1 Regenerative Processes

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Given a stochastic process $\mathbf{X} = \{X(t) : t \geq 0\}$, suppose that there exists a (proper) random time $\tau = \tau_1$ such that $\{X(\tau + t) : t \geq 0\}$ has the same distribution as $\mathbf{X}$ and is independent of the *past*, $C_1 = \{\{X(t) : 0 \leq t < \tau\}, \tau\}$. Then we say that $\mathbf{X}$ *regenerated* at time τ, meaning that it has stochastically "started over again", as if it was time $t = 0$ again, and its future is independent of its past. In particular, $X(\tau)$ has the same distribution as $X(0)$, and is independent of the regeneration time τ. C_1 is called the first *cycle* and $X_1 = \tau_1$ is the first *cycle length.* But if such a τ_1 exists, then, since things start over again as if new, there must be a second such time $\tau_2 > \tau_1$ yielding an identically distributed second cycle $C_2 = \{\{X(\tau_1 + t) : 0 \leq t < \tau_2 - \tau_1\}, \tau_2 - \tau_1\}$. Continuing in this fashion we conclude that there is a renewal process of such times $\{\tau_k : k \geq 1\}$ (with $\tau_0 \stackrel{\text{def}}{=} 0$) with iid cycle lengths $X_k = \tau_k - \tau_{k-1}, \ k \geq 1$, and such that the cycles $C_k = \{\{X(\tau_{k-1} + t) : 0 \leq t < X_k\}, X_k\}$ are iid objects.

By definition, $X(\tau_k)$ is independent of the time τ_k: Upon regeneration the value of the process is independent of what time it is. (This turns out to be of fundamental importance when we later prove "weak convergence" results for regenerative processes.)

The above defines what is called a *regenerative process.* It is said to be *positive recurrent* if the renewal process is so, that is, if $0 < E(X_1) < \infty$. Otherwise it is said to be *null recurrent* if $E(X_1) = \infty$.

We always assume that the paths of $\mathbf{X}$ are right-continuous and have left hand limits. (Think of the forward recurrence time paths $\{A(t) : t \geq 0\}$ of a simple point process.)

1.1 Examples

1. *Any recurrent continuous-time Markov chain is regenerative:* Fix any state i and let $X(0) = i$. Then let $\tau = \tau_{i,i}$ denote the first time that the chain returns back to state i (after first leaving it). By the (strong) Markov property, the chain indeed starts all over again in state i at time τ independent of its past. Since i is recurrent, we know that the chain must in fact return over and over again, and we can let τ_k denote the k^{th} such time. We also know that positive recurrence of the chain, by definition, means that $E(\tau_{i,i}) < \infty$, which is thus equivalent to being a positive recurrent regenerative process.

2. *Age, excess and spread for a renewal process:* Given a renewal process, both $\{B(t) : t \geq 0\}$, $\{A(t) : t \geq 0\}$ and $\{S(t) : t \geq 0\}$ are regenerative processes; they regenerate at the arrival times $\{t_n : n \geq 1\}$; $\tau_n = t_n, \ n \geq 1$. They are positive recurrent when the renewal process is so.

2 Renewal reward theorem applied to regenerative processes

Because $\{\tau_k\}$ is a renewal process and the cycles $\{C_k\}$ are iid, we can apply the renewal reward theorem in a variety of ways to a regenerative process so as to compute various time average quantities of interest. The general result can be stated in words as *the time average is equal to the expected value over a cycle divided by the expected cycle length.*

For example, we can obtain the time-average of the process itself as

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)ds = \frac{E(R)}{E(X)}, \ wp1,$$

where $X = X_1$ and $R = R_1 = \int_0^X X(s)ds$. The proof is derived by letting $N(t)$ denote the number of renewals (regenerations) by time t, and observing that

$$\int_0^t X(s)ds \approx \sum_{j=1}^{N(t)} R_j,$$

where

$$R_j = \int_{\tau_{j-1}}^{\tau_j} X(s)ds.$$

So we are simply interpreting $X(t)$ as the rate at which we earn a reward at time t and applying the renewal reward theorem. The (R_j, X_j) are iid since they are constructed from iid cycles; (R_j, X_j) is constructed from C_j.

2.1 Functions of regenerative processes are regenerative with the same regeneration times

If $\mathbf{X}$ is regenerative with regeneration times τ_k, then it is immediate that for any (measurable) function $f = f(x)$, the process $Y(t) = f(X(t))$ defines a regenerative process with the very same regeneration times. This allows us to obtain

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(s))ds = \frac{E(R)}{E(X)}, \ wp1,$$

where $X = X_1$ and $R = R_1 = \int_0^X f(X(s))ds$.

We next precisely state and prove the main result. The proof mimics quite closely the proof of the renewal reward theorem.

Theorem 2.1 *If $\mathbf{X}$ is a positive recurrent regenerative process, and $f = f(x)$ is a (measurable) function such that $E\Big[\int_0^{X_1} |f(X(s))|ds\Big] < \infty$, then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(s))ds = \frac{E(R)}{E(X)}, \ wp1, \tag{1}$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t E[f(X(s))]ds = \frac{E(R)}{E(X)}, \tag{2}$$

where $R = R_1 = \int_0^{X_1} f(X(s))ds$, and $X = X_1$.

Proof : Let $N(t) = \max\{j : \tau_j \le t\}$ denote the number of renewals (regenerations) by time t. First note that if $f \ge 0$, then since $t_{N(t)} \le t < t_{N(t)} + 1$, by defining $R_j = \int_{\tau_{j-1}}^{\tau_j} f(X(s))ds$, (1) can be derived by using the following sandwiching bounds (upper and lower bounds),

$$\frac{1}{t}\sum_{j=1}^{N(t)} R_j \le \frac{1}{t}\int_0^t f(X(s))ds \le \frac{1}{t}\sum_{j=1}^{N(t)+1} R_j. \tag{3}$$

Both these bounds converge to $E(R)/E(X)$ yielding the result: For the lower bound,

$$\frac{1}{t}\sum_{j=1}^{N(t)} R_j = \frac{N(t)}{t}\frac{1}{N(t)}\sum_{j=1}^{N(t)} R_j;$$

$N(t)/t \to 1/E(X)$ by the elementary renewal theorem, while $\frac{1}{N(t)}\sum_{j=1}^{N(t)} R_j \to E(R)$ by the strong law of large numbers since the R_j are iid and have finite first moment by the theorem's hypothesis.

Similarly, for the upper bound, noting that $(N(t)+1)/t \to 1/E(X)$ (since $1/t \to 0$), and rewriting the upper bound as

$$\frac{N(t)+1}{t}\frac{1}{N(t)+1}\sum_{j=1}^{N(t)+1} R_j,$$

we obtain the same limit for this upper bound, thus completing the proof of (1) in this non-negative case. Note in passing from (3), that because the upper bound converges to the same (finite) limt as the lower bound, it must hold that

$$\frac{R_{N(t)+1}}{t} \to 0, \text{wp1}, \tag{4}$$

because $R_{N(t)+1}/t$ is precisely the difference between the upper and lower bounds.

If f is not non-negative, then first observe that since $|f|$ is non-negative, we can apply the above proof to $\frac{1}{t}\int_0^t |f(X(s))|ds$ to conclude from (4) that $R^*_{N(t)+1}/t \to 0, wp1$, where $R^*_j = \int_{\tau_{j-1}}^{\tau_j} |f(X(s))|ds,\ j \ge 1$. The proof of (1) is then completed by writing

$$\frac{1}{t}\int_0^t f(X(s))ds = \frac{1}{t}\sum_{j=1}^{N(t)} R_j + \frac{1}{t}\int_{t_{N(t)}}^t f(X(s))ds,$$

noting that the first piece on the rhs converges to the desired answer, and the second piece (the error) tends to 0 by (4),

$$\Big|\frac{1}{t}\int_{t_{N(t)}}^t f(X(s))ds\Big| \le \frac{1}{t}\int_{t_{N(t)}}^t |f(X(s))|ds \le \frac{1}{t}\int_{t_{N(t)}}^{t_{N(t)+1}} |f(X(s))|ds = \frac{R^*_{N(t)+1}}{t} \to 0.$$

($E(|R_j|) < \infty$ since $|R_j| \le R^*_j$ and $E(R^*_j) < \infty$ by the theorem's hypothesis.)

To prove (2) we must show that $\{R(t)/t : t \geq 1\}$ is uniformly integrable (UI), where $R(t) \stackrel{\text{def}}{=} \int_0^t f(X(s))ds$. Since $|R(t)| \leq \int_0^t |f(X(s))|ds$, it follows that

$$|R(t)|/t \leq Y(t) = \frac{1}{t}\sum_{j=1}^{N(t)+1} R_j^*. \tag{5}$$

We know that $Y(t) \to E(R^*)/E(X) < \infty$, wp1. Moreover, $N(t)+1$ is a stopping time with respect to $\{X_j, R_j^*\}$, so from Wald's equation $E(Y(t)) = E(N(t)/t+1/t)E(R^*) \to E(R^*)/E(X)$ (via the elementary renewal theorem); thus $\{Y(t) : t \geq 1\}$ is uniformly integrable (UI)[1] and since $|R(t)|/t \leq Y(t)$, $\{R(t)/t : t \geq 1\}$ is UI as well.

■

As an immediate application of Theorem 2.1 take $f(x) = I\{x \leq b\}$ with b constant. Such a function is non-negative and bounded, so the hypothesis of Theorem 2.1 is immediately verified.

Corollary 2.1 *A positive recurrent regenerative process has a limiting distribution: Letting X^* denote a rv with this distribution, we have*

$$P(X^* \leq b) = \lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s) \leq b\}ds = \frac{E(R)}{E(X)}, \ wp1, \ b \in \mathcal{R}$$

where $R = \int_0^{X_1} I\{X(s) \leq b\}ds$, and $X = X_1$. Also,

$$P(X^* \leq b) = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(X(s) \leq b)ds = \frac{E(R)}{E(X)}, \ b \in \mathcal{R}.$$

2.2 Weak convergence to the limiting distribution

Theorem 2.2 *A positive recurrent regenerative process with a non-lattice cycle length distribution, $F(x) = P(X_1 \leq x)$, converges weakly to its limiting distribution: as $t \to \infty$,*

$$X(t) \Longrightarrow X^*,$$

where X^* has the limiting distribution given in Corollary 2.1.

Proof : The proof is a straightforward application of the key renewal theorem, where we condition on the first regeneration time, τ_1, to obtain a renewal equation. Letting f be any bounded continuous function such that $\max_x |f(x)| \leq 1$, we must show that $E(f(X(t)) \to E(f(X^*))$. Conditioning on $\tau_1 > t$ and $\tau_1 = s \leq t$ yields (in the usual manner)

$$E(f(X(t)) = E(f(X(t); \tau_1 > t) + \int_0^t E(f(X(t-s))dF(s).$$

The function $Q(t) = E(f(X(t); \tau_1 > t)$ can be shown to be directly Reimann integrable (DRI) because it is bounded by $P(\tau_1 > t)$ (itself already DRI) and is right-continuous (by the

[1] A collection $\{X(t)\}$ of non-negative rvs for which $X(t) \to X$ wp1 with $E(X) < \infty$ is UI if and only if $E(X(t)) \to E(X)$.

continuity and boundedness of f and the always assumed right-continuity of the paths of $\mathbf{X}$.). Thus

$$E(f(X(t)) \to \{E(\tau_1)\}^{-1} \int_0^\infty E(f(X(t); \tau_1 > t)dt.$$

The limit can be re-written (via Fubini's theorem) as

$$\{E(\tau_1)\}^{-1} E\left\{ \int_0^{\tau_1} f(X(t))dt \right\};$$

the same limit as in Theorem 2.1. ■

2.3 Delayed regenerative processes

Given a regenerative process, $\mathbf{X} = \{X(t) : t \geq 0\}$, it is natural to want to begin observing it starting at some future time $t = s$ (say) (random or deterministic) as opposed to at time $t = 0$; we thus would consider the process $\{X(s+t) : t \geq 0\}$. But if we do that, then the first cycle will be a remaining cycle, and thus not be the same (in distribution) as a typical future iid one. More generally, we can take any regenerative process, and simply replace the first cycle by something different in distribution from (but independent of) the others, and yet it will keep regenerating once the first cycle ends.

A regenerative process in which the first cycle has been replaced by one with a different distribution from (but is still independent of) the iid cycles following it is called a *delayed* regenerative process with delayed *initial cycle* C_0 and delayed *initial cycle length* τ_0. Thus, there is an initial random time $0 \leq \tau_0 < \tau_1$, and the initial cycle is given by $C_0 = \{\{X(t) : 0 \leq t < \tau_0\},\ \tau_0\}$. $X_1 = \tau_1 - \tau_0$, and the next cycle is $C_1 = \{\{X(\tau_0 + t) : 0 \leq t < X_1\},\ X_1\}$, and so on. C_0 is independent of the other iid cycles $\{C_k : k \geq 1\}$, but is allowed to have a different distribution. The renewal process $\{\tau_n : n \geq 0\}$ is a delayed renewal process with initial delay τ_0. (We assume that $P(\tau_0 > 0) > 0$.)

When there is no delay cycle, a regenerative process is called *non-delayed.* Note that once the delay is finished, a delayed regenerative process becomes non-delayed, thus $\{X(\tau_0+t) : t \geq 0\}$ is a regular non-delayed regenerative process, called a *non-delayed version.* A delayed regenerative process is called *positive recurrent* if its non-delayed version is so, that is, if $0 < E(X_1) < \infty$. It is not required that τ_0 have finite first moment, but we of course assume that it is a proper rv, that is, that $P(\tau_0 < \infty) = 1$, so that C_0 will end wp1.

As a common example, given a positive recurrent CTMC and using visits to state 1 as regeneration times, what if it started in state 2 instead? Clearly, when $X(0) = 2$, once the chain enters state 1 (which it will by recurrence), we are back to the non-delayed case; the delay is short lived. In this case $\tau_0 = \tau_{2,1} = \inf\{t > 0 : X(t) = 1\ |X(0) = 2\}$, the hitting time to state 1 given $X(0) = 2$, and C_0 captures the evolution of the chain until it hits state 1.

Since a delay cycle C_0 eventually ends, limiting results such as Theorem 2.1 should and do remain valid with some minor regularity conditions places on C_0 ensuring that its effect is asymptotically negligible as $t \to \infty$. We deal with this next. The reward and cycle used in such results must now be the first iid one, $R = R_1 = \int_{\tau_0}^{\tau_0+X_1} f(X(s))ds$ of length X_1, not of course the delay one, $R_0 = \int_0^{\tau_0} f(X(s))ds$ of length τ_0.

Corollary 2.1 remains valid without any further conditions:

Proposition 2.1 *A delayed positive recurrent regenerative process (with no conditions placed on the delay cycle C_0) has the same limiting distribution as its non-delayed counterpart: Letting X^* denote a rv with this distribution, we have*

$$P(X^* \leq b) = \lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s) \leq b\}ds = \frac{E(R)}{E(X)},\ wp1,\ b \in \mathcal{R}$$

where $R = R_1 = \int_{\tau_0}^{\tau_0+X_1} I\{X(s) \leq b\}ds$, and $X = X_1$. Also,

$$P(X^* \leq b) = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(X(s) \leq b)ds = \frac{E(R)}{E(X)},\ b \in \mathcal{R}.$$

Proof : For $t > t_0$, we have wp1,

$$\frac{1}{t}\int_0^t I\{X(s) \leq b\}ds = \frac{R_0}{t} + \frac{1}{t}\int_{\tau_0}^t I\{X(s) \leq b\}ds,$$

where $R_0 = \int_0^{\tau_0} I\{X(s) \leq b\}ds \leq \tau_0$; thus $\frac{R_0}{t} \to 0$, wp1. The second piece is a non-delayed version, and can be re-written as

$$\frac{t-\tau_0}{t}\frac{1}{t-\tau_0}\int_{\tau_0}^t I\{X(s) \leq b\}ds,$$

hence converges wp1, as $t \to \infty$, to the desired $\frac{E(R)}{E(X)}$ by the first part of Corollary 2.1. Taking expected value of $\frac{1}{t}\int_0^t I\{X(s) \leq b\}ds$, and applying the bounded convergence theorem then yields the second result. ■

In general, some further conditions on C_0 are needed to obtain Theorem 2.1: The only further condition needed for (1) is

$$\int_0^{\tau_0} |f(X(s))|ds < \infty,\ \text{wp1}, \tag{6}$$

and for (2) a sufficient condition is

$$E\Big[\int_0^{\tau_0} |f(X(s))|ds\Big] < \infty. \tag{7}$$

Theorem 2.3 *If $\mathbf{X}$ is a delayed positive recurrent regenerative process, and $f = f(x)$ is a (measurable) function such that $E\Big[\int_{\tau_0}^{\tau_0+X_1} |f(X(s))|ds\Big] < \infty$, and also (6) holds, then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(s))ds = \frac{E(R)}{E(X)},\ wp1, \tag{8}$$

and if additionally (7) holds, then also

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t E[f(X(s))]ds = \frac{E(R)}{E(X)}, \tag{9}$$

where $R = R_1 = \int_{\tau_0}^{\tau_0+X_1} f(X(s))ds$, and $X = X_1$.

Proof : $R_0 = \int_0^{\tau_0} f(X(s))ds$ and the proof of (8) is identical to the proof in Proposition 2.1 with condition (6) ensuring that $P(|R_0| < \infty) = 1$, so that $R_0/t \to 0$.

For (9) we argue UI as we did in proving (2), modifying the upper bound in (5) to a stochastic upper bound

$$|R(t)|/t \le K(t) = \frac{R_0^*}{t} + Y(t),$$

where $R_0^* = \int_0^{\tau_0} |f(X(s))|ds$ is independent of $\{Y(t)\}$, and $Y(t)$ is constructed from a non-delayed version, and hence (as we proved) $E(Y(t)) \to \frac{E(R)}{E(X)}$. Since $E(R_0^*) < \infty$ by (7), it follows that $E(R_0^*)/t \to 0$ and hence $\{K(t)\}$ is UI; thus so is $\{R(t)/t\}$ as was to be shown. ∎

That (7) is not *necessary* for (9) can be seen by Proposition 2.1. That is because whenever f is non-negative and bounded (such as $f(x) = I\{x \le b\}$), (6) is automatically satisfied, but (7) might not be satisfied even though (9) still holds (by applying the bounded convergence theorem to (8)).

Finally Theorem 2.2 remains valid (the proof is the same, we leave it out):

Theorem 2.4 *With no extra conditions placed on the delay distribution $P(\tau_0 \le x)$, a positive recurrent delayed regenerative process with a non-lattice cycle length distribution, $F(x) = P(X_1 \le x)$, weakly converges to its limiting distribution: as $t \to \infty$,*

$$X(t) \Longrightarrow X^*,$$

where X^* has the limiting distribution given in Corollary 2.1.

2.4 Stationary versions of a regenerative processes

Given a positive recurrent regenerative process $\mathbf{X} = \{X(t) : t \ge 0\}$, we know that the 1-dimensional $X(t)$ always has a limiting distribution from Corollary 2.1. But that is only a *marginal* distribution. It turns out that by creating a particular initial delay cycle C_0^*, with its corresponding initial delay t_0^*, we can make the regenerative process a stationary process, called a *stationary version* of the process, denoted by $\mathbf{X}^* = \{X^*(t) : t \ge 0\}$. For every fixed $s \ge 0$ it holds that $X^*(s)$ has the distribution given in Corollary 2.1, but much more is true: For every fixed $s \ge 0$, the entire shifted process $\{X^*(s+t) : t \ge 0\}$ has the same distribution; the same as $\{X^*(t) : t \ge 0\}$. This means that all joint distributions are the same. For example $(X^*(s+t_1), X^*(s+t_2), \ldots, X(s+t_k))$ has the same distribution as $(X^*(t_1), X^*(t_2), \ldots, X(t_k))$ for any $s \ge 0$. In particular, $X^*(s)$ has the same distribution at $X^*(0)$ for all $s \ge 0$, the distribution in Corollary 2.1. As an easy example, consider a positive recurrent CTMC, with stationary probabilities $\{P_n\}$ (solution to the balance equations). Then, as we know, by starting off $X(0)$ randomly distributed as $\{P_n\}$, $P(X(0) = n) = P_n$, the chain is stationary. Taking this stationary version, we can still use visits to a fixed state i as regeneration points; we choose $\tau_0^* = \inf\{t \ge 0 : X(t) = i\}$ and C_0^* is simply the evolution of the chain during $[0, \tau_0^*)$.

In general, though, how do we find C_0^*? First we must recall that the underlying renewal process has a stationary version by using delay $\tau_0 \sim F_e$, and we obtained F_e (as a limiting distribution) by randomly observing the renewal process at a time point way out in the infinite future. We now do the same thing for the entire regenerative process: We observe $\mathbf{X}_s =$

$\{X(s+t) : t \geq 0\}$ in which the origin (time s) is randomly chosen way out in the infinite future. (We choose s uniformly over $(0, t)$ and then let $t \to \infty$.) For each s, let $C_0(s)$ denote the delay cycle associated with $\mathbf{X}_s$. It is precisely the remaining cycle of the one covering time s. (It is a generalization of forward recurrence time.) It has delay length $\tau_0(s) = A(s)$, the forward recurrence time. For each $s \geq 0$, the only part of $\mathbf{X}_s$ that can change in distribution is this delay cycle $C_0(s)$ because all the others are iid regular ones. Thus we get a stationary version of the regenerative process if and only if the distribution of $C_0(s)$ is the same for all $s \geq 0$. (This is the same basic argument we used to produce a stationary version of a renewal process.)

$\tau_0(s) = A(s)$ we already know has F_e as its limiting distribution from our Lectures Notes on renewal theory. It turns out that the entire cycle $C_0(s)$ has a limiting distribution. Letting $C_0^* = \{\{X^*(t) : 0 \leq t < \tau_0^*\},\ \tau_0^*\}$ denote such a limiting delay cycle with this limiting distribution, the length τ_0^* must of course be distributed as F_e. It turns out that the entire distribution of C_0^*, denoted by $P(C_0^* \in \mathcal{B})$ (for appropriate path sets $\mathcal{B}$), is given by

$$P(C_0^* \in \mathcal{B}) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(C_0(s) \in \mathcal{B}) ds = \frac{E(R)}{E(X)}, \tag{10}$$

where $R = \int_0^{X_1} I\{C_0(s) \in \mathcal{B}\} ds$, and $X = X_1$. This is because $\{C_0(s) : s \geq 0\}$ forms a positive recurrent regenerative process, with the same regeneration times as $\mathbf{X}$ but with a more general state space, and the theory remains valid for such general state spaces; thus we are merely applying a more general form of Corollary 2.1. At a regeneration time τ_k, $k \geq 1$, the distribution of $C_0(\tau_k)$ is that of a regular iid cycle C_k.

The same basic "renewal reward" proof works for obtaining (10):

$$\frac{1}{t} \int_0^t I\{C_0(s) \in \mathcal{B}\} ds \approx \frac{1}{t} \sum_{j=1}^{N(t)} R_j,$$

where $R_j = \int_{\tau_{j-1}}^{\tau_j} I\{C_0(s) \in \mathcal{B}\} ds$.

That $C_0^*(u)$ (the delay cycle at time u from $\mathbf{X}^*$) has the same distribution for all $u \geq 0$, follows by the same argument used to show that $A^*(u)$ has the same distribution for all $u \geq 0$: Once out at ∞, going u time units further does not change the distribution;

$$P(C_0^*(u) \in \mathcal{B}) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(C_0(s+u) \in \mathcal{B}) ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(C_0(s) \in \mathcal{B}) ds. \tag{11}$$

2.5 Time averages as expected values under the stationary distribution

A general result is that time averages are equal to the expected value under the stationary distribution. Letting $\mathbf{X}^* = \{X^*(t) : t \geq 0\}$ denote a stationary version, so that $X^*(0)$ has the stationary distribution in Corollary 2.1, we next state

Theorem 2.5 *If $\mathbf{X}$ is a positive recurrent regenerative process, and $f = f(x)$ is a (measurable) function such that $E\Big[\int_0^{X_1} |f(X(s))| ds\Big] < \infty$, then*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s)) ds = E(f(X^*(0))),\ wp1. \tag{12}$$

Let us see some examples: In particular, using $f(x) = x$ yields

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)ds = E(X^*(0)),\ wp1. \tag{13}$$

As an example of this, consider a non-negative positive recurrent CTMC, and suppose we solved the balance equations and found the stationary probabilities $\{P_n\}$. Then (13) says

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)ds = E(X^*(0)) = \sum_n nP_n. \tag{14}$$

To prove (14) directly, recall that wp1,

$$P_n = \lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s) = n\}ds.$$

Now let $I_n(t) = \int_0^t I\{X(s) = n\}ds$; thus $I_n(t)/t \to P_n$. Then

$$\begin{aligned}
\frac{1}{t}\int_0^t X(s)ds &= \frac{1}{t}\sum_{n=0}^{\infty} nI_n(t) \\
&= \sum_n^{\infty} n(I_n(t)/t) \\
&\to \sum_n nP_n.
\end{aligned}$$

The interchange of limit and infinite sum can be justified.

For the more general (13), once again assume non-negativity. Then $E(X^*(0)) = \int_0^\infty P(X^*(0) > b)db$, and we also know that $P(X^*(0) > b) = \lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s) > b\}ds$, we thus obtain (changing the order of integration and limits; allowed by the bounded convergence theorem and Tonelli's theorem):

$$\begin{aligned}
E(X^*(0)) &= \int_0^\infty P(X^*(0) > b)db \\
&= \int_0^\infty \lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s) > b\}dsdb \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_0^\infty I\{X(s) > b\}dbdt \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_0^{X(s)} dbdt \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)ds.
\end{aligned}$$

More generally: From Theorem 2.3, we know that the two versions have the same limit;

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(s))ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(X^*(s))ds =\ wp1. \tag{15}$$

and (assuming some conditions) we can obtain the same limit by taking expected values too. But by stationarity $E(f(X^*(s))) = E(f(X^*(0)),\ s \geq 0$, a constant; thus

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t E(f(X^*(s)))ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t E(f(X^*(0)))ds = E(f(X^*(0)));$$

we have derived (12).

Discrete-time regenerative processes

For a stochastic sequence $\{Y_n : n \geq 0\}$, the definition of regenerative is the same. In this case, $C_1 = \{\{Y_n : 0 \leq n \leq \tau_1 - 1\}, \tau_1\}$, and the renewal process is a discrete time renewal process; $N(n)/n \to 1/E(X)$ as $n \to \infty$. Because of the discrete setting, the k^{th} cycle begins at time τ_{k-1} and ends at time $\tau_k - 1$, and there are X_k values of the Y_n in the cycle. The analog of Theorem 2.1 is

Theorem 2.6 *If* $\mathbf{Y}$ *is a positive recurrent regenerative sequence, and* $f = f(x)$ *is a (measurable) function such that* $E\Big[\sum_{j=0}^{X_1-1}|Y_j|\Big] < \infty$*, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(Y_j) = \frac{E(R)}{E(X)},\ wp1, \tag{16}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} E(f(Y_j)) = \frac{E(R)}{E(X)}, \tag{17}$$

where $R = R_1 = \sum_{j=0}^{X_1-1} f(Y_j)$*, and* $X = X_1$*.*