# 1 Discrete-time renewal processes

Imagine busloads of passengers, where the $i^{th}$ bus contains $H_i$ passengers, and the $\{H_i\}$ are iid with pmf

$$p(k) = P(H = k),\ k \geq 1. \tag{1}$$

Imagine further that the seats on a bus are labeled $1, 2, \ldots, H$, from the back of the bus to the front in one long line. A passenger is said to be in the $j^{th}$ position on their bus if they are in the seat labeled $j$.

We shall assume the expected bus size is positive, $0 < E(H) < \infty$, to avoid trivialities. (We also allow the bus to be of unlimited size, if necessary.) If we "randomly select a passenger way out among all passengers among all buses", then it is of intrinsic interest to determine such quantities as

1. The distribution (and mean) of the position of the passenger in question.
2. The distribution (and mean) of the bus size containing the passenger in question.
3. The distribution (and mean) of the number of passengers in front of (or in back of) this passenger on the bus.

A little thought reveals that to determine the above quantities (and to make this framework precise), we simply need to construct a discrete-time renewal process in which the "interarrival" times are the $H_i$, and $N(n) \stackrel{\text{def}}{=}$ the number of bus arrivals by time $n$ defines the discrete-time counting process$\{N(n) : n \geq 0\}$. The "arrival times" are given by $t_1 = H_1,\ t_2 = H_1 + H_2,\ \ldots, t_n = \sum_{i=1}^{n} H_i,\ n \geq 1$. Thus "time" $n$ denotes the $n^{th}$ passenger (among all infinite of them), and every chosen such passenger is on some bus in some position. Thus our quantities of interest are essentially determined by considering limiting (in distribution) quantities like spread, backward, and forward recurrence times–but in discrete time in insead of continuous time. We can simply use renewal reward arguments (e.g., discrete-time regenerative process results) to obtain our quantities of interest, where the $H_i$ are the cycle lengths.

1. *The distribution (and mean) of the position of the passenger in question:*

   Let $J$ denote the position of a randomly chosen passenger. We define the distribution of $J$ as

   $$P(J = j) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I\{k^{th} \text{ passenger is in position } j \text{ on their bus}\},\ j \geq 1.$$

   From renewal reward, the answer is given by $E(R)/E(H)$ where $R = I\{H \geq j\}$. The point is that $R =$ the number of passengers on the (first) bus who are in position $j$. There can only be at most one such passenger, and there will be one if and only if the bus is at least of size $j$. Thus

   $$P(J = j) = \frac{P(H \geq j)}{E(H)},\ j \geq 1.$$

To compute $E(J)$ we could either directly take the mean of the above distribution, or simply take the corresponding average:

$$E(J) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \{\text{position of the } k^{th} \text{ passenger}\}.$$

From renewal reward, the answer is given by $E(R)/E(H)$ where now

$$R = 1 + \cdots + H = \frac{H(H+1)}{2},$$

the sum of all the positions over the (first) cycle. Thus

$$E(J) = \frac{E(H(H+1))}{2E(H)}, \tag{2}$$

2. *The distribution (and mean) of the bus size containing the passenger in question:* This is simply the *spread*: The total size of the bus containing a randomly chosen customer has distribution given by

$$\frac{jP(H=j)}{E(H)} = \frac{jp(j)}{E(H)} \; j \geq 1, \tag{3}$$

interpreted as the long run proportion of customers who are in a bus of size $j$. Once again, this follows from renewal reward, with $R = jI\{H = j\}$: Buses of size $j$ each contain exactly $j$ customers from a bus of size $j$ (no other buses are relevant). This yields (3). The multiplicative factor of $j$ in $jp(j)$ is a discrete-time analog of the length-biasing cause of the inspection paradox (in which, for a continuous distribution with density function $f(x)$ and mean $\lambda^{-1}$, the density function of the spread is given by $f_s(x) = \lambda x f(x)$): Larger buses contain more customers and thus a randomly chosen customer is more likely to be from a larger bus; the bus size of a randomly chosen customer is *stochastically larger* than a regular bus size $H$.

The mean of the distribution in (3) is given by

$$\frac{E(H^2)}{E(H)}, \tag{4}$$

3. *The distribution (and mean) of the number of passengers in front of (or in back of) this passenger on the bus:*

This is the discrete-time analogue of the forward and backward recurrence time, and is simply the distribution of $J-1$, and $E(J-1)$.

$$P(J-1=j) = P(J=j+1) = \frac{P(H \geq j+1)}{E(H)}, \; j \geq 0.$$

$$E(J-1) = \frac{E(H(H-1))}{2E(H)}. \tag{5}$$

Notice how the mean is slightly different from that of $H_e$, $E(H^2)/2E(H)$; that is due to the discrete nature of $J$. $H_e$ is defined as a continuous distribution, while the distribution of $J$ is discrete.

The above notions are very useful in queueing theory, when arrivals occur in batches (busloads) as opposed to only one at a time. For example, we could consider a model where the buses themselves arrive according to a Poisson process at rate $\lambda$ with counting process $\{N(t)\}$, and then consider the arrival process of customers as

$$X(t) = \sum_{j=1}^{N(t)} H_j, \ t \geq 0,$$

where the $H_i$ are iid and independent of the Poisson process. This is an example of a *compound Poisson process.* $E(X(t)) = \lambda t E(H)$ and $\{X(t)\}$ has both stationary and independent increments. The case when $P(H = 1) = 1$ is the Poisson process. (The $H_i$ need not be positive in the general definition of compound Poisson).

We can still use PASTA too: Although the individual customers do not see time averages, the buses do because they are arriving as a Poisson process!