Overview of Financial Engineering

Why Are Financial Markets Necessary?

Financial markets exist to enable the efficient allocation of resources across time and states of nature.

Example 1

Consider a young worker with a very high salary. What should she do with her earnings: spend and consume them all immediately? No: she would probably want to invest them in the financial markets with various objectives related to retirement, home ownership, children’s education, capital growth etc.

If there were no financial markets available to her, what could or would she do?

Example 2

Q: Assuming that the young worker above has access to financial markets, where exactly does her money go?

A: (Via the stock and bond markets) To corporations and agents who use it to create products / ideas / wealth.

Moreover, her money will often go to those corporations or agents who are best qualified to use it! Why?

Contrast this with the the mechanism used by a dictator or central planner?

Example 3

Consider a farmer who produces wheat. What should the farmer do with his wheat: make bread and then consume it all? No: he should sell it of course. Note that it will be much easier to sell it if there is a market for wheat.

Given that the farmer will sell his wheat, should he sell it in the spot market or the futures market?

Efficient markets are essential for a successful economy. This statement is well supported by historical data.

Question: Can you name another structure or system that is also essential for a successful economy?

Modelling Financial Markets

When we study financial engineering we must decide what type of model we will use for the financial markets. A number of possibilities exist:

- Discrete-time models
  - single-period models
  - multi-period models
- Continuous-time models
In this course we will focus on discrete-time models. The advantage of discrete time models is that less sophisticated mathematics are required to study them. On the other hand, while continuous-time models require more sophisticated mathematics, they often lead to explicit solutions to problems. These problems would generally have to be solved numerically in discrete-time models. All of the important concepts of financial engineering can be studied in discrete-time models, however, and they will be the focus of this course.

### Central Problems of Financial Engineering

There are many problems that may be classified as financial engineering problems. Perhaps the most obvious such problems are: securities pricing, risk management and portfolio optimization.

1. **Securities Pricing**

   We can typically identify two classes of securities: *primitive* securities and *derivative* securities. Examples of primitive securities are stocks and bonds whereas options, futures and swaps are examples of derivative securities. It is probably fair to say that financial economics is more concerned with pricing primitive securities, usually using equilibrium arguments (e.g. supply = demand) to do so. Financial engineering is typically more concerned with pricing derivative securities and uses *arbitrage* arguments to do so. This distinction is not hard, however, and sometimes it is necessary to use equilibrium arguments when pricing derivative securities. Moreover, some models such as the Capital Asset Pricing Model (CAPM), are equilibrium-based models that are of fundamental importance to both financial economists and financial engineers.

2. **Risk Management**

   Risk management is concerned with understanding the *risks* that are inherent to your portfolio of securities. For example, you might be interested in determining \( P(W_T/W_0 \leq .8) \), i.e., the probability that you will have lost more than 20% of your wealth by time \( T \). If this probability is unacceptably high then you need to adjust your portfolio to reduce this probability. There are a number of interesting and challenging questions that are related to risk management.

3. **Portfolio Optimization**

   At the most basic level, portfolio optimization is the problem of choosing a *trading strategy* with the goal of optimizing some objective function that measures the performance of the portfolio. For example, you may wish to solve

   \[
   \max_{\theta} \ E[u(W_T)] \quad \text{subject to various constraints}
   \]

   where \( \theta \) is the (possibly dynamic) trading strategy, \( W_T \) is terminal wealth and \( E[u(W_T)] \) is the objective function. Later in the course we will see how the problems of security pricing and portfolio optimization are intimately related.

There are of course many other problems and applications of financial engineering. They include problems in corporate finance (e.g. structuring deals, real options), accounting, applied mathematics (probability theory, control theory), economics and econometrics, and risk management more generally.

### The No-Arbitrage Assumption

The no-arbitrage assumption is probably the most important assumption in finance theory, and it is the assumption that generally distinguishes finance from the remainder of economics. We will put this assumption to work many times in this course. Basically it states that it is not possible to get something for nothing, or alternatively, “there is no free lunch”.

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*Overview of Financial Engineering*
Example 4 (Arbitrage)

Bank $A$ is willing to borrow and lend money at 6% while bank $B$ is willing to borrow and lend at 8%.

Q: Why is this an opportunity to "get something for nothing"?

Q: Suppose more realistically, that bank $A$ borrows at $r^A_B\%$ and lends at $r^A_L\%$, and that bank $B$ borrows at $r^B_B\%$ and lends at $r^B_L\%$. What conditions must $r^A_B$, $r^A_L$, $r^B_B$ and $r^B_L$ satisfy if no arbitrage opportunity can exist?

Q: Why can arbitrage opportunities not exist (at least for very long) in the real world?

We will define arbitrage more carefully when we study martingale pricing later in the course.

Some Motivating Examples

Example 5 (Risk Aversion and the St. Petersberg Paradox)

Consider a game where a fair coin is tossed repeatedly until the first head appears. If the first head appears on the $n^{th}$ toss, then you will receive $2^n$. How much, $X$, would you be willing to pay in order to play this game?

The expected payoff, $E[P]$, is given by

$$E[P] = \sum_{n=1}^{\infty} 2^n P(1^{st} \text{ head on } n^{th} \text{ toss}) = \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \infty.$$  

Given the result in (1), would you now choose $X = \infty$?

Daniel Bernoulli resolved this paradox by introducing the idea of a utility function, $u(x)$, defined on levels of wealth, $x$. The interpretation of $u(x)$ is that it measures how much utility or benefit someone obtains from holding $x$ units of wealth. Different people will have different utility functions. What properties should $u(.)$ have?

1. Increasing, decreasing or neither?
2. Convex, concave or neither?

The particular utility function that Bernouilli introduced was the log($\cdot$) utility function. In particular, if an individual has log utility then his expected utility is given by

$$E[u(P)] = \sum_{n=1}^{\infty} \log(2^n) P(1^{st} \text{ head on } n^{th} \text{ toss}) = \log(2) \sum_{n=1}^{\infty} \frac{n}{2^n} < \infty.$$ 

Many other utility functions\textsuperscript{1} are of course possible.

Exercise 1 How much do you think the opportunity to play the game is worth to you if you have log utility?

When we study financial engineering problems, we sometimes (though not always) need to understand what the preferences of the decision-maker are. If the decision-maker is an individual, we often assume that he is risk-averse and endow him with an appropriate utility function.

Example 6 (The Binomial Model)

Consider the following example of a financial market. There are 3 periods, one risky asset and one risk-free asset. At any time, $t$, the value of the risky asset, $S_t$, will either increase by a factor, $u$, or decrease by a factor, $d$, over the next period. The possible evolutions of $S_t$ are given in Figure 1 where $u = 1.06$, $d = 1/u$ and $S_0 = 100$. The risk-free asset is a cash account so that $1$ invested in it at $t = 0$ will be worth $(1 + r)^t$ dollars at time $t$, where $r$ is the interest rate per period. A number of interesting questions arise:

\textsuperscript{1}The theory of expected utility was formalized in the 1950’s by John Von Neumann and Oscar Morgenstern. It has been a cornerstone of economics and finance ever since.
1. What is the value of an option that pays \( h(S) := (S - 95)^+ \) at time \( t = 3 \)?
   (a) do we have enough information to answer this question?
   (b) as in Example 5, shouldn’t the price somehow depend on the utility functions of the buyer and seller?
   (c) will the price depend on the probability of an upmove in each period?

2. (Risk management) Suppose you stand to lose a lot of money if at date \( t = 3 \) the risky asset is worth 83.96. On the other hand, you stand to earn a lot of cash if the risky asset is worth 119.10 at \( t = 3 \). Suppose you don’t want this type of exposure. What could you do?

3. Suppose you have \( W_0 = $100,000 \) available for investment at \( t = 0 \). You can invest in both the risky and risk-free assets. What should you do? In particular, what is your optimal investment strategy? What will this strategy depend on?

We will answer these questions and many others during the course.

Example 7 (A More General Tree)

There are two time periods and three securities. We will use \( S_t^{(i)} \) to denote the value of the \( i^{th} \) security on a given node at date \( t \) for \( i = 0, 1, 2 \) and for \( t = 0, 1, 2 \). These values are given in the tree below. For example, the value of the 0th security at date \( t = 2 \) satisfies

\[
S_2^{(0)}(\omega_k) = \begin{cases} 
1.1235, & \text{for } k = 1, 2, 3 \\
1.1025, & \text{for } k = 4, 5 \\
1.0815, & \text{for } k = 6, 7, 8, 9.
\end{cases}
\]

We can pose the questions of Example 6 and others here. We will see later in the course that the answers to these questions depend on (i) the security prices at each node and (ii) the structure of the tree. If we can answer these questions and understand the theory behind these answers then we will have succeeded in understanding most of the important ideas in financial engineering. These ideas will apply even when we work...
with considerably more complicated (e.g. continuous time and continuous space) models than those of Examples 6 and 7. This is not surprising since it should be the case that even the most complicated models can be approximated in some sense by the models of Examples 6 and 7.

**Key:** \([S_t^{(0)}, S_t^{(1)}, S_t^{(2)}]\)

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I_0

[1.05, 1.4346, 1.9692]
I_1

[1, 2.2173, 2.1303] [1.05, 1.9571, 2.2048]

[1.05, 3.4045, 2.4917] I_2

[1.1235, 2, 1] [1.1235, 2, 3] [1.1235, 1, 2] [1.1025, 2, 3] [1.1025, 1, 2] [1.0815, 3, 2] [1.0815, 4, 1] [1.0815, 2, 4] [1.0815, 5, 2]
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\(t = 0\) \(t = 1\) \(t = 2\)

**Challenge Question**

When marketing their equity funds, salesmen often use statements like the following:

"Over any historical 20 year period, stocks have outperformed bonds. Therefore stocks are better securities than bonds for people with long investment horizons."

Suppose the premise of this statement is true (it actually is in the U.S.). How would you respond?