0.1 Uniform integrability

Given a sequence of rvs $\{X_n\}$ for which it is known apriori that $X_n \to X,\ n \to \infty$, wp1. for some r.v. X, it is of great importance in many applications to determine conditions ensuring that $E(X_n) \to E(X),\ n \to \infty$. In other words, conditions ensuring that

$$E(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} E(X_n). \tag{1}$$

To quickly dispense with the notion that conditions might not be needed, we present a simple counterexample: Let U be uniformly distributed over $(0,1)$, and define $X_n = nI\{U \le 1/n\},\ n \ge 1$. Then, wp1., $U > 0$ so for n sufficiently large, $U > 1/n$. Thus $I\{U \le 1/n\} = 0$ for n sufficiently large and we conclude that $X_n \to X = 0,\ n \to \infty$, wp1. But $E(X_n) = 1 \ne 0,\ n \ge 1$. In this section we present the required conditions involving *uniform integrability.* The main result is Theorem 0.2; more applicable forms of the theorem are presented as well.

0.1.1 Monotone convergence theorem

The following basic theorem from probability theory yields some sufficient conditions ensuring (1), more general ones will be given shortly:

Theorem 0.1 (Monotone convergence theorem) *If $0 \le X_n \uparrow X$ wp1 as $n \to \infty$, then $E(X_n) \uparrow E(X)$.*

The point here is that $E(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} E(X_n)$ when the rvs are non-negative and monotone $0 \le X_n \le X_{n+1},\ n \ge 1$. (This theorem remains valid even if the limiting rv X has infinite mean $(E(X) = \infty)$, in which case $E(X_n) \uparrow \infty$.)

Proof : The proof utilizes *Fatou's Lemma*, which asserts that for any sequence of non-negative rvs $\{X_n\}$, it holds that $E(\underline{\lim}\, X_n) \le \underline{\lim}\, E(X_n)$[1]. In the present application, $\underline{\lim}\, X_n = \lim X_n = X$, yielding $E(X) \le \underline{\lim}\, E(X_n)$. Moreover $\underline{\lim}\, E(X_n) \le \overline{\lim}\, E(X_n)$ (always), and since $X_n \le X$, we have $E(X_n) \le E(X),\ n \ge 0$, and hence $\overline{\lim}\, E(X_n) \le E(X)$. Putting all this together yields

$$E(X) \le \underline{\lim}\, E(X_n) \le \overline{\lim}\, E(X_n) \le E(X);$$

$\lim E(X_n) = E(X)$ as was to be shown. ■

Note that for a fixed non-negative rv X with $E(X) < \infty$ we have that $0 \le XI\{X \le x\}$ is monotone increasing in x to X, and hence using the monotone convergence theorem yields $E(XI\{X \le x\}) \to E(X)$ as $x \to \infty$. But since $E(X) = E(XI\{X > x\}) +$

[1] $\underline{\lim}\, x_n \stackrel{\text{def}}{=} \lim_{n\to\infty} y_n$ where $y_n \stackrel{\text{def}}{=} \inf\{x_k : k \ge n\}$; y_n is non-decreasing hence its limit exists (possibly $\pm\infty$). Similarly, $\overline{\lim}\, x_n \stackrel{\text{def}}{=} \lim_{n\to\infty} z_n$,where $z_n \stackrel{\text{def}}{=} \sup\{x_k : k \ge n\}$; z_n is non-increasing hence its limit exists (possibly $\pm\infty$). A simple example to consider is when $\{x_n\} = \{1, -1, 2, -2, 3, -3, \ldots\}$, in which case $\underline{\lim}\, x_n = -\infty$, and $\overline{\lim}\, x_n = +\infty$. In general, $\underline{\lim}\, x_n \le \overline{\lim}\, x_n$, and x_n is said to converge if $\underline{\lim}\, x_n = \overline{\lim}\, x_n$; the limit x is defined as this common value, and we write $\lim_{n\to\infty} x_n = x$, or simply $x_n \to x$.

$E(XI\{X \leq x\})$ we conclude that $E(XI\{X > x\}) \to 0$ yielding that for any rv X such that $E(|X|) < \infty$,

$$E(|X|I\{|X| > x\}) \to 0, \text{ as } x \to \infty. \tag{2}$$

Noting that

$$E(|X|I\{|X| > x\}) = xP(|X| > x) + \int_x^\infty P(|X| > y)dy, \tag{3}$$

we see that $xP(|X| > x)$ and the remainder, $\int_x^\infty P(|X| > y)dy$ converge to 0, as $x \to \infty$, when $E(|X|) < \infty$.

0.1.2 Definition of uniform integrability; the dominated convergence theorem

In general, the needed condition ensuring that $E(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} E(X_n)$ involves (2) and is called uniform integrability :

Definition 0.1 *A collection of rvs $\{X_t : t \in T\}$ is said to be uniformly integrable (UI), if $\sup_{t\in T} E(|X_t|I\{|X_t| > x\}) \to 0$, as $x \to \infty$; that is, (2) holds uniformly over all members of the family.*

From (2) we have (trivially)

every rv X such that $E(|X|) < \infty$ is itself UI.

Our main result is:

Theorem 0.2 *If $X_n \to X$, wp1, and $\{X_n\}$ is UI, then $E(X_n) \to E(X)$ (and $E(|X|) < \infty$).*

We will prove the theorem in the next section, but at first give a useful sufficient condition ensuring UI (Lemma 0.1), and a widely used simpler version of the theorem (Theorem 0.3):

Lemma 0.1 *If $\sup_{t\in T} |X_t| \leq Y$ for a rv Y with $E(Y) < \infty$, then $\{X_t : t \in T\}$ is UI.*

Proof : $|X_t|I\{|X_t| > x\} \leq YI\{Y > x\}$, and $E(YI\{Y > x\}) \to 0$ since $E(Y) < \infty$ (recall (2)); the result follows. ■

Corollary 0.1 *Any finite collection of rvs $X_1, \ldots, X_k$ such that $E(|X_i|) < \infty$, $1 \leq i \leq k$, is UI.*

Proof : Choose $Y = |X_1| + \cdots + |X_k|$ and apply Lemma 0.1. ■

The following important result is a corollary of Theorem 0.2 by way of Lemma 0.1:

Theorem 0.3 (Dominated convergence theorem) *If $X_n \to X$, wp1, and $\sup_n |X_n| \leq Y$ for a rv Y with $E(Y) < \infty$, then $E(X_n) \to E(X)$ and $E(|X|) < \infty$.*

Proof :
By Lemma 0.1, $\{X_n\}$ is UI and so Theorem 0.2 applies. ■

In the special case when Y is bounded, e.g., $Y \leq b$, for some constant b, the above theorem is called the *bounded convergence theorem*.

Definition 0.2 *For a fixed $0 < p < \infty$, a collection of rvs $\{X_t : t \in T\}$ is said to be bounded in L^p if $\sup_{t \in T} E(|X_t|^p) < \infty$. A sequence of rvs $X_n \in L^p$ is said to converge to $X \in L^p$, denoted by $X_n \xrightarrow{L^p} X$, if $E|X_n - X|^p \to 0$, as $n \to \infty$.*

Proposition 0.1 *If $\{X_t : t \in T\}$ is UI, then it is bounded in L^1: $\sup_{t \in T} E(|X_t|) < \infty$.*

Proof : If $\{X_t : t \in T\}$ is UI, then $\exists x > 0$, large, such that $\sup_{t \in T} E(|X_t| I\{|X_t| > x\}) \leq 1$; thus regardless of $t \in T$,

$$E(|X_t|) = E(|X_t| I\{|X_t| \leq x\}) + E(|X_t| I\{|X_t| > x\}) \leq x + 1 < \infty.$$

■

Proposition 0.2 *$\{X_t : t \in T\}$ is UI if and only if it is bounded in L^1 and satisfies: $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\sup_{t \in T} E(|X_t|; A) < \epsilon$ whenever $P(A) < \delta$. (Here, A is an event and $(|X_t|; A)$ denotes $|X_t| I\{A\}$.)*

Proof : If part: Suppose $\{X_t : t \in T\}$ is UI. Then it is bounded in L^1 from Proposition 0.1. For $\epsilon > 0$, choose $x > 0$ large enough so that $\sup_{t \in T} E(|X_t| I\{|X_t| > x\}) < \epsilon/2$. Note that

$$\begin{aligned} |X_t| I\{A\} &= |X_t| I\{A\} I\{|X_t| \leq x\} + |X_t| I\{A\} I\{|X_t| > x\} \\ &\leq x I\{A\} + |X_t| I\{|X_t| > x\}, \end{aligned}$$

yielding $E(|X_t|; A) \leq x P(A) + \epsilon/2$. Choosing $\delta = \epsilon/2x$ we see that for any A with $P(A) < \delta$, $\sup_{t \in T} E(|X_t|; A) < \epsilon$.
Only if part: If bounded in L^1, then by Markov's inequality

$$\begin{aligned} P(|X_t| > x\}) &\leq E(|X_t|)/x && (4) \\ &\leq \sup_{t \in T} E(|X_t|)/x, && (5) \end{aligned}$$

which tends to 0 as $x \to \infty$. For $\epsilon > 0$ fixed, choose $\delta > 0$ such that $\sup_{t \in T} E(|X_t|; A) < \epsilon$ whenever $P(A) < \delta$. Then choosing x large enough yields (via (5)) $P(|X_t| > x\}) < \delta$ for all t, and thus $\sup_{t \in T} E(|X_t| I\{|X_t| > x\}) \leq \epsilon$. Since $\epsilon > 0$ can be chosen arbitrarily small, the result follows. ■

Corollary 0.2 *if $\{X_t : t \in T\}$ is UI, and $Z \geq 0$ is any (proper) rv, then*

$$\sup_{t \in T} E(|X_t| I\{Z > x\}) \to 0,$$

as $x \to \infty$.

Proof : Since $P(Z > x) \to 0$, as $x \to \infty$, the result follows from Proposition 0.2. ■

0.1.3 Proof of Theorem 0.2

*Proof :*It suffices to show that if $X_n \to X$ wp1, and $\{X_n\}$ is UI, then $X_n \xrightarrow{L^1} X$, that is, $E|X_n - X| \to 0$. This is sufficient since $|E(X_n) - E(X)| = |E(X_n - X)| \le E|X_n - X|$. To this end, first note that $\sup E(|X_n|) < \infty$ by Proposition 0.1, hence from Fatou's lemma $E|X| \le \underline{\lim}\, E(|X_n|) \le \sup E(|X_n|) < \infty$; $E(|X|) < \infty$. Continuing: Choosing any $\epsilon > 0$ no matter how small, we re-write

$$
\begin{aligned}
E|X_n - X| &= E(|X_n - X| I\{|X_n - X| \le \epsilon\}) + E(|X_n - X| I\{|X_n - X| > \epsilon\}) && (6)\\
&\le \epsilon + E(|X_n - X| I\{|X_n - X| > \epsilon\}) && (7)\\
&\le \epsilon + E(|X_n| I\{|X_n - X| > \epsilon\}) + E(|X| I\{|X_n - X| > \epsilon\}) && (8)
\end{aligned}
$$

As $n \to \infty$, $P(|X_n - X| > \epsilon) \to 0$ since $X_n \to X$, wp1 (e.g., convergence wp1 implies convergence in probability). Thus $E(|X_n| I\{|X_n - X| > \epsilon\}) \to 0$ as $n \to \infty$ by UI of $\{X_n\}$ via Proposition 0.2. Finally, $E(|X| I\{|X_n - X| > \epsilon\}) \to 0$ as $n \to \infty$ by UI of X (itself) via Proposition 0.2; the proof is complete. ■

Proposition 0.3 *Suppose $X_n \to X$, wp1., as $n \to \infty$, and $E(|X_n|) < \infty$, $n \ge 0$. Then the following are equivalent:*

1. *$\{X_n : n \ge 0\}$ is UI;*
2. *$E(|X|) < \infty$ and $E(|X_n - X|) \to 0$, as $n \to \infty$;*
3. *$E(|X|) < \infty$ and $E(|X_n|) \to E(|X|)$, as $n \to \infty$.*

Proof : We already proved that 1 implies 2 in our proof of Theorem 0.2, so next we prove that 2 *implies* 1 *:* First note that $\sup E|X_n| \le \sup E|X_n - X| + E|X| < \infty$ since $E|X_n - X| \to 0$. Similarly for any A

$$
\begin{aligned}
E(|X_n|; A) &\le E(|X_n - X|; A) + E(|X|; A) && (9)\\
&\le E|X_n - X| + E(|X|; A). && (10)
\end{aligned}
$$

Now fix $\epsilon > 0$ and choose N such that $E|X_n - X| < \epsilon/2$, $n \ge N$. For this N fixed, choose $\delta > 0$ such that both $\sup_{n \le N} E(|X_n|; A) < \epsilon$ and $E(|X|; A) < \epsilon/2$, whenever $P(A) < \delta$. This is possible since (N being finite) $\{X_0, X_1, \ldots, X_N\}$ is UI as is X. Then whenever $P(A) < \delta$, we conclude that $\sup_n E(|X_n|; A) \le \epsilon$.

2 implies 3: From the triangle inequality we get $E|X_n| + E|X_n - X| \le E|X| \le E|X_n| + E|X_n - X|$; taking $n \to \infty$ yields the result.

3 *implies* 1*:* First, for $x > 0$ fixed, construct a non-negative bounded continuous function satisfying:

$$
f(y) = \begin{cases} |y|, & \text{if } |y| \le x \,; \\ \le x, & \text{if } x < y < x+1; \\ 0, & \text{if } y \ge x+1. \end{cases}
$$

It is easy to do so, just connect, in a continuous way, the two points $(-x, |x|)$ and $(x, |x|)$ to $(-(x+1), 0)$ and $(x+1, 0)$ respectively, while keeping below x.

Note that by continuity, $f(X_n) \to f(X)$, wp1, and hence by the bounded convergence theorem, $E(f(X_n)) \to E(f(X))$. Moreover, the shape of $f(y)$ implies that $E|X_n| I\{|X_n| \le x+1\} \ge E(f(X_n))$ and $E(f(X)) \ge E|X| I\{|X| \le x\}$. Thus

$$\underline{\lim}\, E[|X_n| I\{|X_n| \leq x+1\}] \geq E[|X| I\{|X| \leq x\}].$$

Using this together with the assumed $E|X_n| \to E|X|$ yields

$$\overline{\lim}\, E[|X_n| I\{|X_n| > x+1\}] \leq E[|X| I\{|X| > x\}],$$

and this is so for all $x > 0$. The RHS $\downarrow 0$ since $E|X| < \infty$, so for $\epsilon > 0$, choose x_0 large enough so that $E[|X| I\{|X| > x\}] < \epsilon \ \forall x \geq x_0$. For this x_0, choose $N = N(x_0)$ large enough so that (via definition of $\overline{\lim}$) $\sup_{n>N} E[|X_n| I\{|X_n| > x_0 + 1\}] < \epsilon$. Now choose $x' \geq x_0$ large enough so that $\sup_{n \leq N} E[|X_n| I\{|X_n| > x' + 1\}] < \epsilon$, possible since a finite collection of rvs in L^1 is UI. Then, we conclude that $\sup_{n \geq 0} E[|X_n| I\{|X_n| > x + 1\}] < \epsilon$ for x sufficiently large; this completes the proof (that $\{X_n\}$ is UI) since ϵ was arbitrary.
■