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Equivalence of Categories for Affine Varieties

1 Introduction

Recall from last time that every affine algebraic variety $V \subset \mathbb{A}^n$ determines a unique finitely generated, reduced $\mathbb{C}$-algebra, namely, its coordinate ring $\mathbb{C}[V]$. Let us first recall how this coordinate ring was defined.

For a polynomial f in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, we view f as a function from $\mathbb{C}^n$ into $\mathbb{C}$ in the natural way. We define an equivalence relation on the ring $\mathbb{C}[x_1, \ldots, x_n]$ by declaring two polynomials to be equivalent if they restrict to the same function on the affine variety V. We then let $\mathbb{C}[V]$ denote the set of equivalence classes of polynomials under this equivalence relation. Because the zero element of the ring consists precisely of those polynomials which vanish everywhere on V, it is not too hard to believe that we have an isomorphism

$$\mathbb{C}[V] \simeq \mathbb{C}[x_1, \ldots, x_n]/\mathbb{I}(V)$$

where $\mathbb{I}(V)$ denotes the ideal determined by the variety V, namely, the set of polynomials given by

$$\mathbb{I}(V) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for each } x \in V\}.$$

This means that we have a mapping

$$\begin{aligned} \{\text{affine varieties}\} &\longrightarrow \{\text{finitely-generated, reduced } \mathbb{C}\text{-algebras}\} \\ V &\longmapsto \mathbb{C}[V]. \end{aligned}$$

One can hope that this mapping is a bijective mapping, meaning that every finitely-generated, reduced $\mathbb{C}$-algebra arises as the coordinate ring for a unique (up to isomorphism) variety. The amazing fact of algebraic geometry is that this is indeed the case. Furthermore, this correspondence extends beyond a set-theoretic bijection to a category-theoretic equivalence. That is, one can show that there is an equivalence of categories between the category of affine algebraic varieties and the category of finitely-generated, reduced $\mathbb{C}$-algebras.

Let us pause here to make a brief digression on categories so that we can later state the extended category-theoretic version of the correspondence mentioned in the previous paragraph.

2 Categories

The notion of a category makes precise the notion of a class of mathematical objects together with maps between these objects. That is, to specify a category C, we need to specify

(i) A collection of mathematical objects which make up the *objects* of C and

(ii) For each pair of objects x and y of C, a collection $\mathsf{C}(x, y)$ of structure-preserving maps called *morphisms* between these objects.

Furthermore, we will demand some extra structure on this data, such as the existence of identity morphisms and the existence of an associative composition function for these morphisms. More precisely, we have the following definition.

Definition 2.1. A *category* C consists of the following data, equipped with the following structure, subject to the following axioms.

DATA:

(i) A collection $\mathrm{Obj}(\mathsf{C})$ of *objects* (written $x, y, \ldots$).

(ii) For each pair of objects x, y in C, a collection $\mathsf{C}(x, y)$ of *morphisms* from x into y (written $f, g, \ldots$).

STRUCTURE:

(i) For each object x in C, a distinguish morphism 1_x in $\mathsf{C}(x, x)$ called the *identity morphism* for x.

(ii) For each triple of objects x, y, z in C, a function

$$\begin{aligned} \circ : \mathsf{C}(y, z) \times \mathsf{C}(x, y) &\longrightarrow \mathsf{C}(x, z) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

called *composition*.

AXIOMS:

(i) The identity morphism is a unit for composition in the sense that for each pair of objects x, y in C and each morphism $f \in \mathsf{C}(x, y)$, we have $1_y \circ f = f = f \circ 1_x$.

(ii) Composition is associative in the sense that for each quadruple of objects w, x, y, z in C and each triple of morphisms $f \in \mathsf{C}(w, x)$, $g \in \mathsf{C}(x, y)$, $h \in \mathsf{C}(y, z)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Example 2.2. The category Set of sets is formed in the following way.

(i) The objects of Set are sets.

(ii) For two sets S, S', the collection of morphisms $\mathsf{Set}(S, S')$ is just the collection of set-theoretic functions $S \to S'$.

(iii) For each set S, the identity morphisms is the identity function on S.

(iv) Composition of morphisms is given by composition of functions in the usual way.

It is readily checked that the identity morphisms are identities for composition and that composition is associative.

Example 2.3. The category Vect_k of vector spaces over the field k is formed in the following way.

(i) The objects of Vect_k are vector spaces over the field k.

(ii) For two vector spaces V, V', the collection of morphisms $\mathsf{Vect}_k(V, V')$ is given by the collection of all linear maps $V \to V'$.

(iii) For each vector space V, the identity morphisms is the identity function on V (which is a linear map).

(iv) Composition of morphisms is given by composition of functions in the usual way.

Again, it is easy to see that the axioms for a category are satisfied.

Example 2.4. Many mathematical constructions give rise to categories, and we list a few below. We leave it to the interested reader to expand and formulate the precise notions of each of the following categories.

(i) The category Top of topological spaces and continuous maps.

(ii) The category Grp of groups and homomorphisms.

(iii) The category Ab of abelian groups and homomorphisms.

(iv) The category Rng of rings and ring homomorphisms.

(v) The category Alg of $\mathbb{C}$-algebras and morphisms of $\mathbb{C}$-algebras.

(vi) The category RedAlg of finitely-generated reduced $\mathbb{C}$-algebras and morphisms of $\mathbb{C}$-algebras.

(vii) The category Aff of affine algebraic varieties and morphisms of affine algebraic varieties.

Because a category is a type of mathematical object itself, one might ask: Can we form a category Cat of categories? The answer to this question is yes. However, it is not quite clear at this moment what we should take as the structure preserving maps, or *morphisms*, in this category. The next definition makes precise what we mean by a morphism in the category Cat of categories.

Definition 2.5. Let C and C' be categories. A functor from C into C' consists of the following data subject to the following axioms.

DATA:

(i) A function $F : \mathrm{Obj}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{C}')$ from objects of C to objects of C'.

(ii) For each pair of objects x, y in C, a function $F : \mathsf{C}(x, y) \to \mathsf{C}'(Fx, Fy)$.

AXIOMS:

(i) Composition is preserved in the sense that for each triple of objects x, y, z in C and for each pair of morphisms $f \in \mathsf{C}(x, y), g \in \mathsf{C}(y, z)$, we have

$$F(g \circ f) = F(g) \circ' F(f).$$

(ii) Identity morphisms are preserved in the sense that for each object x in C, we have $F(1_x) = 1_{F(x)}$.

We now offer a trivial example of a functor, namely, a "forgetful functor." Although this example is somewhat trivial, we don't offer too many more examples now, since we will see more examples of functors later.

Example 2.6. Construct a functor $\mathsf{Forget} : \mathsf{Grp} \to \mathsf{Set}$ in the following manner.

(i) An object $(G, *, e)$ of Grp is mapped to the underlying set $\mathsf{Forget}(G, *, e); = G$ in Set.

(ii) If $f : G \to H$ is a group homomorphism, we let $\mathsf{Forget}(f) : G \to H$ denote the underlying set-theoretic function.

It is routine to check that this construction indeed defines a functor from $\mathsf{Grp} \to \mathsf{Set}$. This functor is called a *forgetful functor*, because it "forgets" the extra information carried around in the category Grp. Analogously, one may define forgetful functors $\mathsf{Rng} \to \mathsf{Set}$ and $\mathsf{Vect} \to \mathsf{Set}$.

With an appropriate type of morphism between categories, we may form the category Cat of categories and functors. The data of this category are categories as objects and functors as morphisms between these objects. It is routine to construct the additional structure (of composition and identities) and check that the axioms of a category are satisfied.

3 Equivalence of categories

We are now almost poised to formulate the equivalence of algebra and geometry that was mentioned in the first section. What we would like to say is the following: The categories Aff of affine algebraic varieties and RedAlg of finitely-generated reduced $\mathbb{C}$-algebras are equivalent. But first we must ask ourselves: What do we mean by equivalent? The first part of this section will be devoted to answering this question, and upon answering this question, we will be able to state the equivalence we have in mind.

So let us begin by asking our question again.

Question. What do we mean by an equivalence of categories?

A naive guess would be the following. It is surely not the case that the categories Aff and RedAlg are equal "on the nose" because they are comprised of completely different objects and morphisms. So the notion of equivalence should not mean outright equality.

Nevertheless, in mathematics, we are often concerned with classifying objects "up to isomorphism," since the isomorphism class of an object is what really distinguishes it from others in its category. For example, in the category of vector spaces, we know that any two n-dimensional vector spaces are isomorphic, which means that we may regard any two such spaces as having the same essence in the category Vect. However, objects of different dimension should be regarded as different in this category.

Perhaps an analogy is in order. The words "two" and "zwei" seem different at first glance, each is composed of different letters and has a different length. However, they both have the same essence, in the sense that they convey the same concept: both mean the integer $1 + 1$ in different languages (English and German respectively). Because of this, we should regard the words as being essentially the same, because they convey the same concept, although in different languages. In a similar fashion, any two-dimensional vector space conveys the same concept as any other two-dimensional vector space in Vect, despite perhaps being expressed in a different mathematical language.

We can extend the notion of isomorphism class of an object very easily to the category Cat. Indeed, morphisms in this category are just functors, so we may define a functor $F : \mathsf{C} \to \mathsf{D}$ to be an *isomorphism* if there is a functor $G : \mathsf{D} \to \mathsf{C}$ such that $G \circ F = 1_{\mathsf{C}}$ and $F \circ G = 1_{\mathsf{D}}$.

Therefore, a guess would at an answer to our question would be to say that an "equivalence of categories" is just an isomorphism of categories. However, it turns out that this guess is wrong, and we need to be more lax in what we mean by an equivalence of categories.

To see why the notion of isomorphism is perhaps too strong, let us consider an example. Let $\underline{\mathsf{Set}}$ denote the category of all *finite* sets. In this category, the isomorphism class of a set S is completely determined by its cardinality $|S|$, which is a natural number. This suggests that we should be able to construct an equivalent category N of natural numbers, which contains all of the isomorphism class data of the category $\underline{\mathsf{Set}}$. Indeed, we can let N denote the category whose objects are sets of the form $[n] = \{1, \ldots, n\}$ for natural numbers n and whose morphisms are just set-theoretic maps $[n] \to [n']$. Then the category N contains all the desired data of the category $\underline{\mathsf{Set}}$ in the sense that every object of $\underline{\mathsf{Set}}$ is isomorphic to some object of N.

However, we claim that there is no isomorphism $F : \underline{\mathsf{Set}} \to \mathsf{N}$ in the category Cat (where we view $\underline{\mathsf{Set}}$ and N as objects in this category). The reason there can be no isomorphism is because the category $\underline{\mathsf{Set}}$ is *much* bigger than the category N. Indeed, if we were to try to construct an inverse $G : \mathsf{N} \to \underline{\mathsf{Set}}$, the image of the corresponding map $G : \mathrm{Obj}(\mathsf{N}) \to \mathrm{Obj}(\underline{\mathsf{Set}})$ on objects would have a countable image, but there are uncountably many objects in $\underline{\mathsf{Set}}$! (In fact, for each n, there are uncountably many sets with cardinality n.) This means that we could never find an inverse G for F.

Thus, we see that the notion of isomorphism of categories is too strong in the sense that it fails to explain the behavior we want even in the most basic of examples. Hence, we must rethink what we mean by an "equivalence of categories," since isomorphism is not correct.

To see how we should introduce the correct notion of equivalence of categories, let us try to "correct" what went wrong in our example above. The problem arose in trying to construct a map $G : \mathrm{Obj}(\mathsf{N}) \to \mathrm{Obj}(\underline{\mathsf{Set}})$ that was surjective. But surjective is perhaps too strong of a condition anyway. What we would really like to say is that G is "essentially surjective," in the sense that for each finite set S, there is an object $[n] \in \mathsf{N}$ such that $G([n])$ is isomorphic to S. Indeed, we can do this easily. We just send each object $[n]$ to itself (viewed as an object in $\underline{\mathsf{Set}}$). We could also get an "essentially surjective" functor $F : \underline{\mathsf{Set}} \to \mathsf{N}$ going the other way as well, by declaring $F(S) = [|S|]$, where $|S|$ denotes the cardinality of S. In this way, we obtain a pair of functors F and G which are almost inverses to each other.

Let us explain that last sentence a little further. Note that on the one hand, we certainly have that the composition $F \circ G$ is the identity functor 1_{N} on N. On the other hand, what can we say about the composition $G \circ F : \underline{\mathsf{Set}} \to \underline{\mathsf{Set}}$? Well, for a finite set S with cardinality n, we see that $(G \circ F)(S) = [n]$. Thus $(G \circ F)$ is not quite the identity morphism $1_{\underline{\mathsf{Set}}}$ on objects. However, it is close to being the identity morphism in the sense that the resulting object is isomorphic to the one that we started with, that is, $S \simeq [n]$. More precisely, note that for each object S in $\underline{\mathsf{Set}}$, we have an isomorphism $\eta_S : S \to (G \circ F)(S)$. This observation hints at a more general behavior and motivates for us the notion of a natural transformation of functors.

Definition 3.1. Let C and D be categories and let F and G be two functors from C into D. A *natural transformation from F into G* consists of an assignment $x \mapsto \eta_x$ of a morphism $\eta_x \in \mathsf{D}(F(x), G(x))$ to each object x in C such that the following property holds: for each pair of objects x, y in C and each morphism $f \in \mathsf{C}(x, y)$, we have $G(f) \circ \eta_x = \eta_y \circ F(f)$, that is, the following diagram commutes

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"] & F(y) \arrow[d, "\eta_y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd} .$$

In such a case, we write that $\eta : F \to G$ is a natural transformation from F into G. The natural transformation η is called a *natural isomorphism* if for each object x in C, the morphism η_x is an isomorphism.

Continuing the previous example, we can see that the functor $G \circ F : \underline{\mathsf{Set}} \to \underline{\mathsf{Set}}$ is naturally isomorphic to the identity. Indeed, before we noted that for each object S in $\underline{\mathsf{Set}}$, we have an isomorphism $\eta_S : S \to (G \circ F)(S)$. The assignment

$S \mapsto \eta_S$ defines the natural isomorphism from $1_{\underline{\mathsf{Set}}}$ to $G \circ F$. We leave it to the reader to check that the necessary diagrams commute.

Exercise 3.2. Show that for $F : \underline{\mathsf{Set}} \to \mathsf{N}$ and $G : \mathsf{N} \to \underline{\mathsf{Set}}$ as above, the assignment $S \mapsto \eta_S$ where η_S is an isomorphism from S onto $[n]$ defines a natural isomorphism from $1_{\underline{\mathsf{Set}}}$ to $G \circ F$.

Because this example motivates what our notion of equivalence of categories should be, we are led to introduce the following definition.

Definition 3.3. Let C and D be categories. An *equivalence* of C and D consists of the following data.

(i) A functor $F : \mathsf{C} \to \mathsf{D}$,

(ii) A functor $G : \mathsf{D} \to \mathsf{C}$,

(iii) A natural isomorphism $\eta : 1_{\mathsf{C}} \to GF$, and

(iv) A natural isomorphism $\epsilon : FG \to 1_{\mathsf{D}}$.

If such an equivalence exists, we say that the categories C and D are *equivalent*.

Exercise 3.4. The notion of equivalence defines an equivalence relation on the set of objects in Cat.

4 Equivalence between algebra and geometry

Now that we have an appropriate notion of equivalence for categories, we can formulate the equivalence of algebra and geometry mentioned in the first section. What we would like to say is that the categories Aff of affine algebraic varieties and RedAlg are equivalent. To formulate this equivalence, we would need to construct, in particular, a functor $F : \mathsf{Aff} \to \mathsf{RedAlg}$. We already have a candidate for one such functor in mind, namely, the functor which assigns to each variety V its coordinate ring $\mathbb{C}[V]$. Moreover, I discussed in a previous meeting how a morphism $f : V \to W$ of affine varieties induces a morphism $f^* : \mathbb{C}[W] \to \mathbb{C}[V]$ of coordinate rings, called the *pullback*. However—and here is where the subtlety lies—note that this does not define a functor in the strict sense: the pullback morphism $f^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is going the wrong way! Nevertheless, we would still like to say that the above assignments define a functor. To do so, we simply introduce the opposite category Aff^{op}, which simply amounts to reversing all of the morphisms in the category Aff.

Definition 4.1. Let C be a category. The *opposite category* C^{op} is the category which has the same objects of C, but whose morphisms are given in the following way: whenever $f : x \to y$ is a morphism in C, then we get a morphism $f^{op} : y \to x$ in C^{op}.

Exercise 4.2. Check that the opposite category C^{op} indeed defines a category.

Using this notion, we can state the main result.

Theorem 4.3. *The categories* Aff^{op} *and* RedAlg *are equivalent.*

To prove this theorem, there are many things to show. In particular, we must construct two functors $F : \mathsf{Aff}^{op} \to \mathsf{RedAlg}$ and $G : \mathsf{RedAlg} \to \mathsf{Aff}^{op}$ and natural isomorphisms $\eta : 1_{\mathsf{Aff}^{op}} \to GF$ and $\epsilon : FG \to 1_{\mathsf{RedAlg}}$. We outline all of these constructions below, and we leave some of the details to the reader.

Lemma 4.4. *The assignments given below describe a functor* $F : \mathsf{Aff}^{op} \to \mathsf{RedAlg}$.

(i) To each affine variety $V \subset \mathbb{A}^n$*, we assign its coordinate ring*

$$F(V) := \mathbb{C}[V] \simeq \mathbb{C}[x_1, \dots, x_n]/\mathbb{I}(V).$$

(ii) To each morphism $f : V \to W$ *of affine varieties, we assign its pullback* $f^* : \mathbb{C}[W] \to \mathbb{C}[V]$*. The pullback* $f^* : \mathbb{C}[W] \to \mathbb{C}[V]$ *is defined in the following manner. Say that* V *is a subset of* $\mathbb{A}^m$ *and* W *is a subset of* $\mathbb{A}^n$*. For a polynomial* $g \in \mathbb{C}[x_1, \dots, x_m]$*, let* $[g]$ *denote its equivalence class in* $\mathbb{C}[W]$*. Then the pullback* $f^* : \mathbb{C}[W] \to \mathbb{C}[V]$ *is the map defined by*

$$f^*([g]) = [g \circ f] \quad \text{for } g \in \mathbb{C}[x_1, \dots, x_m].$$

To prove this lemma, we would need to show many things. In particular, we would need to show

(i) the pullback f^* is a well-defined map,

(ii) the pullback f^* is a morphism in the category RedAlg,

(iii) composition is preserved, and

(iv) identity morphisms are preserved.

We leave it to the reader to check these things.

Exercise 4.5. Check that $F : \mathsf{Aff} \to \mathsf{RedAlg}$ indeed defines a functor.

Lemma 4.6. *The assignments given below describe a functor* $G : \mathsf{RedAlg} \to \mathsf{Aff}^{op}$.

(i) To each finitely-generated reduced $\mathbb{C}$*-algebra* R*, we associate an affine variety* $G(R)$ *formed in the following way. We choose a generating set* $r_1, \dots, r_n$ *for* R *as an algebra, and this induces a surjective* $\mathbb{C}$*-algebra homomorphism* $\phi : \mathbb{C}[x_1, \dots, x_n] \to R$*. We then set* $G(R)$ *to be the variety determined by the kernel of this homomorphism* $G(R) = \mathbb{V}(\ker \phi)$.

(ii) To each morphism $f : R \to S$ of finitely-generated reduced $\mathbb{C}$-algebras, we associate a morphism $G(f) : G(S) \to G(R)$ of affine varieties formed in the following way. Choose presentations for R and S of the form

$$\mathbb{C}[x_1, \ldots, x_n]/I \xrightarrow{f} \mathbb{C}[y_1, \ldots, y_m]/J$$

where I and J are radical ideals. For each j satisfying $1 \leq j \leq n$, let $f_j^\#$ denote any polynomial in $\mathbb{C}[y_1, \ldots, y_m]$ representing the image $f(x_j)$. Define a polynomial map

$$\begin{aligned} \widetilde{f} : \mathbb{A}^m &\longrightarrow \mathbb{A}^n \\ a = (a_1, \ldots, a_m) &\longmapsto (f_1^\#(a), \ldots, f_n^\#(a)). \end{aligned}$$

Then let $G(f) = f^\#$ denote the restriction of $\widetilde{f}$ to $G(S)$.

Proof. We check first that $G(f)$ is a map from $G(S)$ to $G(R)$. Let $V = G(S) = \mathbb{V}(J)$ and let $W = G(R) = \mathbb{V}(I)$. Let a be an element of V. We want to show that $f^\#(a)$ is an element of W. To show this, it suffices to show that $f^\#(a)$ belongs to the zero set of any polynomial g in the ideal I. Using the definition of $f^\#$, we see that for each $g \in I$, we have

$$\begin{aligned} g(f^\#(a)) &= g(f_1^\#(a), \ldots, f_n^\#(a)) \\ &= g((f(x_1))(a), \ldots, (f(x_n))(a)) \\ &= (f(g))(a). \end{aligned}$$

Since g belongs to the ideal I, it represents the zero class in $\mathbb{C}[x_1, \ldots, x_n]/I$ and because f is a ring homomorphism, the image $f(g)$ must represent the zero class in $\mathbb{C}[y_1, \ldots, y_m]/J$. This means that $f(g)$ lies in the ideal J, and thus vanishes for any point $a \in V = \mathbb{V}(J)$. Hence, we have shown that $f^\#(a)$ belongs to W for any point $a \in V$.

To complete the proof, it remains to show that composition of morphisms is preserved and that identities are preserved. We leave it to the reader to show this. □

Corollary 4.7. *Let R be a finitely-generated reduced $\mathbb{C}$-algebra. Suppose that we may present R in two different forms:*

$$R \simeq \mathbb{C}[x_1, \ldots, x_m]/J \simeq \mathbb{C}[y_1, \ldots, y_m]/I.$$

Then the varieties $\mathbb{V}(J)$ and $\mathbb{V}(I)$ are isomorphic.

Proof. This follows from the fact that $G : \mathsf{RedAlg} \to \mathsf{Aff}$ is a functor, and hence preserves isomorphisms. □

Remark 4.8. Note that the functor G depends on many choices, and as such, is somewhat of a bad functor. In particular, we had to choose for each object R in RedAlg, a corresponding generating set $r_1, \ldots, r_n$. Moreover, to define the morphism $G(f)$ in the proof, we had to choose representatives $f_j^{\#}$ of the image $f(x_j)$ in the quotient ring $\mathbb{C}[y_1, \dots, y_m]/J$. Choosing any of these things differently might result in a different functor G! However, one can show that any of these resulting functors would be naturally isomorphic to each other, and so up to natural isomorphism, just considering the functor G will suffice.

Exercise 4.9. Show that if G' is another functor resulting from a different choice of generating set for objects of RedAlg, then G' is naturally isomorphic to G. On the other hand, show that if G'' is another functor defined the same way as G on objects, but the morphisms $G''(f)$ are determined by choosing different representatives for the images $f(x_j)$ in $\mathbb{C}[y_1, \dots, y_m]/J$, then G'' is *equal* to G.

Exercise 4.10. Let V and W be affine algebraic varieties. Show that we have a one-to-one correspondence

$$\{\text{morphisms } V \to W\} \longleftrightarrow \{\mathbb{C}\text{-algebra homomorphisms } \mathbb{C}[W] \to \mathbb{C}[V]\}$$

described by $f \mapsto f^*$.

Proof. To prove this claim, it suffices to construct an inverse for the mapping $f \mapsto f^*$. The inverse is precisely the assignment of the previous lemma. More precisely, suppose that

$$\mathbb{C}[x_1, \dots, x_n]/\mathbb{I}(W) \xrightarrow{g} \mathbb{C}[y_1, \dots, y_m]/\mathbb{I}(V)$$

is a morphism of coordinate rings. For each j satisfying $1 \leq j \leq n$, choose a representative $g_j^{\#}$ of the image $g(x_j)$ in $\mathbb{C}[y_1, \dots, y_m]/\mathbb{I}(V)$. Define a polynomial map

$$\begin{aligned} \widetilde{g} : \mathbb{A}^m &\longrightarrow \mathbb{A}^n \\ a &\longmapsto (g_1^{\#}(a), \dots, g_n^{\#}(a)) \end{aligned}$$

and let $g^{\#}$ denote the restriction of $\widetilde{g}$ to V. Then we claim that the assignment $g \mapsto g^{\#}$ defines an inverse to $f \mapsto f^*$.

Let $g : \mathbb{C}[W] \to \mathbb{C}[V]$ be a morphism of coordinate rings. Consider the pullback $(g^{\#})^*$ of $g^{\#} : W \to V$. For any $h \in \mathbb{C}[x_1, \dots, x_n]$, by the definition of the pullback, we have

$$(g^{\#})^*([h]) = [h \circ g^{\#}]$$

where $[h \circ g^{\#}]$ denotes the restriction of the map $h \circ g^{\#} : \mathbb{A}^m \to \mathbb{A}^n$ to the variety V. This restriction is precisely $h \circ g^{\#}$ itself. So we conclude that $(g^{\#})^*([h]) = h \circ g^{\#}$. Evaluating this on a point $a \in V$, we find that

$$\begin{aligned} (g^{\#})^*([h])(a) &= (h \circ g^{\#})(a) \\ &= h(g_1^{\#}(a), \dots, g_n) \\ &= h((g(x_1))(a), \dots, (g(x_n))(a)) \\ &= (g([h]))(a). \end{aligned}$$

This means that the functions $(g^\#)^*([h])$ and $g([h])$ agree everywhere on V and hence define the element in $\mathbb{C}[V]$. This shows that the morphisms $(g^\#)^*$ and g are equal. Hence the assignment $f \mapsto f^*$ is a left inverse to the assignment $g \mapsto g^\#$.

To conclude the proof, we also need to show that the assignment $f \mapsto f^*$ is also a right inverse to the assignment $g \mapsto g^\#$. We leave this as an exercise. $\square$

Theorem 4.11. *The categories Aff^{op} and RedAlg are equivalent.*

Proof. We are required to find natural isomorphisms $\eta : 1_{\mathsf{Aff}^{op}} \to GF$ and $\epsilon : FG \to 1_{\mathsf{RedAlg}}$.

Let us first consider how to construct $\eta : 1_{\mathsf{Aff}^{op}} \to GF$. Note that if V is an affine variety, then $F(V)$ it is coordinate ring. Now, to form $G(F(V))$, we choose a presentation of $F(V)$ of the form

$$F(V) \simeq \mathbb{C}[x_1, \ldots, x_n]/I$$

and we let $G(F(V)) = \mathbb{V}(I)$. Note that the presentation of $F(V)$ may not be the same as the presentation of the coordinate ring, that is, we may not have $I \simeq \mathbb{I}(V)$. However, the presentation of $F(V)$ is isomorphic to the coordinate ring $\mathbb{C}[V]$, and so Corollary 4.7 asserts that $G(F(V))$ is isomorphic to V. We let $\eta_V : V \to G(F(V))$ denote such an isomorphism.

To see how to construct $\epsilon : FG \to 1_{\mathsf{RedAlg}}$, note that for a finitely-generated reduced $\mathbb{C}$-algebra R, we form the variety $G(R)$ by choosing a presentation for R of the form

$$R \simeq \mathbb{C}[y_1, \ldots, y_m]/J$$

and setting $G(R) = \mathbb{V}(J)$. Because the ideal J is radical, Hilbert's Nullstellensatz guarantees that $\mathbb{I}(\mathbb{V}(J)) = J$, which implies that $F(G(R))$ is isomorphic to $\mathbb{C}[y_1, \ldots, y_m]/J$ and consequently R. Thus we get an isomorphism $\epsilon_R : F(G(R)) \to R$ for each object R in RedAlg. $\square$

Let us conclude our discussion by noting that the previous theorem is less than ideal. Indeed, there are two concerns with this theorem. First, the choice of functor $G : \mathsf{RedAlg} \to \mathsf{Aff}$ relied upon far too many choices in its construction. Although different choices result in functors which are naturally isomorphic to each other, it is somewhat bad practice to use constructions that depend on choices in mathematics, because such constructions are not very formulaic. Secondly, the domain the functor G was very specific: we were only concerned with those $\mathbb{C}$-algebras that are finitely-generated and reduced. One could hope to eliminate the "reduced" clause and find a more general categories of varieties which are equivalent to generalizations of the category RedAlg. This motivates us to introduce the notion of the *spectrum of a ring* and the notion of a *scheme.*