1 Non-stationary Poisson processes and Compound (batch) Poisson processes

Assuming that a Poisson process has a fixed and constant rate λ over all time limits its applicability. (This is known as a time-stationary or time-homogenous Poisson process, or just simply a stationary Poisson process.) For example, during rush hours, the arrivals/departures of vehicles into/out of Manhattan is at a higher rate than at (say) 2:00AM. To accommodate this, we can allow the rate $\lambda = \lambda(t)$ to be a deterministic function of time t. For example, consider time in hours and suppose $\lambda(t) = 100$ per hour except during the time interval (morning rush hour) $(8, 9)$ when $\lambda(t) = 200$, that is

$\lambda(t) = 200,\ t \in (8, 9),\ \lambda(t) = 100,\ t \notin (8, 9).$

In such a case, for a given rate function $\lambda(s)$, the expected number of arrivals by time t is thus given by

$$m(t) \stackrel{\text{def}}{=} E(N(t)) = \int_0^t \lambda(s)ds. \tag{1}$$

For a compound such process such as buses arriving: If independently each bus holds a random number of passengers (generically denoted by B) with some probability mass function $P(k) = P(B = k),\ k \geq 0$, and mean $E(B)$. Letting $B_1, B_2, \ldots$ denote the iid sequential bus sizes, the number of passengers to arrive by time t, $X(t)$ is given by

$$X(t) = \sum_{n=1}^{N(t)} B_n, \tag{2}$$

where $N(t)$ is the counting process for the non-stationary Poisson process; $N(t) =$ the number of buses to arrive by time t. This is known as a *compound* or *batch* non-stationary Poisson arrival process. We have $E(X(t)) = E(N(t))E(B) = m(t)E(B)$.

We have already learned how to simulate a stationary Poisson process up to any desired time t, and next we will learn how to do so for a non-stationary Poisson process.

1.1 The non-stationary case: Thinning

In general the function $\lambda(t)$ is called the *intensity* of the Poisson process, and the following holds:

For each $t > 0$, the counting random variable $N(t)$ is Poisson distributed with mean

$$m(t) = \int_0^t \lambda(s)ds.$$

$$\begin{aligned} E(N(t)) &= m(t) \\ P(N(t) = k) &= e^{-m(t)} \frac{m(t)^k}{k!},\ k \geq 0. \end{aligned}$$

More generally, the increment $N(t+h) - N(t)$ has a Poisson distribution with mean $m(t+h) - m(t) = \int_t^{t+h} \lambda(s)ds$.

Furthermore, $\{N(t)\}$ has independent increments;

If $0 \le a < b < c < d$, then $N(b) - N(a)$ is independent of $N(d) - N(c)$.

We shall assume that the intensity function is bounded from above: There exists a $\lambda^* > 0$ such that

$$\lambda(t) \le \lambda^*, \; t \ge 0.$$

(In practice, we would want to use the smallest such upper bound.) (Note that if $\lambda(t)$ was unbounded, then with very mild further assumptions (such as continuity, etc.), it would be bounded over any finite time interval $(0, T)$ and hence our method could be used over any finite time interval anyhow.)

Then the simulation of the Poisson process is accomplished by a "thinning" method: First simulate a stationary Poisson process at rate λ^*. For example, sequentially generate iid exponential rate λ^* interarrival times and use the recursion $v_{n+1} = v_n + [-(1/\lambda^*)\ln(U_{n+1})]$, to obtain the arrival times which we are denoting by v_n. The rate λ^* is larger than needed for our actual process, so for each arrival time v_n, we independently flip a coin to decide wether to keep it or reject it. The sequence of accepted times we denote by $\{t_n\}$ and forms our desired non-stationary Poisson process. To make this precise: for each arrival time v_n, we accept it with probability $p_n = \lambda(v_n)/\lambda^*$, and reject it with probability $1 - p_n$. Thus for each v_n we generate a uniform U_n and if $U_n \le p_n$ we accept v_n as a point, otherwise we reject it.

The thinning algorithm for simulating a non-stationary Poisson process with intensity $\lambda(t)$ that is bounded by λ^*

Here is the algorithm for generating our non-stationary Poisson process up to a desired time T to get the $N(T)$ arrival times $t_1, \dots t_{N(T)}$.

1. $t = 0$, $N = 0$
2. Generate a U
3. $t = t + [-(1/\lambda^*)\ln(U)]$. If $t > T$, then stop.
4. Generate a U.
5. If $U \le \lambda(t)/\lambda^*$, then set $N = N + 1$ and set $t_N = t$.
6. Go back to 2.

Note that when the algorithm stops, the value of N is $N(T)$ and we have sequentially simulated all the desired arrival times $t_1, t_2 \dots$ up to $t_{N(T)}$.

Here is a proof that this thinning algorithm works:

Proof :[Thinning algorithm] Let $\{M(t)\}$ denote the counting process of the rate λ^* Poisson process. First note that $\{N(t)\}$ has independent increments since $\{M(t)\}$ does and the thinning is done independently. So what is left to prove is that for each $t > 0$, $N(t)$ constructed by this thinning has a Poisson distribution with mean $m(t) = \int_0^t \lambda(s)ds$. We know that for each $t > 0$, $M(t)$ has a Poisson distribution with mean $\lambda^* t$. We will partition $M(t)$ into $N(t)$ (the accepted ones), and $R(t)$ (the rejected ones), and conclude that $N(t)$ has the desired Poisson distribution. To this end recall that conditional on $M(t) = n$, we can treat the n unordered arrival times as iid unif $(0, t)$ rvs. Thus a typical arrival, denoted by $V \sim Unif(0, t)$, will be accepted with conditional probability $\lambda(v)/\lambda^*$, conditional on $V = v$. Thus the unconditional probability of acceptance is $q(t) = E[\lambda(V)/\lambda^*] = (1/\lambda^*)(1/t)\int_0^t \lambda(s)ds$, and we conclude from partitioning[1] that $N(t)$ has a Poisson distribution with mean $\lambda^* t q(t) = m(t)$, as was to be shown.

■

1.2 Simulating a compound Poisson process

Suppose that we wish to simulate a non-stationary compound Poisson process at rate $\lambda(t) \leq \lambda^*$ with iid B_i distributed as (say) G (could be continuous or discrete). Suppose that we already have an algorithm for generating from G.

Here is the algorithm for generating our compound Poisson process up to a desired time T to get $X(T)$:

1. $t = 0$, $N = 0$, $X = 0$.
2. Generate U
3. $t = t + [-(1/\lambda^*)\ln(U)]$. If $t > T$, then stop.
4. Generate U.
5. If $U \leq \lambda(t)/\lambda^*$, then set $N = N + 1$ and generate B distributed as G and set $X = X + B$.
6. Go back to 2.

[1] A general result in elementary probability is known as *partitioning of a Poisson rv* : Suppose X is a rv with a Poisson distribution with mean α. Suppose that for all $n \geq 1$, conditional on $X = n$, each of the n is independently labeled as being of type 1 or 2 with probability p, $1 - p$ respectively, and let X_i denote the number of type $i, i = 1, 2$ (in particular $X = X_1 + X_2$). Then the two X_i are independent rvs, and X_1 has a Poisson distribution with mean αp and X_2 has a Poisson distribution with mean $\alpha(1 - p)$.