IEOR 6711: Introduction to Renewal Theory II

Here we will present some deeper results in renewal theory such as a central limit theorem for counting processes, stationary versions of renewal processes, renewal equations, the key renewal theorem, weak convergence.

1 Central limit theorem for counting processes

Consider a renewal process $\{t_n : n \geq 1\}$ with iid interarrival times $X_n = t_n - t_{n-1},\ n \geq 0$, such that $0 < E(X) = 1/\lambda < \infty$ and $0 < \sigma^2 = Var(X) < \infty$. Let $N(t) = \max\{n : t_n \leq t\},\ t \geq 0$, denote the counting process. $\Longrightarrow$ denotes convergence in distribution (weak convergence). Z denotes a rv with a $N(0,1)$ distribution (standard unit normal);

$$P(Z \leq x) = \Theta(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy,\ x \in \mathbb{R}.$$

Because $t_n = \sum_{j=1}^{n} X_j$, and the X_j are iid, the central limit theorem applies: as $n \to \infty$,

$$Z_n \stackrel{\text{def}}{=} \frac{t_n - n/\lambda}{\sigma\sqrt{n}} \Longrightarrow Z \sim N(0,1); \tag{1}$$

$$P(Z_n \leq x) \to \Theta(x),\ x \in \mathbb{R}. \tag{2}$$

Because our $\{X_n\}$ here are non-negative, we can also obtain an "inverse" version corresponding to the counting process $\{N(t) : t \geq 0\}$. Just as t_n is asymptotically normal as $n \to \infty$, it turns out that $N(t)$ is asymptotically normal as $t \to \infty$:

Theorem 1.1 (central limit theorem for counting processes) *As $t \to \infty$,*

$$Z(t) \stackrel{\text{def}}{=} \frac{N(t) - \lambda t}{\sigma\sqrt{\lambda^3 t}} \Longrightarrow Z \sim N(0,1).$$

Before we prove this let us observe what it indicates: As $t \to \infty$, $N(t)$ becomes normally distributed and both $E(N(t)) \sim \lambda t$ and $Var(N(t)) \sim \sigma^2\lambda^3 t$. The "first order" result, $E(N(t)) \sim \lambda t$ we already proved earlier directly from the elementary renewal theorem, $E(N(t))/t \to \lambda$, so what this new theorem here provides are some new higher order results.

Proof :[Theorem 1.1] The key is in using the fact that $P(N(t) < n) = P(t_n > t)$, so that we can convert a probability involving $Z(t)$ into one involving Z_n and then use the familiar form of the central limit theorem in (1) and (2). To this end, fix x and define (for t large)

$n(t) = \lfloor \lambda t + x\sqrt{\sigma^2\lambda^3 t} \rfloor$, where $\lfloor y \rfloor$ = the greatest integer $\leq y$. Then (algebra),

$$
\begin{aligned}
P(Z(t) < x) &= P(N(t) < n(t)) && (3)\\
&= P(t_{n(t)} > t) && (4)\\
&= P\left(\frac{t_{n(t)} - n(t)/\lambda}{\sigma\sqrt{n(t)}} > \frac{t - n(t)/\lambda}{\sigma\sqrt{n(t)}}\right) && (5)\\
&= P\left(Z_{n(t)} > \frac{t - n(t)/\lambda}{\sigma\sqrt{n(t)}}\right) && (6)\\
&= P\left(Z_{n(t)} > \frac{-x}{\sqrt{1 + (x\sigma)/\sqrt{t/\lambda}}}\right). && (7)
\end{aligned}
$$

As $t \to \infty$, $Z_{n(t)} \Longrightarrow Z$ from (1), while the (non-random) value in the right-hand side of the inequality in (7) converges to the constant $-x$ (because the denominator tends to 1). Thus (via formally using *Slutsky's theorem*)

$$P(Z(t) < x) \to P(Z > -x) = \Theta(x),$$

and the proof is complete. ■

Remark 1 It is worth noting that we do not need the point process to be a renewal process in order to obtain the result in Theorem 1.1: As long as the point process $\{t_n\}$ satisfies a central limit theorem (1) for some $\lambda > 0$ and some $\sigma > 0$, Theorem 1.1 follows by the same proof. Moreover, one can use the argument in reverse to obtain the converse; we conclude that there is an equivalence between the two kinds of central limit theorems.

There are many useful examples in applications of point processes that indeed would satisfy (1) but are not renewal processes, such as when $\{X_n\}$ forms a positive recurrent Markov chain or is the departure process from a complex queueing model.

Remark 2 We know from the elementary renewal theorem that $E(N(t)) = \lambda t + o(t)$ (e.g., $o(t)/t \to 0$ as $t \to \infty$.) It can further be proved that

$$Var(N(t)) = \sigma^2\lambda^3 t + o(t).$$

Both such results remain valid for delayed renewal processes.

2 Delayed renewal processes: the stationary version

Unlike a Poisson process, the counting process $\{N(t) : t \geq 0\}$ of a renewal process generally does not have stationary increments. But we can make it have them by allowing the first interarrival time $t_1 = X_1$ to be distributed as the equilibrium distribution F_e of F. We will proceed now to show why this is so.

A point process $\psi = \{t_n : n \geq 1\}$ has for each $s \geq 0$ a *shifted by time* s version $\psi_s = \{t_n(s) : n \geq 1\}$, which is the point process obtained by moving the origin to be time $t = s$ and then counting only the points to the right of s. If we denote the counting process of ψ_s by $\{N_s(t) : t \geq 0\}$, then $N_s(t) = N(s + t) - N(s)$. We thus observe that *if ψ_s has the same distribution for all $s \geq 0$, then $\{N(t) : t \geq 0\}$ has stationary increments:*

Definition 2.1 *A point process for which ψ_s has the same distribution for all $s \geq 0$ is called a (time) stationary point process. Its counting process has stationary increments.*

Let us now focus on the case when ψ is a (positive recurrent) renewal process and look more closely at the structure of ψ_s. Specifically we note that, $t_1(s) = t_{N(s)+1} - s = A(s)$, the forward recurrence time, and $t_2(s) = t_{N(s)+2} - s = A(s) + X_{N(s)+2}$, and in general $t_n(s) = t_{N(s)+n} - s = A(s) + X_{N(s)+2} + \cdots + X_{N(s)+n}$, for any $n \geq 2$. But the interarrival times $X_{N(s)+n},\ n \geq 2$, are again iid distributed as F and independent of $A(s)$. (Recall that only the spread $X_{N(s)+1}$ is "biased" via the inspection paradox since it is the only one covering the time s; after that none of the others are biased.) Thus ψ_s is (in distribution) the same renewal process but with its first point $t_1(s)$, independently, distributed as $A(s)$ instead of distributed as F. This is an example of a *delayed* renewal process:

Definition 2.2 *A delayed renewal process is a renewal process in which the first arrival time, $t_1 = X_1$, independently, is allowed to have a different distribution $P(X_1 \leq x) = F_1(x),\ x \geq 0$, than F, the distribution of all the remaining iid interarrival times $\{X_n : n \geq 2\}$. t_1 is then called the delay. When there is no such delay, that is, when $t_1 \sim F$ as usual, the renewal process is said to be a non-delayed version.*

With $\lambda = E(X_2)^{-1}$, it is easily seen that the elementary renewal theorem remains valid (the same proof goes through) for a delayed renewal process; both $N(t)/t \to \lambda$ wp1, and $E(N(t))/t \to \lambda$. (F_1 does *not* need to have finite first moment.)

Anyhow, as s varies, the only part of the delayed renewal process ψ_s that can change in distribution is the independent delay $t_1(s) = A(s)$. We conclude that *ψ will be a stationary renewal process if and only if $A(s)$ has the same distribution for all $s \geq 0$.* For example, the Poisson process is stationary because $A(s) \sim exp(\lambda)$ for all $s \geq 0$ by the memoryless property. In any case, as $s \to \infty$ we know that the limiting distribution of $A(s)$ is in fact F_e, that is,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P(A(s) \leq x)ds = F_e(x) = \lambda \int_0^x \overline{F}(y)dy,\ x \geq 0.$$

This is the distribution of $A(s)$ obtained by randomly choosing the shift time s, way out in the infinite future. We conclude that as $s \to \infty$, ψ_s has a limiting distribution represented by a delayed version of ψ, denoted by $\psi^* = \{t_n^* : n \geq 1\}$, in which the delay $t_1^* \sim F_e$. Denoting its counting process by $\{N^*(t) : t \geq 0\}$, and its forward recurrence time process by $\{A^*(t) : t \geq 0\}$, where $A^*(t) = t_{N^*(t)+1} - t$, it will be shown next that $A^*(u) \sim F_e,\ u \geq 0$, that is, it has the same distribution for all $u \geq 0$ and hence that ψ^* is indeed a stationary renewal process, called a *stationary version of ψ*. To see this, observe that for any $u \geq 0$, the distribution of $A^*(u)$ is obtained as the limiting distribution, as $s \to \infty$, of $A(s+u)$ which of course is the same as just the limiting distribution of $A(s)$, namely F_e: for any $u \geq 0$,

$$P(A^*(u) \leq x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(A(s+u) \leq x)ds = \lim_{t\to\infty} \frac{1}{t}\int_u^{t+u} P(A(s) \leq x)ds = F_e(x).$$

The point is that we obtained ψ^* by selecting ψ_s at a time s that was chosen randomly way out in the infinite future. By going further into the future another u times units we are still in the infinite future; nothing changes.

Proposition 2.1 *The stationary version ψ^* of a renewal process at rate $\lambda = E(X)^{-1}$ satisfies $E(N^*(t)) = \lambda t,\ t \geq 0$, and $\lambda = E(N^*(1))$.*

Proof : Let $U(t) = E(N^*(t))$. First we show that $U(t) = tU(1)$ for all $t \geq 0$, then we show that $U(1) = \lambda$. For any integer $n \geq 1$,

$$N^*(n) = \sum_{i=1}^{n} N^*(i) - N^*(i-1),$$

and taking expected values while using the stationary increments, yields $U(n) = nU(1)$. By a similar argument, $U(1) = nU(1/n)$, and so $U(1/n) = (1/n)U(1)$. Thus, continuing in this spirit, we see that for any rational number $r = n/m$, $U(r) = rU(1)$. For any $t > 0$ irrational, choosing a decreasing sequence of rationals $r_k \downarrow t$ implies by the right-continuity and non-decreasing properties of counting processes that $N^*(r_k) \downarrow N^*(t)$. Thus $U(r_k) \downarrow U(t)$ by the dominated convergence theorem since $0 \leq N^*(r_k) \leq N^*(r_1),\ k \geq 1$. Thus $U(t) = tU(1)$ since on the other hand $U(r_k) = r_k U(1) \downarrow tU(1)$. By the elementary renewal theorem $U(t)/t \to \lambda$, but here $U(t) = tU(1)$ and we conclude that $U(1) = \lambda$. ■

Example

1. (*Deterministic renewal process:*) Consider the case when $P(X = c) = 1$ for some $c > 0$. Thus $t_n = nc,\ n \geq 1$. F_e is thus the uniform distribution over $(0, c)$. Letting U denote a unform$(0, c)$ rv we can explicitly construct a stationary version $\{t_n^*\}$ of this renewal process: Define $t_n^* = U + (n-1)c,\ n \geq 1$.

Remark 3 While the stationary version of a (simple) renewal process has stationary increments, it of course does not also have independent increments unless it is a Poisson process.

3 Renewal equations

For a (non-delayed) renewal process $\{t_n : n \geq 1\}$ with $P(X \leq x) = F(x)$, consider computing $P(A(t) > x)$ for a given fixed $t > 0$ and $x > 0$. If we condition on $t_1 = X_1$, the time of the first arrival, we can obtain an equation as follows: $P(A(t) > x \mid X_1 = s \leq t) = P(A(t-s) > x)$ because time t is exactly $t - s$ units of time past the first renewal, and the distribution of $\{A(t) : t \geq 0\}$ starts over again at each renewal with a new iid cycle length and is independent of the past. Meanwhile if $X_1 > t$, then $A(t) = X_1 - t$ yielding in this case that $P(A(t) > x \mid X_1 > t) = P(X_1 - t > x \mid X_1 > t) = \overline{F}(x+t)/\overline{F}(t)$. Thus $P(A(t) > x; X_1 = s \leq t) = P(A(t-s) > x)dF(s)$ and $P(A(t) > x; X_1 > t) = \overline{F}(x+t)$ yielding a *renewal equation*:[1]

$$P(A(t) > x) = \overline{F}(x+t) + \int_0^t P(A(t-s) > x)dF(s). \tag{8}$$

[1] An integral of the form $\int g(s)dF(s)$ is an example of a *Riemann-Stieljies integral.* If F represents a continuous rv, with density function $f(t)$ then the integral reduces to $\int g(s)f(s)ds$. For any non-negative non-decreasing function $J(t)$, such an integral $\int g(s)dJ(s)$ can be defined (in fact it is only required that $J(t)$ be of bounded variation meaning that it can be expressed as $J(t) = J_1(t) - J_2(t)$, where each of the J_i are non-decreasing functions). In our application here to the renewal equation, we will need to use the non-decreasing function $m(t) = E(N(t))$ and consider integrals of the form $\int g(s)dm(s)$.

Letting $H(t) = P(A(t) > x)$ and $Q(t) = \overline{F}(x+t)$, we can re-write (8) more generally in the form of

$$H(t) = Q(t) + H \star F(t), \tag{9}$$

where $\star$ denotes convolution. We view this as an equation for solving for the unknown H, in terms of the known distribution F and known function Q.

Letting $F^{\star n}(t) = P(X_1 + X_2 + \cdots + X_n \leq t) = P(t_n \leq t),\ n \geq 1$, denote the n^{th}-fold convolution of F, we can solve this equation iteratively (by consecutively replacing H by $H = Q + H \star F$):

$$\begin{aligned}
H &= Q + H \star F && (10)\\
&= Q + (Q + H \star F) \star F && (11)\\
&= Q + Q \star F + H \star F^{\star 2} && (12)\\
&= Q + Q \star F + (Q + H \star F) \star F^{\star 2} && (13)\\
&= Q + Q \star F + Q \star F^{\star 2} + H \star F^{\star 3} && (14)\\
&\vdots && (15)\\
&= Q + \sum_{j=1}^{n} Q \star F^{\star j} + H \star F^{\star (n+1)}. && (16)
\end{aligned}$$

But for fixed t, $F^{\star(n+1)}(t) = P(t_{n+1} \leq t) \to 0$, as $n \to \infty$, since $t_n \to \infty$, wp1, and thus $H \star F^{\star(n+1)}(t) \to 0$.

Thus we conclude, by taking $n \to \infty$ that

$$H = Q + Q * \sum_{n=1}^{\infty} F^{\star n}.$$

Letting $m(t) = E(N(t))$ and observing that

$$m(t) = E(N(t)) = E\left(\sum_{n=1}^{\infty} I\{t_n \leq t\}\right) = \sum_{n=1}^{\infty} F^{\star n}(t). \tag{17}$$

Thus we can elegantly write the solution as

$$H = Q + Q * m, \tag{18}$$

meaning that

$$H(t) = Q(t) + \int_0^t Q(t-s)dm(s).$$

It is rare that we can explicitly compute the integral $\int_0^t Q(t-s)dm(s)$, hence rare that we ever get a solution explicitly. But in the case when $Q(t) \to 0$ as $t \to \infty$, it is often possible to do so in the limit as $t \to \infty$, as long as certain regularity conditions are met. The two conditions needed are as follows:

Non-lattice and directly Reimann integrable conditions

1. The distribution F must be *non-lattice*, meaning that there does not exist a $\delta > 0$ and an initial point $a \geq 0$ such that $P(X \in \{a + n\delta : n \geq 0\}) = 1$. In other words the mass of the distribution is not concentrated on a countable equidistant lattice of points. Of course any continuous distribution is non-lattice and it helps to assume continuity in the rest of our analysis in order to understand the essentials without getting tied up in unnecessary technical details.

2. The function $Q(t)$ must be *directly Reimann integrable (DRI)*, meaning that its integral $\int_0^\infty Q(t)dt$ must exist when it is defined on all of the interval $(0, \infty)$ *at once* in the Riemann sense (upper and lower sums defined on all of $(0, \infty)$ converging to the same thing). The point here is that it is *not* sufficient that the integral exist only in the "improper integral" way we learn in calculus, as $\lim_{b\to\infty} \int_0^b Q(s)ds$, where each integral $\int_0^b$ is defined in the classic Riemann sense. DRI is a stronger condition on Q than the improper integral approach.

   Here is a precise definition. Assume that $Q \geq 0$. Divide all of $\mathbb{R}_+ = [0, \infty)$ into subintervals of length $h > 0$ (small), $I(n, h) = (nh, (n + 1)h]$, $n \geq 0$. Define $\overline{q}(n) = \sup\{Q(t) : t \in I(n, h)\}$, $n \geq 0$, and $\underline{q}(n) = \inf\{Q(t) : t \in I(n, h)\}$, $n \geq 0$. The upper Riemann sum is given by

   $$\overline{U}(h) = \sum_{n\geq 0} h\overline{q}(n),$$

   and the lower Riemann sum is given by

   $$\underline{U}(h) = \sum_{n\geq 0} h\underline{q}(n).$$

   Q is said to be DRI if $\overline{U}(h)$ is finite for some (hence all) $h > 0$ and $\overline{U}(h) - \underline{U}(h) \to 0$ as $h \to 0$. Then $\int_0^\infty Q(t)dt \stackrel{\text{def}}{=} \lim_{h\to 0} \overline{U}(h) = \lim_{h\to 0} \underline{U}(h)$. In the general case when Q is not non-negative, Q is said to be DRI if both Q^+ and Q^- are, in which case $\int_0^\infty Q(t)dt \stackrel{\text{def}}{=} \int_0^\infty Q^+(t)dt + \int_0^\infty Q^-(t)dt$.

In the next section we will offer the formal limiting result, known as the *key renewal theorem.*

3.1 The key renewal theorem and weak convergence

Theorem 3.1 (Key renewal theorem) *Suppose that a renewal equation (9) holds for a given non-lattice F with mean $1/\lambda$, and a given function Q that is DRI. Then the solution (18) for $H(t)$ holds and*

$$\lim_{t\to\infty} H(t) = \lambda \int_0^\infty Q(t)dt. \tag{19}$$

We will not prove this theorem, it is very technical and beyond the scope of these Lecture Notes.

But we will discuss here what is its main consequence and use. Let us return to our original renewal equation (8) for determining $P(A(t) > x)$. Assuming that F is non-lattice with finite mean, $1/\lambda$, and assuming that the function $Q(t) = \overline{F}(x+t),\ t \geq 0$, is DRI for any x (it is), we conclude from (19) that for all $x \geq 0$,

$$\lim_{t\to\infty} P(A(t) > x) = \lambda \int_x^\infty \overline{F}(t)dt = \overline{F}_e(x).$$

Thus all we have gained is that our already existing "time-average" convergence to the limiting distribution

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P(A(s) > x)ds = \overline{F}_e(x),$$

has been strengthened to *weak convergence* (see definition below). The point is that we already had the limiting distribution for $A(t)$ from the renewal reward theorem, and no special conditions were ever needed beyond requiring that F have finite first moment.

There is nothing special about $P(A(t) > x)$; more generally it can be proved that any positive recurrent *regenerative* process $\{X(t) : t \geq 0\}$ converges weakly to its limiting distribution (which already exists anyhow as a time average) as long as the cycle-length distribution is non-lattice. A regenerative process is one for which there exists a sequence of random times $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots$ forming a renewal process with iid cycle lengths $X_n = \tau_n - \tau_{n-1},\ n \geq 1$, such that at any such time the stochastic process "starts over distributionally" and is independent of the past: $\{X(\tau_n + t),\ t \geq 0\}$ has the same distribution as $\{X(t) : t \geq 0\}$ and is independent of $\{X(s) : 0 \leq s < \tau_n,\ \{\tau_1, \ldots, \tau_n\}\}$. (That it be independent of τ_n is required; $X(\tau_n)$ is independent of τ_n). The τ_n are called regeneration times, the stochastic process is said to *regenerate* at those times, and the process can thus be broken up into iid *cycles*, $\mathcal{C}_n = \{X(\tau_n + t) : 0 \leq t < X_n,\ X_n\}$, describing what the process does during its n^{th} cycle length X_n (and what the length X_n is). The process is called positive recurrent if $E(X) < \infty$ and null recurrent if $E(X) = \infty$.

An easy example is any recurrent continuous-tme Markov chain. Fix any state i, and start off with $X(0) = i$. Then define τ_n as the n^{th} return time to this fixed state i. By the (strong) Markov property, every time the chain revisits state i, the future is independent of the past and proceeds distributionally just as it did at time $t = 0$. Another example is the forward recurrence time process $\{A(t) : t \geq 0\}$ for a renewal process. It starts over again at each arrival time.

Suppose for a positive recurrent regenerative process we wish to find the limiting distribution,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P(X(s) \leq x)ds,\ x \in \mathbb{R}. \tag{20}$$

We can use the renewal reward theorem with reward rate function $r(s) = I\{X(s) \leq x\},\ s \geq 0$, and conclude that the limit exists and is given by $E(R)/E(X)$ where for $X = X_1,\ R = \int_0^X I\{X(s) \leq x\}$. We will study regenerative processes in more detail later; suffice to say:

> *Every positive recurrent regenerative process has a limiting distribution defined as in (20), and given by*
>
> $$\frac{E\Big\{\int_0^X I\{X(s) \leq x\}\Big\}}{E(X)}.$$

If in addition, the cycle length distribution $F(x) = P(X \leq x)$ is non-lattice, then the process also converges weakly to the limiting distribution.

Definition of weak convergence

Definition 3.1 *Given a sequence of rvs $\{X_n\}$ and a rv X (all on $\mathbb{R}$), we say that the distribution of X_n converges weakly to the distribution of X, as $n \to \infty$, denoted by $X_n \Longrightarrow X$, if it holds for all continuous bounded functions $f = f(x)$, that $E(f(X_n)) \to E(f(X))$. (Equivalently, if it holds for all continuous functions f which are bounded by 1, that is, for which $\max_x |f(x)| \leq 1$.)*

It can be shown that if the limiting rv X is continuous, then weak convergence is equivalent to $P(X_n \leq x) \to P(X \leq x)$ for all $x \in \mathbb{R}$; that is, the cdfs converge pointwise.

If X is a discrete rv taking values in $\mathbb{Z}$, then every function f mapping $\mathbb{Z}$ to $\mathbb{Z}$ is continuous, and weak convergence is equivalent to $P(X_n = k) \to P(X = k)$ for all $k \in \mathbb{Z}$; the pmfs converge pointwise.

Remark 4 The key renewal theorem remains valid for delayed renewal processes. To see how this works out, let $F_0 = F$ and then let H_0, Q_0, m_0 denote the non-delayed quantities for a derived renewal equation, that is, we re-write (9) as

$H_0 = Q_0 + H_0 \star F_0$, so as to emphasize the non-delayed situation; the solution is then written as $H_0 = Q_0 + Q_0 * m_0$.

Letting F_1 denote the delay distribution, the renewal equation for the delayed version would change to the form

$$H_1 = Q_1 + H_0 \star F_1. \tag{21}$$

For example (8) becomes

$$P(A_1(t) > x) = \overline{F}_1(x + t) + \int_0^t P(A_0(t - s) > x) dF_1(s).$$

The solution to (21) is similarly derived to be

$$H_1 = Q_1 + Q_0 \star m_1, \tag{22}$$

where $m_1(t) = E(N_1(t))$ is for the delayed version.

The key renewal theorem then becomes:

Theorem 3.2 *If F_0 is non-lattice and Q_0 is DRI, then*

$$\lim_{t \to \infty} Q_0 \star m_1(t) = \lim_{t \to \infty} Q_0 \star m_0(t) = \lambda \int_0^\infty Q_0(t) dt. \tag{23}$$

So if in (21), we have $Q_1(t) \to 0$, then $H_1(t) \to \lambda \int_0^\infty Q_0(t)dt$ yielding the same limit as if the renewal process was nondelayed from the start. For example, even in the delayed case we get (when F is non-lattice) that $A(t)$ converges weakly to F_e. The delay has no effect on the limiting distribution, nor on the weak convergence to it.