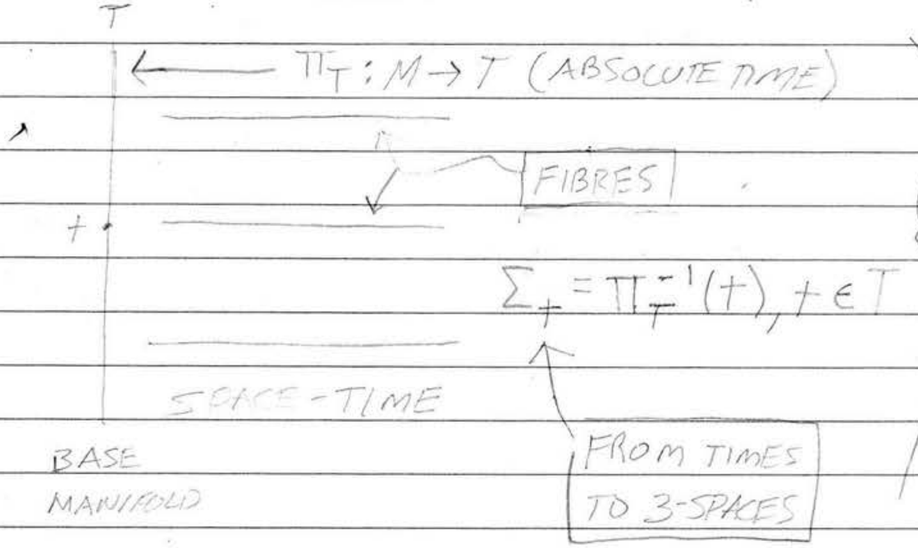
$$\underset{\sim}{F} = m\underset{\sim}{a}$$

  - NOTE: THIS LAW DOES NOT DEFINE ANY OF THE COMPONENT TERMS.

  - NOTE: AS WITH ARISTOTLE'S LAWS, THE FIRST IS NOT REDUNDANT EVEN THOUGH $\underset{\sim}{F}=0$ IMPLIES $\underset{\sim}{a}=0$. AGAIN, THE FIRST LAW DEFINES A PRIVILEGED CLASS OF FRAMES – THE INERTIAL FRAMES – WITH RESPECT TO WHICH $\underset{\sim}{F}=m\underset{\sim}{a}$.

- NEWTON ASSUMED THAT TIME WAS "ABSOLUTE" NOT JUST IN THAT ABSOLUTE SIMULTANEITY MAKES SENSE, BUT IN THAT TIME IS INDEPENDENT OF ALL MEASUREABLE PHYSICAL SYSTEMS.

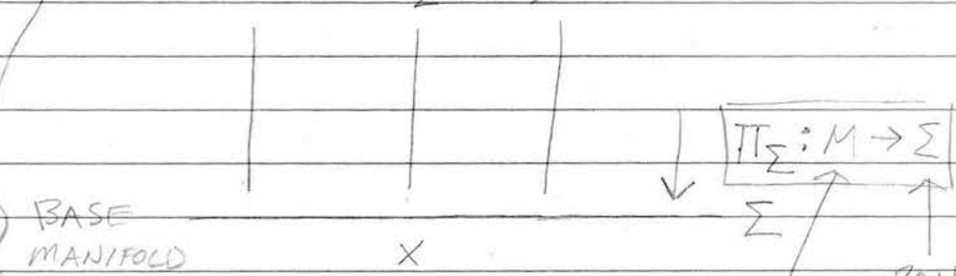
- NEWTON'S THEORY SUPPORTS A DIVISION OF THE SPACE-TIME MANIFOLD INTO ABSOLUTE TIME SLICES, AS DID ARISTOTLE'S.



- NOTE: UNLIKE ARISTOTELIAN SPACE, THERE IS NO PRIVILEGED PROJECTION. $\pi_\Sigma : T \to \Sigma$

$\pi_\Sigma^{-1}(x)$

NO NATURAL FIBER STRUCTURE WITH $\mathbb{R}^3$ AS BASE, BUT THERE IS ONE WITH $\mathbb{R}^1$ AS BASE



• NOTE: ANY INERTIAL PATH GIVES A PROJECTION $\pi_\Sigma : M \to \Sigma$ WHICH COUNTS ITS OBSERVER AT REST. BUT NEWTON'S THEORY, AS FORMULATED, REQUIRED A PRIVILEGED PROJECTION, LIKE ARISTOTLE'S.

• GRAVITY

• WE CAN DISTINGUISH CONCEPTUALLY BETWEEN "GRAVITATIONAL MASS", m_p, WHICH DETERMINES THE FORCE THE BODY FEELS IN A GRAVITATIONAL FIELD, FROM THE "INERTIAL MASS", m_i, THE RESPONSE A BODY FEELS TO A FORCE. SO WE CAN WRITE:

$$\boxed{m_p g = m_i a}$$

GRAVITATIONAL FIELD STRENGTH

FOR SUITABLE UNITS, $g = a$. SO, THE FORCE A BODY FEELS FROM GRAVITY IS INDEPENDENT OF ITS INERTIAL MASS.

• THIS IS GALILEO'S "EQUIVALENCE OF GRAVITATIONAL AND INERTIAL MASS". TO DETERMINE THE ACCELERATION DUE TO GRAVITY AND HOW IT IS FIXED, WE APPEAL TO NEWTON'S LAW OF UNIVERSAL GRAVITATION:

$$\boxed{\underset{\sim}{F} = -\frac{G m_1 m_2}{r^2} \frac{\underset{\sim}{r}}{r}}$$

DIRECTION OF FORCE

SEPARATION BETWEEN BODIES

· WE CAN REWRITE NEWTON'S LAW USING THE CONCEPT OF POTENTIAL, WHERE $\phi(\mathbf{r})$ IS THE POTENTIAL AT $\mathbf{r}$ FOR A GIVEN DISTRIBUTION OF GRAVITATING MASSES. THIS IS THE WORK DONE ON A UNIT MASS IN BRINGING IT FROM INFINITY TO $\mathbf{r}$. THE FORCE ON MASS M AT $\mathbf{r}$ IS:

$$\boxed{\mathbf{F} = -m\,\nabla\phi}$$

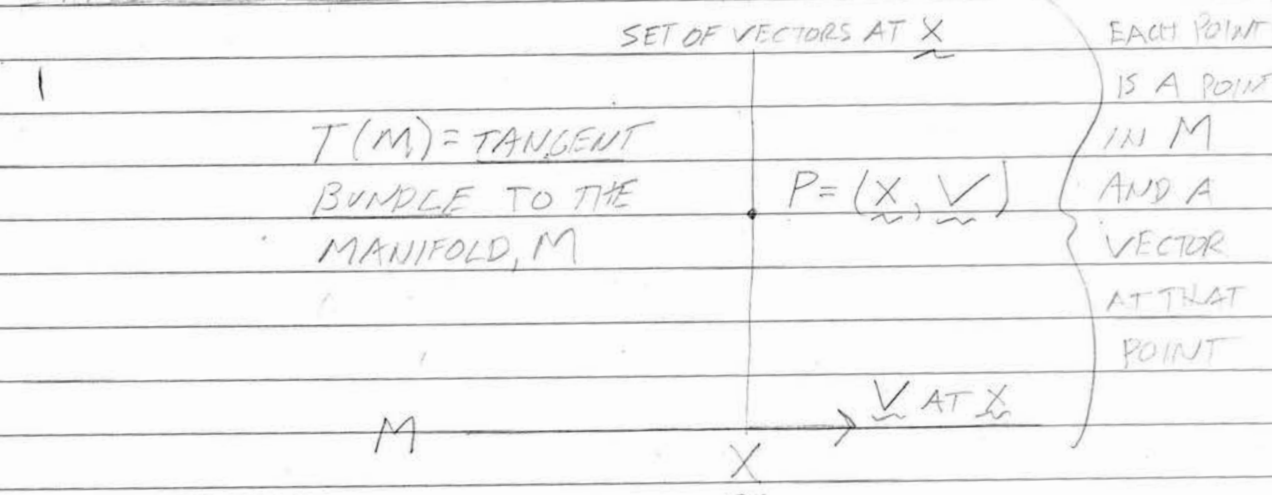
AND FOR A DISTRIBUTION OF MASS WITH WITH DENSITY $\rho(\mathbf{r})$:

$$\boxed{\phi(\mathbf{r}) = -\int \frac{G\rho(\mathbf{r}')\,d^3r'}{|\mathbf{r}-\mathbf{r}'|}}$$

OR IN MORE FAMILIAR FORM:

$$\boxed{\text{POISSON'S EQUATION: } \nabla^2\phi = -4\pi G\rho}$$

· THE LAGRANGIAN, L, IS A REAL-VALUED FUNCTION OF THE COORDINATES AND VELOCITIES OF THE OBJECTS IN THE SYSTEM. THE SPACE ON WHICH IT IS DEFINED - THAT OF ALL COORDINATES AND TANGENT VECTORS - IS CALLED THE "TANGENT BUNDLE" TO THE "CONFIGURATION SPACE."

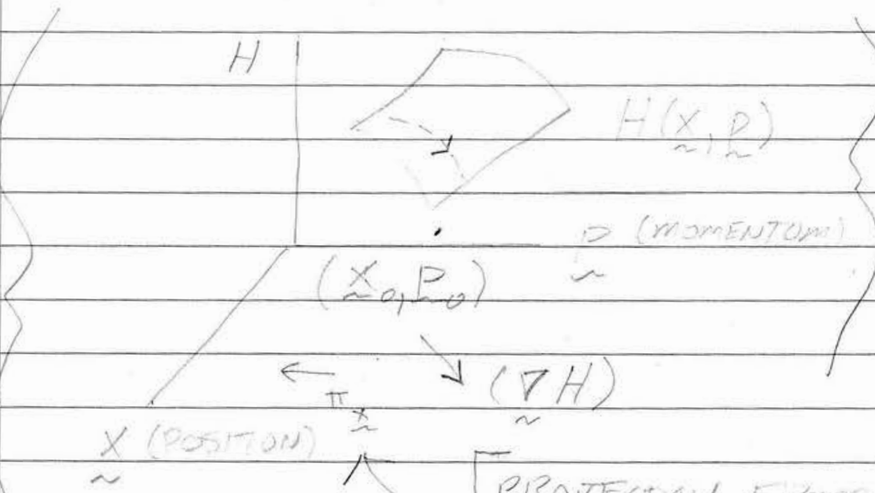


"TOTAL ENERGY" (IN TERMS OF COORDINATES & MOMENTA)

A.K.A. "PHASE SPACE"

- A RELATED FUNCTION IS THE HAMILTONIAN DEFINED ON THE SPACE OF COORDINATES AND MOMENTA, ALSO KNOWN AS THE "COTANGENT BUNDLE" TO THE CONFIGURATION SPACE. H GENERATES A VECTOR FIELD IN CONFIGURATION SPACE.

MOTION IN TERMS OF VECTOR FIELDS (∇H) IN PHASE SPACE $(\underset{\sim}{x}, \underset{\sim}{p})$



VECTOR FIELD REPRESENTS POSSIBLE PARTICLE MOTIONS

PROJECTION FROM COTANGENT BUNDLE (PHASE SPACE) INTO CONFIGURATION SPACE GIVES PARTICLE TRAJECTORIES IN SPACE

- WHEREAS ARISTOTLE'S KINEMATICS SPECIFIES MOTION VIA VECTOR FIELDS ON M, NEWTON'S - IN A MODERN FORMULATION - SPECIFIES MOTION VIA VECTOR FIELDS ON THE PHASE SPACE OF M.

ELLIS [1971]

- NOTE: CONTRARY TO POPULAR BELIEF A RIGOROUS CONTEMPORARY FORMULATION OF NEWTONIAN MECHANICS IS NOT DETERMINISTIC, SIMILAR AS WE WILL SEE TO GR.

- LEIBNIZ AND BERKELEY

- LEIBNIZ AND BERKELEY EACH OBJECTED TO NEWTON'S ABSOLUTE SPACE AND TIME. THE LATTER'S ARGUMENTS PREFIGURED THE VERIFICATIONIST ARGUMENTS OF THE LOGICAL POSITIVISTS, AND MACH.

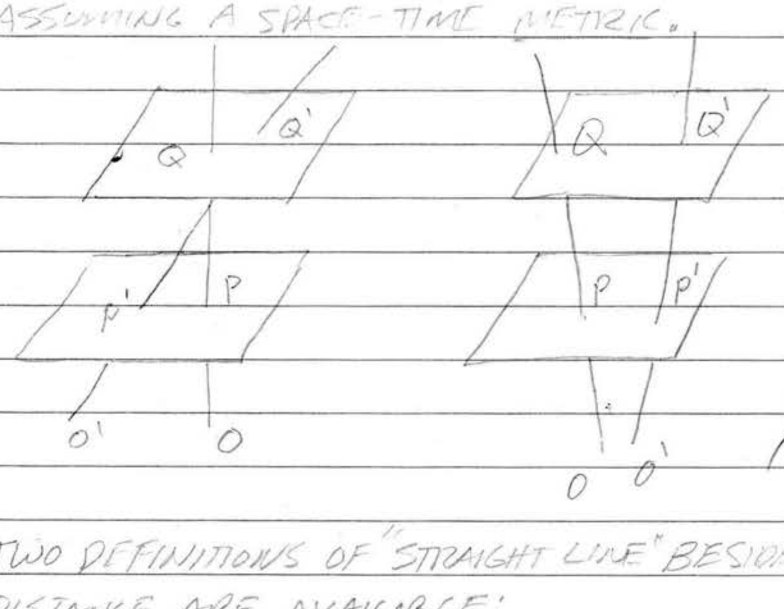
- A STRUCTURE APPROPRIATE TO THEIR "RELATIONALISM" DOES INVOLVE AN ABSOLUTE TIME (ALBEIT WITHOUT A METRIC). SINCE THEY NEED A NOTION OF OBJECTIVELY COEXISTENT EVENTS. THIS GIVES A PRIVILEGED PROJECTION Π_T.

- HOWEVER, THEIR RELATIONALISM IS PARTLY SUSTAINED BY DENYING AN ABSOLUTE SPACE PROJECTION, Π_Σ (WHICH WOULD DEFINE PREFERRED WORLD-LINES, $\Pi^{-1}(\underset{\sim}{x}),\ \underset{\sim}{x} \in \Sigma$).

- MODERN FORMULATION OF NEWTON'S THEORY (W/OUT GRAVITY)

  - WE HAVE SEEN THE NEED FOR A SPACE-TIME STRUCTURE INTERMEDIATE BETWEEN NEWTON'S ABSOLUTE SPACE AND LEIBNIZ'S AND BERKELEY'S RELATIONAL SPACE.

  - ABSENT AN ABSOLUTE SPACE, THE APPROPRIATE CONTEXT FOR DYNAMICS IS SPACE-TIME. THE PROBLEM IS TO DEFINE THE "STRAIGHT LINES" OF NEWTON'S FIRST LAW WITHOUT ASSUMING A SPACE-TIME METRIC.



- BY SLIDING THE TIME SLICES OVER EACH OTHER THE TIME SEPARATION $t(Q') - t(P')$ CAN BE MADE TO EQUAL $t(Q) - t(P)$, TRIVIALIZING THE NOTION OF SPACE-TIME DISTANCE.

- TWO DEFINITIONS OF "STRAIGHT LINE" BESIDES SHORTEST DISTANCE ARE AVAILABLE:

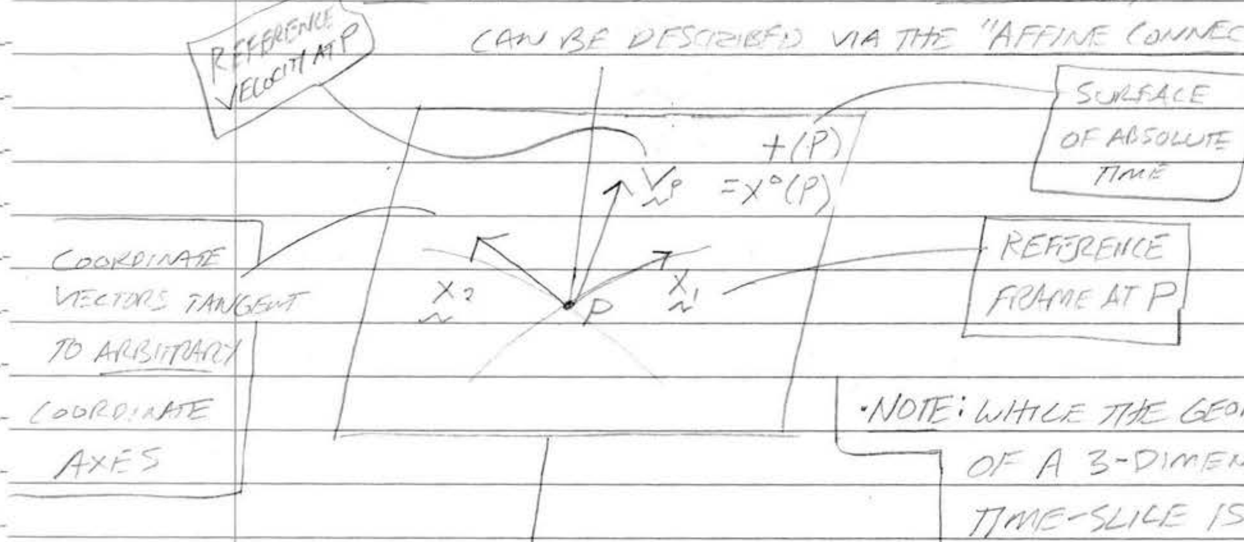
  (a) A PATH THE TANGENT VECTOR TO WHICH REMAINS PARALLEL TO ITSELF.

(b) A PATH WHICH IN CARTESIAN COORDINATES, X^i $(i=1,2,3)$, AND ABSOLUTE TIME, X^0, SATISFIES:

$$\boxed{\frac{d^2 X^\mu}{dt^2} = 0} \qquad (\mu = 0,1,2,3)$$

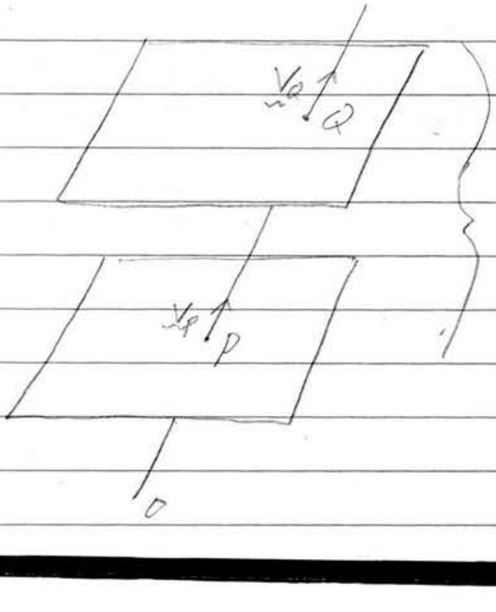
FOR SOME PARAMETER, t.

- NOTE: (a) AND (b) ARE ACTUALLY EQUIVALENT, AND CAN BE DESCRIBED VIA THE "AFFINE CONNECTION"



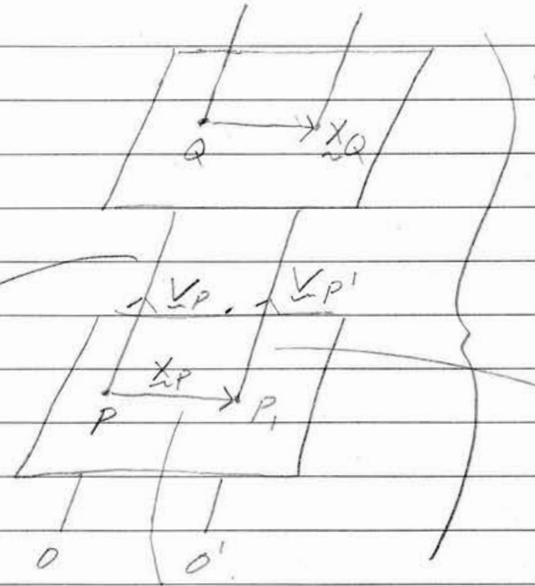
- NOTE: WHILE THE GEOMETRY OF A 3-DIMENSIONAL TIME-SLICE IS EUCLIDEAN (AND HENCE LETS US CONSTRUCT VECTORS ANYWHERE PARALLEL TO ANOTHER), THAT OF NEWTONIAN SPACE-TIME IS NOT.

- FOR THE SPECIAL CASE OF PATHS OF INERTIAL PARTICLES, IT IS EASY TO DEFINE PARALLEL VECTORS AT DIFFERENT POINTS IN TIME:



- V_Q IS TANGENT TO V_P BECAUSE IT IS TANGENT TO THE WORLDLINE OF AN INERTIAL OBSERVER PASSING THROUGH BOTH POINTS.

• WE NOW USE THE INERTIAL OBSERVER TO JUDGE OTHER VECTORS PARALLEL OR NOT.

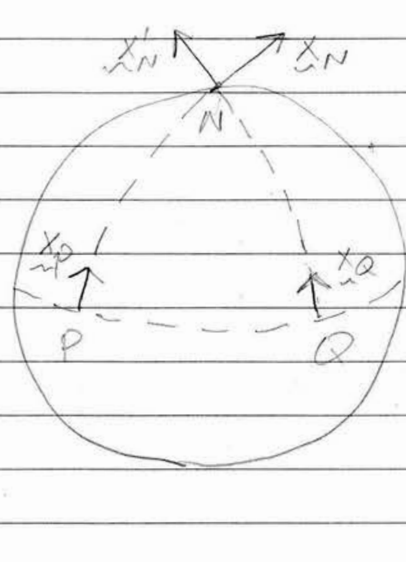


• IF O AND O' ARE INERTIAL OBSERVERS AT RELATIVE REST, THEN $\underset{\sim}{X}_Q$ IS PARALLEL TO X_{P_1}, SINCE $\underset{\sim}{V}_{P'}$ IS PARALLEL TO $\underset{\sim}{V}_P$.

WORLDLINE OF PARTICLE AT RELATIVE REST

DISTANCE AND DIRECTION OF PARTICLE

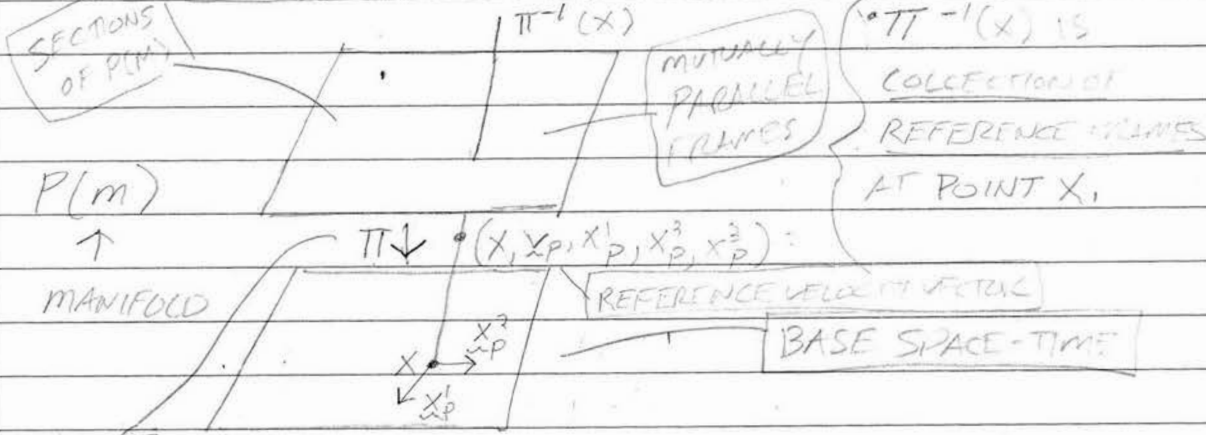
• NOTE: IF PARALLELISM IS TRANSITIVE, THEN WE SHOULD BE ABLE TO TALK ABOUT PARALLELISM AT DIFFERENT POINTS IN SPACETIME INDEPENDENT OF PATH. GIVEN NEWTON'S FIRST LAW, THIS IS INDEED THE CASE. THIS IS NOT THE CASE IN A SPACE-TIME FORMULATION OF NEWTONIAN GRAVITY, AS WE WILL SEE.



FAILURE OF TRANSITIVITY

• $\underset{\sim}{X}_P$ IS PARALLEL TO $\underset{\sim}{X}_Q$, AND $\underset{\sim}{X}_P$ IS PARALLEL TO $\underset{\sim}{X}_N$, AND $\underset{\sim}{X}_Q$ IS PARALLEL TO $\underset{\sim}{X}'_N$. HOWEVER, X'_N IS NOT PARALLEL TO $\underset{\sim}{X}_N$.

• THE GLOBAL STRUCTURE WE HAVE SKETCHED IS GIVEN BY THE PRINCIPAL BUNDLE, $P(M)$, THE BUNDLE OF FRAMES OVER SPACE-TIME, CONSISTING OF POINTS OF SPACE-TIME TOGETHER WITH A FRAME OF REFERENCE AT EACH POINT.



NATURAL PROJECTION

• NOTE: $P(M)$ IS ITSELF A DIFFERENTIABLE MANIFOLD, LETTING US SPEAK OF SMOOTH CURVES AND TANGENT VECTORS.

• $P(M)$ IS THE PRODUCT SPACE OF SPACE-TIME WITH THE SET OF ALL REFERENCE FRAMES AT A POINT IN SPACE-TIME. THIS PROVIDES GLOBAL CARTESIAN COORDINATES FOR M.

• UPSHOT: ALTHOUGH, CONTRA NEWTON, HIS SPACE-TIME IS NOT ITSELF A PRODUCT SPACE, THE BUNDLE OF FRAMES OF IT IS A PRODUCT SPACE.

NEWTON'S FIRST LAW

• AFFINE CONNECTION

• IF V_P IS A VECTOR IN AN ARBITRARY COORDINATE PATCH, THE COORDINATES OF A NEIGHBORING PARALLEL VECTOR $\underset{\sim}{V}_Q$ IN THE SAME COORDINATES WILL GENERALLY DIVERGE. IF V^α ARE THOSE COMPONENTS, THE DIVERGENCE, δV^α, IS GIVEN BY:

AFFINE CONNECTION

$$\boxed{\delta V^{\alpha} = -\sum_{\beta,\gamma} \Gamma^{\alpha}_{\beta\gamma} V^{\beta} \delta x^{\gamma}}$$

OR, USING EINSTEIN SUMMATION CONVENTION:

$$\boxed{\delta V^{\alpha} = -\Gamma^{\alpha}_{\beta\gamma} V^{\beta} \delta x^{\gamma}}$$

- UPSHOT: SPECIFYING PARALLEL VECTORS IS THE SAME AS SPECIFYING THE COMPONENTS OF AN AFFINE CONNECTION.

- LETTING V^{α} BE A VELOCITY VECTOR, WE GET:

$$\boxed{\delta\left(\frac{dx^{\alpha}}{dt}\right) = -\Gamma^{\alpha}_{\beta\gamma} \frac{dx^{\beta}}{dt} \delta x^{\gamma}}$$

WHICH IS TO SAY:

$$\boxed{\frac{d^2 x^{\alpha}}{dt^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dx^{\beta}}{dt} \frac{dx^{\gamma}}{dt} = 0}$$

- THIS IS THE EQUATION FOR GEODESIC CURVES, $x^{\alpha}(t)$.

- NOTE: IN A FRAME, (y^{α}), IN WHICH THE COMPONENTS OF Γ VANISH, THE EQUATION BECOMES:

$$\boxed{\frac{d^2 y^{\alpha}}{dt^2} = 0}$$

WHICH IS JUST THE EQUATION FROM (b) FOR AN ORDINARY STRAIGHT LINE. SO, GEODESICS ARE SIMPLY INERTIAL MOTIONS.

• FINALLY, IF $y^\alpha = y^\alpha(x^\beta)$, THEN:

$$\frac{dy^\alpha}{dt} = \frac{\partial y^\alpha}{\partial x^\beta}\frac{dx^\beta}{dt}$$

SO,

$$\boxed{\frac{d^2 y^\alpha}{dt^2} = 0 = \frac{\partial y^\alpha}{\partial x^\beta}\frac{d^2 x^\beta}{dt^2} + \frac{\partial^2 y^\alpha}{\partial x^\gamma \partial x^\beta}\frac{dx^\gamma}{dt}\frac{dx^\beta}{dt}}$$

• USING:

$$\frac{\partial y^\alpha \partial x^\gamma}{\partial x^\beta \partial y^\alpha} = 1 \text{ IF } \beta = \gamma \text{ AND } 0 \text{ IF } \beta \neq \gamma.$$

WE GET THE FOLLOWING EXPRESSION FOR Γ:

$$\boxed{\Gamma^\alpha_{\beta\gamma} = \frac{\partial x^\alpha}{\partial y^\delta}\frac{\partial^2 y^\delta}{\partial x^\gamma \partial x^\beta}}$$

• NOTE: WHEN THE SPACE IN QUESTION HAS A METRIC, $\Gamma^\alpha_{\beta\gamma}$ CAN BE WRITTEN IN TERMS OF IT. HOWEVER, NEWTONIAN SPACE-TIME LACKS A METRIC (THOUGH ITS SPACE AND TIME INDIVIDUALLY HAVE METRICS).

• SINCE PHYSICAL REALITY IS INDEPENDENT OF HOW WE COORDINATE IT, ANY SUCH THEORY SHOULD BE GENERALLY COVARIANT. WHEN A THEORY SEEMS TO REQUIRE A PRIVILEGED SET OF COORDINATES (e.g. NEWTON'S SECOND LAW), IT INDICATES ADDITIONAL UNDERLYING GEOMETRICAL STRUCTURE WHICH CAN BE MADE EXPLICIT.

- HOWEVER, CERTAIN CLASSES OF COORDINATES TRANSFORMATIONS REPRESENT SYMMETRIES. IN THE CASE OF NEWTON'S FIRST LAW, WE COULD TRANSFORM ONLY BETWEEN COORDINATE SYSTEMS WITH VANISHING AFFINE CONNECTIONS. THESE WOULD AGREE ABOUT THE INERTIAL PATHS.

- SUCH TRANSFORMATIONS TAKE ONE SET OF COORDINATES THAT DISPLAY THE PRODUCT STRUCTURE OF THE BUNDLE TO ANOTHER. THE CORRESPONDING SYMMETRY IS THE INVARIANCE OF NEWTON'S LAWS UNDER GALILEAN TRANSFORMATION (CONSTANT VELOCITY BOOSTS).

- NOTE: THE NOTION OF A PHYSICAL SYMMETRY IS ONLY AS CLEAR AS THE DISTINCTION BETWEEN LAWS AND ACCIDENTAL GENERALIZATIONS.

- MODERN FORMULATION OF NEWTONIAN GRAVITY

- GRAVITY IS EXCEPTIONAL AMONG NATURE'S FORCES IN THAT IT IS APPARENTLY UNIVERSAL AND APPARENTLY AFFECTS ALL BODIES EQUALLY, INDEPENDENT OF THEIR NATURE.

- THE LATTER INSPIRES THE "UNIVERSALITY OF FREE FALL" POSTULATE, ACCORDING TO WHICH THE SPACE-TIME TRAJECTORY OF A SMALL BODY SUBJECT TO ONLY GRAVITATIONAL FORCE IS THE SAME AS THAT OF ANY NEIGHBORING ONE WITH THE SAME INITIAL VELOCITY. EINSTEIN CALLED THIS THE "WEAK PRINCIPLE OF EQUIVALENCE".

- QUESTION: GIVEN GRAVITY'S EXCEPTIONAL CHARACTERISTICS, CAN WE DETERMINE A LOCAL INERTIAL FRAME BY LOCAL DYNAMICAL OBSERVATIONS?

- PROBLEM: ANY LOCAL DYNAMICAL EXPERIMENT IS CONSISTENT WITH THE HYPOTHESIS THAT WE ARE IN AN A NON-INERTIAL REFERENCE FRAME ACCELERATING UPWARD AT g!

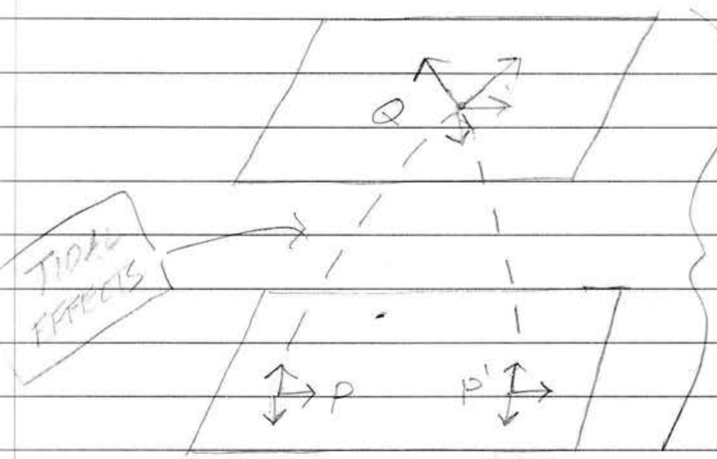
- IT WOULD SEEM THAT LOCALLY GRAVITY AND ACCELERATION ARE INDISTINGUISHABLE DYNAMICALLY. HENCE, IN THE PRESENCE OF GRAVITY, THERE IS NO LOCAL DYNAMICAL DISTINCTION BETWEEN INERTIAL AND NON-INERTIAL FRAMES. THIS SUGGESTS THAT A CORRECT THEORY OF GRAVITY WILL PRIVILEGE NO SPECIAL CLASS OF FRAMES.

- HOWEVER, THIS ALL ASSUMES THAT GRAVITY IS A FORCE LIKE ANY OTHER. WE MIGHT INSTEAD TAKE "FREE FALL" TO BE THE PRIVILEGED STATE OF MOTION. ONLY FREE FALLERS SEE "GRAVITY DISAPPEAR".

- UPSHOT: GIVEN A DISTRIBUTION OF GRAVITATING OBJECTS THERE IS AN OBJECTIVE DISTINCTION BETWEEN PATHS THAT CANNOT BE FOLLOWED ABSENT NON-GRAVITATIONAL FORCES, AND THOSE THAT CAN. IT IS THE LATTER THAT ARE NOW PRIVILEGED.

- PROBLEM: FREE FALL FRAMES DO NOT GIVE RISE TO A HORIZONTAL PROJECTION OF THE FRAME BUNDLE (AS INERTIAL FRAMES DID IN NEWTONIAN PHYSICS WITHOUT GRAVITY). SO, THE BUNDLE IS NO LONGER A PRODUCT SPACE.

  - IN A GRAVITATIONAL FIELD, THERE DOES NOT EXIST A COORDINATE TRANSFORMATION MAKING ALL FREE FALL TRAJECTORIES STRAIGHT.



- FRAMES P AND P' ARE EACH PARALLEL TO FRAMES AT Q. BUT THOSE FRAMES AT Q ARE NOT PARALLEL TO EACH OTHER.

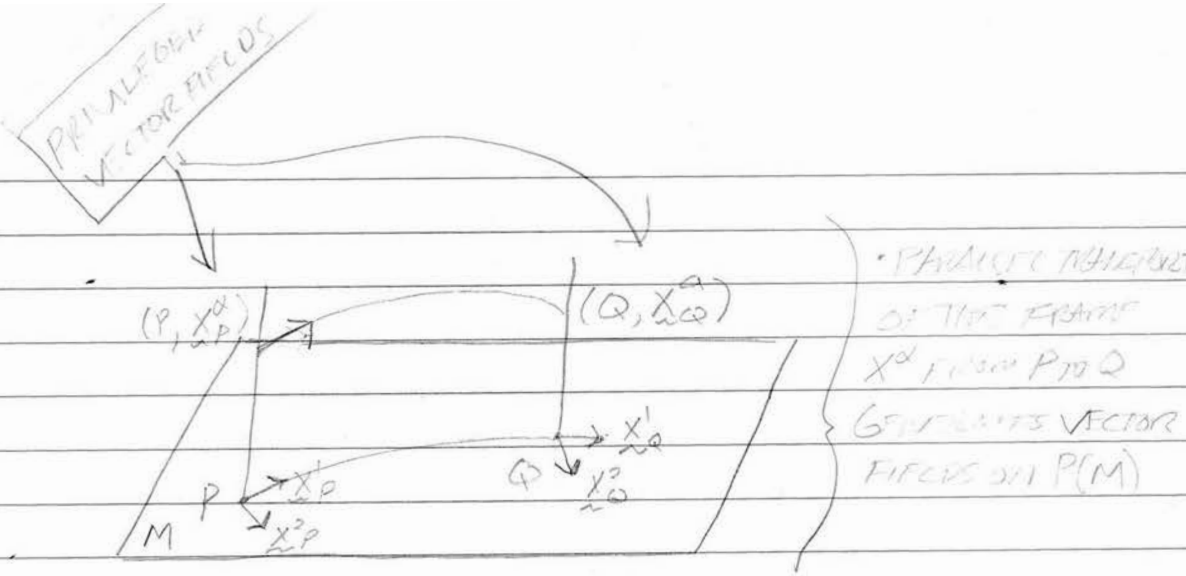
- NOTE: THE PROBLEM STEMS FROM THE NON-UNIFORMITY OF THE GRAVITATIONAL FIELD (HENCE THE RESTRICTION OF THE WEAK EQUIVALENCE PRINCIPLE TO "LOCAL" REGIONS).

- IN LIGHT OF THESE COMPLICATIONS, WE NOW DEFINE THE PARALLEL TRANSPORT OF A VECTOR ALONG A FREE FALL TRAJECTORY FROM P TO Q:

$$\underbrace{A^{\mu}(P) - A^{\mu}(Q)}_{\text{CHANGE IN COMPONENTS OF VECTOR IN SOME COORDINATE SYSTEM}} = \Gamma^{\mu}_{\rho\sigma} A^{\rho}(P) \underbrace{\delta x^{\sigma}}_{\text{INFINITESIMAL COORDINATE DIFFERENCE BETWEEN P AND Q.}}$$

- NOTE! THERE IS NOW NO COORDINATE SYSTEM IN WHICH THE Γ's VANISH EVERYWHERE (SINCE OTHERWISE WE WOULD AGAIN HAVE TRANSITIVITY OF PARALLELISM).

- THE SPECIFICATION OF AN AFFINE CONNECTION IS EQUIVALENT TO THE SPECIFICATION OF FOUR HORIZONTAL VECTOR FIELDS ON P(M):



· SUMMARY: IN PASSING FROM ARISTOTELIAN SPACE-TIME TO NEWTONIAN (GALILEAN), WE GIVE UP THE PRODUCT STRUCTURE ON THE MANIFOLD, AND IN PASSING FROM THE LATTER TO NEWTONIAN GRAVITATION THEORY WE GIVE UP THE PRODUCT STRUCTURE ON THE FRAME BUNDLE.

· RELATIVE TO A FREELY FALLING FRAME, THE NEWTONIAN INERTIAL TRAJECTORY (e.g. OF AN APPLE ON A TREE) IS GIVEN BY:

$$\frac{d^2 y^{\alpha}}{dt^2} = \frac{\partial \phi}{\partial y^{\alpha}}$$

GRAVITATIONAL POTENTIAL OF POISSON'S EQUATION

RELATIVE TO THE NEWTONIAN FRAME, THE FREE FALL FRAME DUE TO GRAVITY IS GIVEN BY:

$$\frac{d^2 x^{\alpha}}{dt^2} + \frac{\partial \phi}{\partial x^{\alpha}} = 0$$

· USING THE EQUATION FOR FREE FALL, WE OBTAIN:

$$\Gamma^{\alpha}_{00} = \frac{\partial \phi}{\partial x^{\alpha}},$$ WITH ALL OTHER COMPONENTS EQUAL TO 0.

• NOTE! THERE ARE MANY FRAMES IN WHICH Γ HAS THIS FORM.

• QUESTION: HOW DO WE DISTINGUISH THE TIDAL EFFECTS OF "TRUE" GRAVITY (AS OPPOSED TO CURVED COORDINATES)?

• ANSWER: THE CURVATURE TENSOR, R.

• USING COORDINATES IN WHICH A FREE FALL OBSERVER HAS COORDINATES $x^i = 0$, AND THOSE OF A NEIGHBORING FREE FALLER HAVE COORDINATES $(\delta x^i(t), t)$, WE REQUIRE:

BY TAYLOR'S THEOREM

$$0 = \frac{d^2 \delta x^i}{dt^2} + \Gamma^i_{00}(\delta x^k) = \frac{d^2 \delta x^i}{dt^2} + \underbrace{\delta x^k \left[\frac{\partial \Gamma^i_{00}}{\partial x^k}\right]_{\underline{x}=0}}_{\text{SUMMED OVER } k}$$

• WRITING ε^i FOR THE FINITE GEODESIC SEPARATION VECTOR PARALLEL TO δx^i, AND R^i_{0k0} FOR $\partial_k \Gamma^i_{00}$, WE GET:

• GEODESIC DEVIATION EQUATION: $\boxed{0 = \ddot{\varepsilon}^i + R^i_{0k0}\varepsilon^k}$
(FOR NEWTONIAN THEORY)

WHICH IS UNCHANGED UNDER THE COORDINATE TRANSFORMATION:

PRESERVES EUCLIDEAN SPACE METRIC

$$z^i = \sum_j R^i_j x^j + a^i(t), \quad z^0 = x^0 = ct, \quad \phi \to \phi + \sum \frac{d^2 a^i}{dt^2} x^i$$

R^i_j: CONSTANT RIGID ROTATION MATRIX

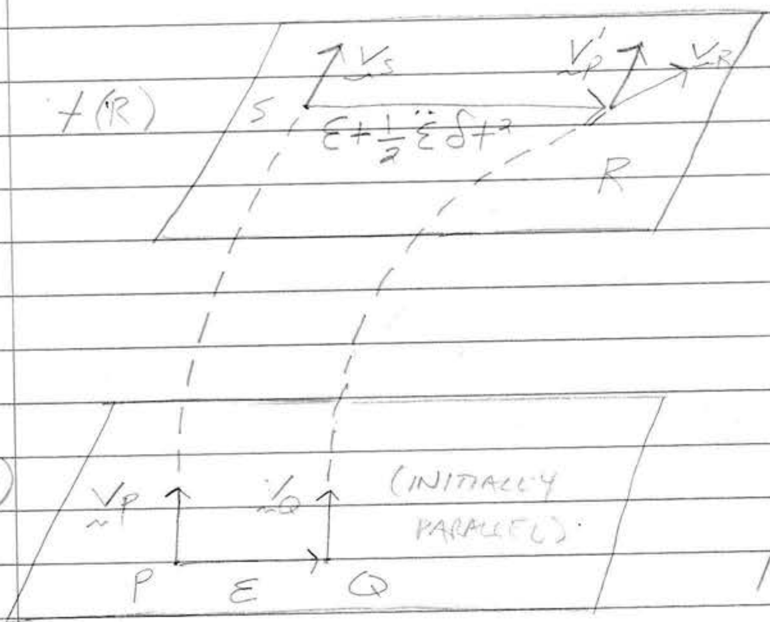
$a^i(t)$: ARBITRARY FUNCTIONS

• NOTE: IN A FRAME WHERE $\Gamma^i_{00} = \frac{\partial \phi}{\partial x^i}$ WITH OTHER COMPONENTS 0,

$$R^{i}_{\;0k0} = \frac{\partial^2 \phi}{\partial x_i}\,\partial x^k$$

WHERE, R, IS THE CURVATURE OF SPACE-TIME.

COMPUTE PARALLEL TRANSPORT OF V_P AROUND PQRSP



• PS AND QR ARE FREE-FALL TRAJECTORIES WITH TANGENT VECTORS V_P, V_S, V_Q, V_R. V_P' IS OBTAINED FROM V_P BY PARALLEL TRANSPORT ALONG PSR

DIFFERENCE BETWEEN VELOCITIES V_P AND V_Q (IF $\ddot{\varepsilon} = 0$, V_P AND V_Q ARE PARALLEL)

R MEASURES CHANGE IN VECTOR THRU PARALLEL TRANSPORT

$(\delta\ddot{\varepsilon})\,\delta t$ MEASURES CHANGE IN INITIALLY PARALLEL VECTORS, i.e., $V_R - V_P' = V_R - V_S$.

• THE CHANGE IN V_P AROUND AN INFINITESIMAL CURVE IS GIVEN BY:

$$\Delta V_P^{\lambda} = (V_R - V_P')^{\lambda} = -\frac{1}{2} R^{\lambda}_{\;\mu\nu\rho} V^{\mu} (\delta A)^{\nu\rho}$$

• UPSHOT: THE SPACETIME OF NEWTONIAN GRAVITATION — i.e., ITS AFFINE GEOMETRY — IS CURVED.

• WE CAN NOW REWRITE POISON'S EQUATION IN MANIFESTLY COVARIANT FORM AS FOLLOWS:

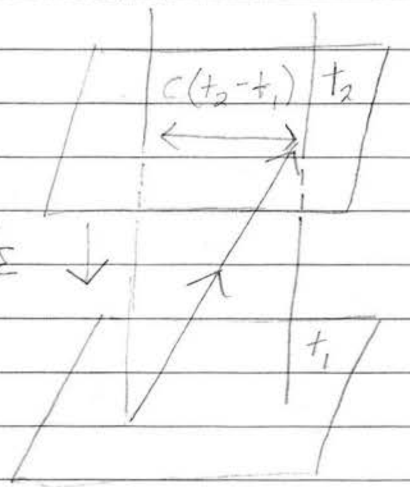
$$R'_{uiv} = \frac{-4\pi G}{c^2 \rho} V_u V_v$$

- UPSHOT: THIS RESULTS IN A COVARIANT FORMULATION OF NEWTONIAN GRAVITY (WHICH IS NOT STRICTLY EQUIVALENT TO THE ORIGINAL BUT IS OPERATIONALLY SO).

- MINKOWSKI SPACETIME

  - IN ORDER TO CONSISTENTLY ACCOUNT FOR ELECTROMAGNETIC PHENOMENA ALONG WITH DYNAMICS, THE NEWTONIAN SPACETIME STRUCTURE MUST BE MODIFIED.

  - NEWTON'S THEORY IN TANDEM WITH MAXWELL'S FIX A SLICING OF SPACETIME, REINSTATING ABSOLUTE SPACE:



- AGAINST THIS, EINSTEIN GENERALIZED GALILEAN RELATIVITY FROM MECHANICAL PHENOMENA TO ALL PHENOMENA.

- PRINCIPLE OF RELATIVITY: NO LOCAL PHYSICAL EXPERIMENT CAN DISTINGUISH REST FROM UNIFORM MOTION.

- IN ORDER TO LEAVE MAXWELL'S EQUATIONS UNCHANGED BETWEEN INERTIAL OBSERVERS, WE USE THE LORENTZ TRANSFORMATIONS:

"MIX" TIME AND SPACE

VELOCITY ALONG X-AXIS WITH $t=0=t'$

$$\boxed{x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \boxed{t' = \frac{t - vx/c^2}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \boxed{y'=y} \quad \boxed{z'=z}$$

$$\Leftarrow \boxed{\gamma = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}}}$$

- NOTE: EVENTS IN THE PRIMED FRAME THAT ARE ASSIGNED THE SAME TIME, $t'=0$, DO NOT OCCUR AT THE SAME TIME IN THE UNPRIMED FRAME. THIS IS THE RELATIVITY OF SIMULTANEITY.

- UPSHOT: ANY INERTIAL OBSERVER HAS THEIR OWN TIME Π_T AND SPACE Π_Σ PROJECTIONS.

- THIS GIVES THE INVARIANT RELATION:

$$\boxed{c^2(\Delta\tau)^2 = c^2(\Delta t)^2 - (\Delta x)^2}$$

PROPER TIME: TIME MEASURED BY A CLOCK MOVING INERTIALLY BETWEEN SPACETIME POINTS, ALONG X-AXIS.

FOR PROPER LENGTH, WE NEED $\Delta\tau = 0$ AND $\Delta\underset{\sim}{\varepsilon} \neq 0$: $(\Delta\underset{\sim}{\varepsilon})^2 = -c^2(\Delta t)^2 + (\Delta\underset{\sim}{x})^2$

- SO, MORE GENERALLY, THE PROPER TIME INTERVAL IS:

$$\boxed{c^2(\Delta\tau)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2}$$

- IF A CLOCK TRACES OUT A CURVE IN SPACETIME PARAMETERIZED BY λ (WHICH MAY OR MAY NOT BE PROPER TIME), THEN THE TANGENT VECTOR OF THE CURVE AT POINT P IS $(dt/d\lambda, d\underset{\sim}{x}/d\lambda)$ AT VALUE λ_P IN THE $(t, \underset{\sim}{x})$ COORDINATE SYSTEM, AND $(d\tau/d\lambda)$ IN THAT OF THE CLOCK. HENCE:

$$-c^2\left(\frac{d\tau}{d\lambda}\right)^2 = -c^2\left(\frac{dt}{d\lambda}\right) + \left(\frac{dx}{d\lambda}\right)^2$$

EVALUATED AT P IS THE NEGATIVE OF THE SQUARE OF THE TANGENT VECTOR'S "LENGTH" AT P.

- IN GENERAL, FOR ANY VECTOR AT A POINT WITH COMPONENTS (V^0, V^1, V^2, V^3), THE SQUARE OF ITS LENGTH IS:

$$\left| -c^2 (V^0)^2 + (V^1)^2 + (V^2)^2 + (V^3)^2 \right|$$

THIS AMOUNTS TO IMPOSING A METRIC ON SPACETIME.

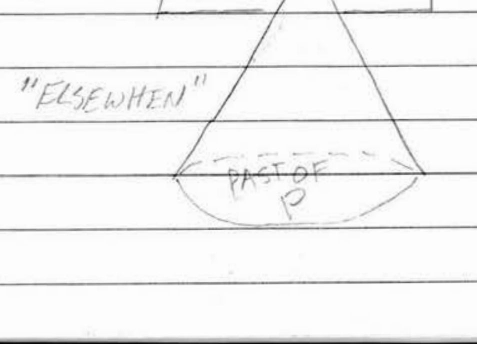
- NOTE: $\sqrt{\left| -c^2 (dt/d\lambda)^2 + (d\underset{\wedge}{x}/d\lambda)^2 \right|}\, d\lambda$ IS THUS THE DISTANCE BETWEEN $P(\lambda)$ AND $Q(\lambda + d\lambda)$. INTEGRATING ALONG THE CURVE THEN GIVES THE "LENGTH".

- IN ORDER TO SAY WHAT COUNTS AS "STRAIGHT LINES" IN THIS GEOMETRY, AND, HENCE, INERTIAL MOTION WE NEED TO SPECIFY AN AFFINE CONNECTION.

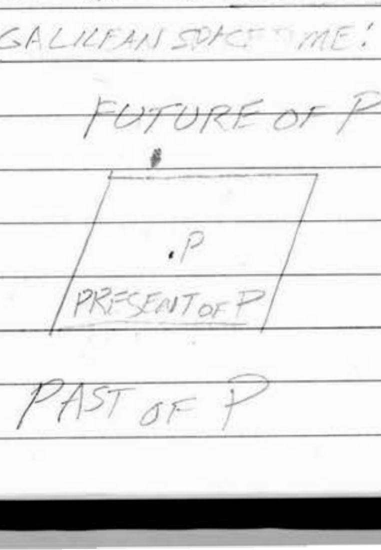
- FORTUITOUSLY, THE AFFINE CONNECTION THAT FOLLOWS NATURALLY FROM THE MINKOWSKI METRIC IS APPROPRIATE TO THE FIRST LAW OF RELATIVISTIC MECHANICS.

IF SPEED OF LIGHT WERE INFINITE

CAUSAL STRUCTURE IN SPECIAL RELATIVITY (SR)



- CONTRAST THIS STRUCTURE WITH THAT OF NEWTONIAN / GALILEAN SPACETIME:



• NOTE: WHILE THE CAUSAL STRUCTURE OF CLASSICAL DYNAMICS IS INDEPENDENT OF ITS METRIC STRUCTURE, THAT OF RELATIVISTIC DYNAMICS IS DERIVED FROM IT.

• NOTE: WHILE SR IMPLIES THAT NO BODY CAN BE ACCELERATED TO A SUPERLUMINAL SPEED, IT DOES NOT ASSERT THAT NOTHING CAN TRAVEL FASTER THAN LIGHT.

• THE ORDER OF TIMELIKE SEPARATED IS INVARIANT FOR ALL INERTIAL OBSERVERS CONNECTED BY LORENTZ TRANSFORMATIONS. THIS MARKS A SPECIAL ROLE FOR LIGHT SIGNALS.

• SPECIAL RELATIVITY STANDS TO NEWTONIAN THEORY WITHOUT GRAVITY AS GENERAL RELATIVITY STANDS TO NEWTONIAN THEORY WITH GRAVITY.

• GENERAL RELATIVISTIC SPACETIME

• SCHEMATICALLY WE CAN SAY:

$$\boxed{\text{SPECIAL RELATIVITY}} = \left\{ \begin{array}{l} \text{NEWTONIAN DYNAMICS MODIFIED} \\ \text{TO SATISFY MAXWELL'S ELECTRO-} \\ \text{MAGNETIC THEORY} \end{array} \right.$$

$$\boxed{\text{GENERAL RELATIVITY}} = \left\{ \begin{array}{l} \text{SPECIAL RELATIVITY MODIFIED} \\ \text{TO SATISFY NEWTON'S THEORY} \\ \text{OF GRAVITY} \end{array} \right.$$

• FROM THIS PERSPECTIVE, WE NEED ONLY MODIFY THE NEWTONIAN THEORY OF GRAVITY SO AS TO COHERE WITH SPECIAL RELATIVITY.

• RECALL THAT IN SR, AS IN NEWTONIAN DYNAMICS, BODIES SUBJECT TO NO FORCES FOLLOW STRAIGHT TRAJECTORIES. IN INERTIAL COORDINATE SYSTEMS SUCH PATHS ARE SOLUTIONS TO THE EQUATIONS:

$$\boxed{\frac{d^2 x^\mu}{ds^2} = 0} \quad (v = ct, \underset{\sim}{x})$$

COMPONENTS OF AFFINE CONNECTION VANISH, AS IN NEWTON'S THEORY

CHANGE IN PROPER TIME ALONG PATH

• IN ARBITRARY COORDINATES, WE HAVE:

$$\boxed{\frac{d^2 y^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{dy^\nu}{ds}\frac{dy^\rho}{ds} = 0}$$

WHERE TRANSFORMING FROM X TO Y COORDINATES, WE FIND:

$$\boxed{\Gamma^\eta_{\mu\nu} = \frac{\partial y^\eta}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial y^\mu \partial y^\nu}}$$

← LIKE THE NON-RELATIVISTIC CASE

• ASSUMING THE WEAK PRINCIPLE OF EQUIVALENCE (i.e., THE UNIVERSALITY OF FREE FALL), GRAVITY MUST BE DESCRIBED BY AN AFFINE CONNECTION, AS IN THE CLASSICAL THEORY.

• HOWEVER, THE UNIFICATION OF METRICAL AND AFFINE STRUCTURES CHARACTERISTIC OF SR IS STRENGTHENED BY:

• STRONG PRINCIPLE OF EQUIVALENCE: ALL NON-GRAVITATIONAL LAWS OF PHYSICS ARE THE SAME IN EVERY LOCAL FREELY FALLING FRAME (i.e., A FRAME IS FREELY-FALLING IF/F IT IS A LOCAL LORENTZ FRAME).

THIS LEADS TO A CURVED SPACETIME THEORY OF GRAVITY.