1 Capital Asset Pricing Model (CAPM)

We now assume an idealized framework for an open market place, where all the risky assets refer to (say) all the tradeable stocks available to all. In addition we have a risk-free asset (for borrowing and/or lending in unlimited quantities) with interest rate r_f. We assume that all information is available to all such as covariances, variances, mean rates of return of stocks and so on. We also assume that everyone is a risk-averse rational investor who uses the same financial engineering mean-variance portfolio theory from Markowitz. A little thought leads us to conclude that since everyone has the same assets to choose from, the same information about them, and the same decision methods, *everyone has a portfolio on the same efficient frontier, and hence has a portfolio that is a mixture of the risk-free asset and a unique efficient fund F (of risky assets).* In other words, everyone sets up the same optimization problem, does the same calculation, gets the same answer and chooses a portfolio accordingly.

This efficient fund used by all is called the *market portfolio* and is denoted by M. The fact that it is the same for all leads us to conclude that it should be computable without using all the optimization methods from Markowitz: The market has already reached an *equilibrium* so that the weight for any asset in the market portfolio is given by its capital value (total worth of its shares) divided by the total capital value of the whole market (all assets together). If, for example, asset i refers to shares of stock in Company A, and this company has 10,000 shares outstanding, each worth \$20.00, then the captial value for asset i is $V_i = (10,000)(\$20) = \$200,000$. Computing this value for each asset and summing over all n (total number of assets) yields the total capital value of the whole market, $V = V_1 + \cdots + V_n$, and the weight $w_i = V_i/V$ is the weight used for asset i in the market portfolio. The general idea is that assets under high demand will fetch high prices and yield high expected rates of return (and vice versa); repeated trading of the assets over time adjusts the various prices yielding an equilibrium that reflects this; the optimal weights w_i for the market portfolio are thus governed by supply and demand.

In the end, we don't need to use the optimization methods nor any of the detailed data (covariances, variances, mean rates, nor even the risk-free rate r_f) to determine the market portfolio; we only need all the V_i values.

Note in passing that all the weights w_i are positive (no shorting exists in the market portfolio). This implies that even if apriori we ruled out shorting of the assets in our framework, the same equilibrium market portfolio M would arise. Also note that since presumably all risky assets available on the open market have a non-zero capital value, all risky assets are included in the portfolio (although some have very small weights).

The above equilibrium model for portfolio analysis is called the *Capital Asset Pricing Model (CAPM).* [1]

1.1 Capital market line and CAPM formula

Let $(\sigma_M, \overline{r}_M)$ denote the point corresponding to the market portfolio M. All portfolios chosen by a rational investor will have a point $(\sigma, \overline{r})$ that lies on the so-called *capital market line*

$$\overline{r} = r_f + \frac{\overline{r}_M - r_f}{\sigma_M}\sigma, \tag{1}$$

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[1]In the literature however, CAPM sometimes refers to results (formulas) that follow from using this model. As we shall see at the end of this lecture, this model leads to a formula for pricing assets which is why the word *pricing* is used in its name.

the efficient frontier for investments. It tells us the expected return of any efficient portfolio, in terms of its standard deviation, and does so by use of the so-called *price of risk*

$$\frac{\overline{r}_M - r_f}{\sigma_M}, \tag{2}$$

the slope of the line, which represents the change in expected return $\overline{r}$ per one-unit change in standard deviation σ.

If an individual asset i (or portfolio) is chosen that is not efficient, then we learn nothing about that asset from (1). It would seem useful to know, for example, how $\overline{r}_i - r_f$, the *expected excess rate of return* is related to M. The following formula involves just that, where $\sigma_{M,i}$ denotes the covariance of the market portfolio with individual asset i:

Theorem 1.1 (CAPM Formula) *For any asset i*

$$\overline{r}_i - r_f = \beta_i(\overline{r}_M - r_f),$$

where

$$\beta_i = \frac{\sigma_{M,i}}{\sigma_M^2},$$

is called the beta of asset i. This beta value serves as an important measure of risk for individual assets (portfolios) that is different from σ_i^2; it measures the nondiversifiable part of risk.

More generally, for any portfolio $p = (\alpha_1, \ldots, \alpha_n)$ of risky assets, its beta can be computed as a weighted average of individual asset betas:

$$\overline{r}_p - r_f = \beta_p(\overline{r}_M - r_f),$$

where

$$\beta_p = \frac{\sigma_{M,p}}{\sigma_M^2} = \sum_{i=1}^{n} \alpha_i \beta_i.$$

Before proving the above theorem, we point out a couple of its important consequences and explain the meaning of the beta. For a given asset i, σ_i^2 tells us the risk associated with its own fluctuations about its mean rate of return, but not with respect to the market portfolio. For example if asset i is uncorrelated with M, then $\beta_i = 0$ (even presumably if σ_i^2 is huge), and this tells us that there is no risk associated with this asset (and hence no high expected return) in the sense that the variance σ_i^2 can be diversified away (recall our example in Lecture Notes 4 of diversification of n uncorrelated assets). The idea would be to collect a large number of uncorrelated (and uncorrelated with M) assets and form a portfolio with equal proportions thus dwindling the variance to 0 so it becomes like the risk-free asset with deterministic rate of return r_f. So, in effect, in the world of the market you are not rewarded (via a high expected rate of return) for taking on risk that can be diversified away. So we can view β_i as a measure of *nondiversifiable risk*, the correlated-with-the-market part of risk that we can't reduce by diversifying. This kind of risk is sometimes called *market* or *systematic* risk. It is not true in general that higher beta value β_i implies higher variance σ_i^2, but of course a higher beta value does imply a higher expected rate of return: you are rewarded (via a high expected rate of return) for taking on risk that can't be diversified away; everyone must face this kind of risk.

Types of assets that would be expected to have high beta values (today) would be those which are deeply related to the general market such as stocks in big name companies like General Electric, Cisco Systems, Coca-Cola, IBM, Procter & Gamble and so on.

Proof :[CAPM formula]

We form a portfolio of asset i and the market portfolio M; $(\alpha, 1 - \alpha)$, with $\alpha \in [0, 1]$. The rate of return is thus $r(\alpha) = \alpha r_i + (1 - \alpha)r_M$. Asset i is assumed not efficient; it's point lies in the feasible region but not on the efficient frontier. Thus as α varies this portfolio's point traces out a smooth curve in the feasible $(\sigma, \bar{r})$ region $\alpha \to (\sigma(\alpha), \bar{r}(\alpha))$ parametrized by α, where

$$
\begin{aligned}
\bar{r}(\alpha) &= \alpha \bar{r}_i + (1 - \alpha)\bar{r}_M \\
&= \alpha(\bar{r}_i - \bar{r}_M) + \bar{r}_M, \tag{3} \\
\sigma(\alpha) &= \sqrt{\alpha^2 \sigma_i^2 + (1 - \alpha)^2 \sigma_M^2 + 2\alpha(1 - \alpha)\sigma_{M,i}}. \\
&= \sqrt{\alpha^2(\sigma_i^2 + \sigma_M^2 - 2\sigma_{M,i}) + 2\alpha(\sigma_{M,i} - \sigma_M^2) + \sigma_M^2}. \tag{4}
\end{aligned}
$$

When $\alpha = 0$, $(\sigma(0), \bar{r}(0)) = (\sigma_M, \bar{r}_M)$ and when $\alpha = 1$, $(\sigma(0), \bar{r}(0)) = (\sigma_i, \bar{r}_i)$. Thus the curve touches the captital market line at the market point $(\sigma_M, \bar{r}_M)$, but otherwise remains off the captital market line but (of course) within the feasible region where it also hits the point $(\sigma_i, \bar{r}_i)$. We conclude that the curve is tangent to the capital asset line at point $(\sigma_M, \bar{r}_M)$ and therefore when $\alpha = 0$ the curve's derivative

$$
\frac{d\bar{r}(\alpha)}{d\sigma(\alpha)}\bigg|_{\alpha=0}
$$

is identical to the slope of the capital asset line at point M. The slope (the price of risk) of the line is given by (2). We thus conclude that when $\alpha = 0$

$$
\frac{d\bar{r}(\alpha)}{d\sigma(\alpha)} = \frac{\bar{r}_M - r_f}{\sigma_M}. \tag{5}
$$

The composition rule from calculus yields

$$
\frac{d\bar{r}(\alpha)}{d\sigma(\alpha)} = \frac{d\bar{r}(\alpha)/d\alpha}{d\sigma(\alpha)/d\alpha}
$$

Differentiating (3) and (4) and evaluating at $\alpha = 0$ yields

$$
\frac{d\bar{r}(\alpha)/d\alpha}{d\sigma(\alpha)/d\alpha} = \frac{\bar{r}_i - \bar{r}_M}{(\sigma_{M,i} - \sigma_M^2)/\sigma_M}.
$$

From (5) we deduce that

$$
\frac{\bar{r}_i - \bar{r}_M}{(\sigma_{M,i} - \sigma_M^2)/\sigma_M} = \frac{\bar{r}_M - r_f}{\sigma_M}.
$$

Solving for $\bar{r}_i$ then yields the CAPM formula for asset i, $\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f)$, as was to be shown. For the more general portfolio result, observe that

$$
\begin{aligned}
r_p - r_f &= -r_f + \sum_{i=1}^{n} \alpha_i \bar{r}_i \\
&= \sum_{i=1}^{n} \alpha_i(\bar{r}_i - r_f) \\
&= \sum_{i=1}^{n} \alpha_i \beta_i(\bar{r}_M - r_f) \ \textit{(CAPM formula for single asset i)} \\
&= (\bar{r}_M - r_f)\sum_{i=1}^{n} \alpha_i \beta_i.
\end{aligned}
$$

3

Note that when $\beta_p = 1$ then $\bar{r}_p = \bar{r}_M$; the expected rate of return is the same as for the market portfolio. When $\beta_p > 1$, then $\bar{r}_p > \bar{r}_M$; when $\beta_p < 1$, then $\bar{r}_p < \bar{r}_M$.

Also note that if an asset i is negatively correlated with M, $\sigma_{M,i} < 0$, then $\beta_i < 0$ and $\bar{r}_i < r_f$; the expected rate of return is less than the risk-free rate. Effectively, such a negatively correlated asset serves as insurance against a drop in the market returns, and might be viewed by some investors as having enough such advantages so as to make it worth the low return. Investing in gold is thought to be such an example at times.

1.2 Estimating the market portfolio and betas

In the real open market place where the number of assets is enormous, trying to actually construct the market portfolio would be an awsome and unrealistic task for any financial analyst. Thus so-called *index funds* (or *mutual funds*) have been created as an attempt to approximate the market portfolio. Such an index is a smaller portfolio made up of what are viewed as the markets' most dominant assets, that captures the essence of M. The most well-known such index is the Standard & Poor's 500-stock index (S&P), made up of 500 stocks. A beta for a given asset is then estimated by using the S&P in replace of M, and then collecting past data for both rates of return. For example consider an asset A for which we wish to estimate its beta. These estimates are computed using sample means, variances and covariances as follows:

We choose N time points such as the end of each of the last 10 years $k = 1, 2, \ldots, N$. r_{Ak} and $r_{S\&Pk}$ denote the kth such sample values for rate of return yielding estimates

$$\hat{\bar{r}}_A \;=\; \frac{1}{N} \sum_{k=1}^{N} r_{Ak}$$

$$\hat{\bar{r}}_{S\&P} \;=\; \frac{1}{N} \sum_{k=1}^{N} r_{S\&Pk}.$$

The variance $\sigma_{S\&P}^2$ is then estimated by

$$Y = \frac{1}{N-1} \sum_{k=1}^{N} (r_{S\&Pk} - \hat{\bar{r}}_{S\&P})^2,$$

and the covariance $\sigma_{S\&P,A}$ is estimated by

$$X = \frac{1}{N-1} \sum_{k=1}^{N} (r_{S\&Pk} - \hat{\bar{r}}_{S\&P})(r_{S\&Pk} - \hat{\bar{r}}_A).$$

The beta value is then estimated by taking the ratio X/Y.

1.3 More on systematic risk

With the CAPM formula in mind, for a given asset i let us express r_i as

$$r_i = r_f + \beta_i(r_M - r_f) + \epsilon_i, \tag{6}$$

where

$$\epsilon_i = r_i - r_f - \beta_i(r_M - r_f), \tag{7}$$

a random variable, is the error term. Note how we replaced deterministic $\overline{r}_M$ with random r_M in the CAPM to do this. Our objective is to determine some properties of the error term.

It is immediate from the CAPM formula that $E(\epsilon_i) = 0$. Moreover $Cov(\epsilon_i, r_M) = 0$ as is seen by direct computation and the definition of β_i:

$$
\begin{aligned}
cov(\epsilon_i, r_M) &= cov(r_i - r_f - \beta_i(r_M - r_f), r_M) \\
&= cov(r_i - \beta_i(r_M - r_f), r_M) \\
&= cov(r_i, r_M) - \beta_i cov(r_M - r_f, r_M) \\
&= cov(r_i, r_M) - \beta_i cov(r_M, r_M) \\
&= \sigma_{M,i} - \frac{\sigma_{M,i}}{\sigma_M^2}\sigma_M^2 \\
&= \sigma_{M,i} - \sigma_{M,i} \\
&= 0.
\end{aligned}
$$

So the error term has mean 0 and is uncorrelated with the market portfolio. It then follows by taking the variance of both sides of (6) that

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + var(\epsilon_i). \tag{8}$$

We conclude that the variance of asset i can be broken into two orthogonal components. The first, $\beta_i^2 \sigma_M^2$, is called the *systematic* risk and represents that part of the risk (of investing in asset i) associated with the market as a whole. The other part, $var(\epsilon_i)$, is called the *nonsystematic* risk. The nonsystematic risk can be reduced (essentially eliminated) by diversification, but the systematic risk, when $\beta_i^2 > 0$, can not be diversified away.

Solving for σ in terms of $\overline{r}$ in the capital market line yields the inverse line for efficient portfolios

$$\sigma = \frac{\overline{r} - r_f}{\overline{r}_M - r_f}\sigma_M.$$

Using the CAPM formula $\overline{r}_p - r_f = \beta_p(\overline{r}_M - r_f)$ and plugging into this inverse line formula, we conclude that for any efficient portfolio p, $\sigma_p = \beta_p \sigma_M$; there is only systematic risk for efficient portfolios, no nonsystematic risk. This makes perfect sense: Clearly M has only systematic risk since $\beta_M = 1$. We know that all efficient portfolios are a mixture of M and the risk-free asset (e.g., they all have points on the capital market line). Since the risk-free asset is deterministic it does not contribute to either the variance or the beta of the portfolio, only M does; so all of the risk is contained in M which we just pointed out is pure systematic risk.

1.4 Pricing assets

Consider an asset with price $P = X_0$ at time $t = 0$, payoff (random) $Q = X_1$ and expected payoff $\overline{Q} = E(X_1)$ at time $t = 1$. Then by definition $\overline{r} = (E(X_1) - X_0)/X_0 = (\overline{Q} - P)/P$. Solving for P yields

$$P = \overline{Q}/(1 + \overline{r}),$$

which expresses the price as the discounted payoff (present value) if using $\overline{r}$ as the discount rate. But since $\overline{r} = r_f + \beta(\overline{r}_M - r_f)$ from the CAPM formula, we conclude that

$$P = \frac{\overline{Q}}{1 + r_f + \beta(\overline{r}_M - r_f)} \qquad \text{(Pricing version of CAPM formula)} \tag{9}$$

We can re-express this formula by using $r = (Q - P)/P = (Q/P) - 1$, then computing

$$\begin{aligned}
Cov(r, r_M) &= Cov((Q/P) - 1, r_M) \\
&= Cov((Q/P), r_M) \\
&= \frac{1}{P} Cov(Q, r_M);
\end{aligned}$$

obtaining

$$\beta = \frac{1}{P}(Cov(Q, r_M)/\sigma_M^2).$$

Plugging into the pricing version of CAPM formula yields

$$P = \frac{\overline{Q}}{1 + r_f + \frac{1}{P}(Cov(Q, r_M)/\sigma_M^2)(\overline{r}_M - r_f)}.$$

Solving for P yields

$$P = \frac{\overline{Q} - \frac{Cov(Q, r_M)(\overline{r}_M - r_f)}{\sigma_M^2}}{1 + r_f}. \tag{10}$$

This equation is interesting since it expresses the price as the present value of an "adjusted" expected payoff. If the asset is uncorrelated with the market, then the price is exactly $\overline{Q}/(1+r_f)$, the present value of the expected payoff. The adjustment

$$-\frac{Cov(Q, r_M)(\overline{r}_M - r_f)}{\sigma_M^2}$$

yields a lower price if the asset is positively correlated with the market, and a higher price if the asset is negatively correlated with the market.