1 Uniform integrability

Given a sequence of rvs $\{X_n\}$ for which it is known apriori that $X_n \to X,\ n \to \infty$, wp1. for some r.v. X, it is of great importance in many applications to determine conditions ensuring that $E(X_n) \to E(X),\ n \to \infty$. In other words, conditions ensuring that

$$E(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} E(X_n). \tag{1}$$

To quickly dispense with the notion that conditions might not be needed, we present a simple counterexample: Let U be uniformly distributed over $(0,1)$, and define $X_n = nI\{U \le 1/n\},\ n \ge 1$. Then, wp1., $U > 0$ so for n sufficiently large, $U > 1/n$. Thus $I\{U \le 1/n\} = 0$ for n sufficiently large and we conclude that $X_n \to X = 0,\ n \to \infty$, wp1. But $E(X_n) = 1 \neq 0,\ n \ge 1$. In this section we present the required conditions involving *uniform integrability.* The main result is Theorem 1.4; more applicable sufficient conditions are presented as well. For the most part, proofs are omitted.

1.1 Monotone convergence theorem

The following basic theorem from probability theory yields some sufficient conditions ensuring (1), more general ones will be given shortly:

Theorem 1.1 (Monotone convergence theorem) *If $0 \le X_n \uparrow X$ wp1 as $n \to \infty$, then $E(X_n) \uparrow E(X)$.*

The point here is that $E(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} E(X_n)$ when the rvs are non-negative and monotone $0 \le X_n \le X_{n+1},\ n \ge 1$. (This theorem remains valid even if the limiting rv X has infinite mean ($E(X) = \infty$), in which case $E(X_n) \uparrow \infty$.)

1.2 Dominated convergence theorem, and bounded convergence theorem

The following is perhaps the most useful sufficient condition ensuring (1):

Theorem 1.2 (Dominated convergence theorem) *If $X_n \to X$, wp1, and $\sup_n |X_n| \le Y$ for a rv Y with $E(Y) < \infty$, then $E(X_n) \to E(X)$ and $E(|X|) < \infty$.*

A very special case of the above theorem is when the rv Y is bounded by a constant $b > 0$, that is, $P(Y \le b) = 1$. In this case the theorem is called the *bounded convergence theorem*:

Theorem 1.3 (Bounded convergence theorem) *If $X_n \to X$, wp1, and $\sup_n |X_n| \le b$ for a constant $b > 0$ then $E(X_n) \to E(X)$ and $E(|X|) \le b < \infty$.*

A nice application of the bounded convergence theorem is when $b = 1$: Suppose that for a stochastic process $\{Z_n\}$ it holds (for a given fixed integer i) that as $n \to \infty$,

$$X_n \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=1}^{n} I\{Z_k = i\} \to \alpha(i), \text{ wp1},$$

where $\alpha(i)$ is a constant. Then since $0 \le X_n \le 1 = b$, it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P(Z_k = i) = \alpha(i).$$

1.3 Uniform Integrability

In general, the needed condition ensuring that $E(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} E(X_n)$ is called uniform integrability :

Definition 1.1 *A collection of rvs $\{X_t : t \in T\}$ is said to be uniformly integrable (UI), if $\sup_{t\in T} E(|X_t|I\{|X_t| > x\}) \to 0$, as $x \to \infty$.*

Main Theorem

We state without proof

Theorem 1.4 *If $X_n \to X$, wp1, and $\{X_n\}$ is UI, then $E(X_n) \to E(X)$ and $E(|X|) < \infty$. It also holds (stronger) that $E|X_n - X| \to 0$.*

To motivate the notion of UI, consider

Lemma 1.1 *For any rv X such that $E(|X|) < \infty$,*

$$E(|X|I\{|X| > x\}) \to 0, \text{ as } x \to \infty. \tag{2}$$

Thus every rv X such that $E(|X|) < \infty$ is by itself UI.

Proof : $0 \le |X|I\{|X| \le x\}$ is monotone increasing in x to $|X|$, and hence using the monotone convergence theorem yields $E(|X|I\{|X| \le x\}) \to E(|X|)$ as $x \to \infty$. But since

$$E(|X|) = E(|X|I\{|X| > x\}) + E(|X|I\{|X| \le x\}),$$

and by assumption $E(|X|) < \infty$, we conclude that $E(|X|I\{|X| > x\}) \to 0$. ■

Noting that

$$E(|X|I\{|X| > x\}) = xP(|X| > x) + \int_x^\infty P(|X| > y)dy, \tag{3}$$

we see that $xP(|X| > x)$ and the remainder, $\int_x^\infty P(|X| > y)dy$, both converge to 0, as $x \to \infty$, when $E(|X|) < \infty$.

The following says that UI implies that the expected values are bounded:

Proposition 1.1 *If* $\{X_t : t \in T\}$ *is UI, then* $\sup_{t\in T} E(|X_t|) < \infty$.

Proof : If $\{X_t : t \in T\}$ is UI, then $\exists x > 0$, large, such that $\sup_{t\in T} E(|X_t|I\{|X_t| > x\}) \leq 1$; thus regardless of $t \in T$,

$$E(|X_t|) = E(|X_t|I\{|X_t| \leq x\}) + E(|X_t|I\{|X_t| > x\}) \leq x + 1 < \infty.$$

■

The following explains why the dominated convergence theorem is a consequence of Theorem 1.4:

Lemma 1.2 *If* $\sup_{t\in T} |X_t| \leq Y$ *for a rv* Y *with* $E(Y) < \infty$*, then* $\{X_t : t \in T\}$ *is UI.*

Proof : $|X_t|I\{|X_t| > x\} \leq YI\{Y > x\}$, and $E(YI\{Y > x\}) \to 0$ since $E(Y) < \infty$ (recall (2)); the result follows. ■

Remark 1.1 If $\{X_n\}$ are non-negative with $E(X_n) < \infty$ for each n, and $X_n \to X$ wp1 then $\{X_n\}$ is UI if and only if $E(X_n) \to E(X)$ and $E(X) < \infty$. In general if the rvs are not necessarily non-negative, but $E(|X_n|) < \infty$ for each n, and $X_n \to X$ wp1, then UI is a bit stronger than $E(X_n) \to E(X)$ and $E|X| < \infty$, for it also implies that $E|X_n - X| \to 0$ and that $E(|X_n|) \to E(|X|)$.

Remark 1.2 Lemma 1.2 is frequently used, in practice, in a "stochastic" sense: One shows that there exists a tail $\overline{F}(x)$ of a distribution (such as C/x^k), such that $P(|X_t| > x) \leq \overline{F}(x)$, $x \geq 0$, for all t (or all $t \geq t_0$ some t_0). The point is that one need not actually produce a rv Y, one only needs a tail that could serve as the tail of such a Y. For example $\overline{F}(x) = C/x^k$ is the tail of a Pareto distribution.