Lectures on Linear and Nonlinear Dispersive Waves
DRAFT IN PROGRESS

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Contents

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1 Some Basic Analysis

1.1 Function spaces

Definition 1.1 *Schwartz class, $\mathcal{S}(\mathbb{R}^n)$, is the vector space of functions which are C^∞ and which, together with all their derivatives, decay faster than any polynomial rate. Specifically, if $f \in \mathcal{S}$, then for any $\alpha, \beta \in \mathbb{N}_0^n$, there exists a constant $C_{\alpha,\beta}$ such that*

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)| \ \le \ C_{\alpha,\beta}$$

$L^p(\mathbb{R}^n)$ spaces, $1 \le p \le \infty$
$\mathcal{S}$ is dense in L^p

Theorem 1.1 *(Approximation of the identity) Let $K(x) > 0$ and $\int K(x)dx \ = \ 1$. Define $K_N(x) = N^n K(Nx)$, $N \ge 1$.*
Let f be a bounded and continuous function on $\mathbb{R}^n$ and consider the convolution

$$K_N \star f(x) \ = \ \int K_N(x-y)\ f(y)\ dy \tag{1.1}$$

Then, $K_N \star f(x) \ \rightarrow \ f(x)$ uniformly on any compact subset, C, of $\mathbb{R}^n$ as $N \uparrow \infty$, i.e. $\max_{x \in C} |\ [K_N \star f](x) - f(x)| \rightarrow 0$.

1.2 Linear operators

Definition 1.2 *A linear transformation, T, between vector spaces X_1 and X_2...*

Definition 1.3 *A bounded linear transformation...*

Example 1.1 *Symmetric matrix n by n*

Theorem 1.2 *Let $T : E \rightarrow Y$ be a BLT.*

1.3 Fourier Transform

For $f \in \mathcal{S}$ define the *Fourier transform*, $\hat{f}$ or $\mathcal{F}f$ by

$$\hat{f}(\xi) \ = \ \mathcal{F}f(\xi) \ = \ \int e^{-2\pi i x \cdot \xi} f(x)\ dx \tag{1.2}$$

Proposition 1.1 *Assume $f \in \mathcal{S}$.*

(a) $\hat{f} \in C^\infty$ *and* $\partial^\beta \hat{f}(\xi) \ = \left[\ (-2\pi i x)^\beta f(x)\ \right]\hat{}\,(\xi)$

(b) $\widehat{\partial^\beta f}(\xi) \ = \ (2\pi i \xi)^\beta\ \hat{f}(\xi)$

(c) $\hat{f} \in \mathcal{S}$*. Thus, the Fourier transform maps* $\mathcal{S}$ *to* $\mathcal{S}$*.*

Theorem 1.3 *Riemann-Lebesgue Lemma*
$f \in L^1(\mathbb{R}^n) \implies \lim_{\xi\to\infty} \hat{f}(\xi) = 0.$

Proof: Approximate by step functions, for which the result can be checked. Note: no rate of decay of the Fourier transform is implied by $f \in L^1$.

Theorem 1.4 *Let $f(x) \equiv e^{-\pi a|x|^2}$. Then,*

$$\hat{f}(\xi) = a^{-\frac{n}{2}}\, e^{-\pi\frac{|\xi|^2}{a}} \tag{1.3}$$

Proof: Write out definition of $\hat{f}$. Note that the computation factors into computing the Fourier transform of n independent one dimensional Gaussians. For the one-dimensional Gaussian, complete the square in the exponent, deform the contour using analyticity of the integrand, and finally use that $\int_{\mathbb{R}} e^{-\pi y^2} dy = 1$.

1.4 Fourier inversion on $\mathcal{S}$ and L^2

Definition: For $g \in \mathcal{S}$ define $\check{g}$

$$\check{g}(x) \equiv \int e^{2\pi i\xi\cdot x} g(\xi)\, d\xi = \hat{g}(-x)$$

Proposition 1.2

$$\int \hat{f}\ g\ dx = \int f\ \hat{g}\ dx \tag{1.4}$$

Proof: Interchange orders of integration (Fubini's Theorem).

Theorem 1.5 *Fourier inversion formula Assume $f \in \mathcal{S}$. Then,*

$$f \in \mathcal{S} \implies \check{g}(x) = f(x), \quad \text{where } g(\xi) = \hat{f}(\xi) \tag{1.5}$$

Proof: We shall prove Fourier inversion in the following sense.

$$\lim_{\varepsilon\to 0} \int e^{-\pi\varepsilon^2|\xi|^2}\ e^{2\pi i x\cdot\xi}\ \hat{f}(\xi)\ d\xi = f(x)$$

For any $\varepsilon > 0$ define

$$\phi_\varepsilon(\xi) = e^{2\pi i x\cdot\xi\ -\ \pi\varepsilon^2|\xi|^2}$$

whose Fourier transform is

$$\hat{\phi}_\varepsilon(y) = \frac{1}{\varepsilon^n}\, e^{-\pi\frac{|x-y|^2}{\varepsilon^2}} = \frac{1}{\varepsilon^n} g\left(\frac{x-y}{\varepsilon}\right) \equiv g_\varepsilon(x-y)$$

Now,

$$\begin{aligned}\int e^{-\pi\varepsilon^2|\xi|^2}\, e^{2\pi i x\cdot\xi}\, \hat f(\xi)\, d\xi &= \int \phi_\varepsilon(\xi)\, \hat f(\xi)\, d\xi \\ &= \int \hat\phi_\varepsilon(y)\, f(y)\, dy \\ &= \int g_\varepsilon(x-y)\, f(y)\, dy \;\to\; f(x)\end{aligned}$$

as $\varepsilon \downarrow 0$ because g_ε is an *approximation of the identity*; see Theorem 1.1

Theorem 1.6 *(Plancherel Theorem) Assume* $f \in \mathcal{S}$*. Then,*

$$\|\hat f\|_2 \;=\; \|f\|_2$$

Therefore, the Fourier transform preserves the L^2 *norm on* $\mathcal{S}$*. Furthermore, for any* $f, g \in \mathcal{S}$

$$\int f\, \overline{g}\, dx \;=\; \int \mathcal{F}[f]\, \overline{\mathcal{F}[g]}$$

Corollary 1.1 *The Fourier transform can be extended to a unitary operator defined on all* L^2 *such* $\|\hat f\|_2 \;=\; \|f\|_2$.

Proof: A BLT argument. $\mathcal{S}$ is dense in L^2. If $f \in L^2$, there exists a sequence $f_j \in \mathcal{S}$ such that $\|f_j - f\|_2 \to 0$. Define $\hat f = \lim_{j\to\infty} \hat f_j$.

1.5 Sobolev spaces on $\mathbb{R}^n$

Definition 1.4 *(1) For any* $s \in \mathbb{R}$*, define the* $H^s(\mathbb{R}^n)$ *norm for* $u \in \mathcal{S}(\mathbb{R}^n)$ *by*

$$\|u\|^2_{H^s} \;=\; \int \;|\hat u(\xi)|^2\, (1+|\xi|^2)^s\, d\xi. \tag{1.6}$$

(2) The Sobelev space $H^s = H^s(\mathbb{R}^n)$ *is defined to be the completion of* $\mathcal{S}(\mathbb{R}^n)$ *with respect to the norm* $\|\cdot\|_{H^s}$*.*

The following estimate shows the important connection between the control of derivatives of a function in L^2 with the *pointwise* behavior of a function.

Theorem 1.7 *(Sobelev Lemma) Let* k *be a non-negative integer. Let* $u \in H^s(\mathbb{R}^n)$*, where* $s > k + n/2$*. Then,* u *is almost everywhere (in the measure theoretic sense) equal to a function of class* C^k*. Moreover, for any* $\alpha \in N^n$ *with* $|\alpha| = k$*, we have* $\partial^\alpha u \in C_\downarrow(\mathbb{R}^n)$ *and there exists a constant* $C > 0$ *depending only on* s, α, n*, but not on* u*, such that*

$$\|\partial^\alpha u\|_{L^\infty} \;\le\; C\, \|u\|_{H^s}.$$

The connection between the general L^q behavior of a function and that of its Sobelev space regularity also plays an important role, quite often in nonlinear problems. We will use the following special case of the Sobelev-Nirenberg-Gagliardo estimate:

Theorem 1.8 *Let f be in H^1. Then, f is almost everywhere equal to an L^q function for $2 \leq q < \frac{2q}{n-2}$ if $n \geq 3$ and all $q \geq 2$ if $n = 1$ or $n = 2$. Furthermore,*

$$\begin{aligned} \| \ f \ \|_{L^q} \leq \ C_{q,n} \ \|f\|_{H^1}, \ \text{and in fact} \\ \| \ f \ \|_{L^q} \leq \ C_{q,n} \ \|\nabla f\|_{L^2}^{n\frac{q-2}{2q}} \ \ \|f\|_{L^2}^{1-n\frac{q-2}{2q}} \end{aligned} \tag{1.7}$$

1.6 Notes and references for section 1

A good short and user-friendly introduction to basic functional analysis appears in Chapter 0 of G.B. Folland's text on PDEs [1]. Volume 1 of Reed-Simon's series, Functional Analysis [3], contains a discussion, going considerably further.

2 Linear dispersive PDEs - introduction

In this section we introduce the notion of *dispersion* and give numerous examples. A good reference is [6].

2.1 Dispersion relations, examples

Consider a system of m linear, constant coefficient, homogeneous partial differential equations:

$$P(i\partial_t, -i\partial_{x_1}, \ldots, -i\partial_{x_n})u(x,t) \ = \ 0 \tag{2.1}$$

Here, for simplicity, we take $P(\tau, k_1, \ldots, k_n)$ to be an n by m matrix, whose entries are polynomials in τ and ξ_k. A *plane wave solution* is a solution of the form $e^{i(k\cdot x - \omega t)}w$, where w is a constant vector in $\mathbb{R}^n$. Substitution into (2.1) yields the system of algebraic equations for v:

$$P(\omega, k_1, \ldots, k_n) \ w \ = \ 0$$

A non-trivial plane wave solution exists if and only if

$$G(\omega, k) \ \equiv \ \det P(\omega, k) \ = \ 0. \tag{2.2}$$

The relation (2.2) is called the dispersion relation of the system of PDEs. We assume it defines m real-valued branches of the form

$$\omega \ = \ \omega(k), \ \ \text{for which } G(\omega(k), k) = 0 \tag{2.3}$$

Example 2.1 *Transport equation*

$$\partial_t u + v \cdot \nabla u \ = \ 0 \tag{2.4}$$

Dispersion relation: $\omega(k) = v \cdot k$.

Example 2.2 *One dimensional wave equation*

$$c^{-2}\partial_t^2 u \;=\; \partial_x^2 u \tag{2.5}$$

Dispersion relation: $c^{-2}\omega^2(k) - k^2 = 0$, *defining two branches* $\omega_+(k) = ck$ *and* $\omega_-(k) = -ck$.

Example 2.3 *One dimensional Klein-Gordon equation*

$$c^{-2}\partial_t^2 u \;=\; \left(\partial_x^2 \;-\; m^2\right) u,\;\; m > 0. \tag{2.6}$$

Dispersion relation: $c^{-2}\omega^2(k) - (m^2 + k^2) = 0$, *defining two branches* $\omega_+(k) = c\sqrt{m^2+k^2}$ *and* $\omega_-(k) = -c\sqrt{m^2+k^2}$.

Example 2.4 *Free Schrödinger equation*

$$i\hbar\partial_t\ \psi = \;-\frac{\hbar^2}{2m}\ \Delta\psi \tag{2.7}$$

Dispersion relation: $\omega(k) \;=\; \frac{\hbar}{2m}|k|^2$

Example 2.5 *One dimensional beam equation*

$$\partial_t^2 u \;+\; \gamma^2\partial_x^4 u. \tag{2.8}$$

Dispersion relation: $\omega_+(k) = \gamma k^2$ *and* $\omega_-(k) = -\gamma k^2$.

Exercise 2.1 *Show that if* $\psi = U + iV$, *where* U *and* V *are real, satisfies the free Schrödinger equation, then* U *and* V *each statisfy the beam equation.*

Example 2.6 *Linearized KdV aka Airy equation*

$$\partial_t u \;+\; c\partial_x u \;+\; \partial_x^3 u \;=\; 0. \tag{2.9}$$

Dispersion relation: $\omega(k) = ck - k^3$.

Example 2.7 *Linearized BBM*

$$\partial_t u \;+\; c\partial_x u \;-\; \partial_x^2\partial_t u \;=\; 0. \tag{2.10}$$

Dispersion relation: $\omega(k) = \frac{ck}{1+k^2}$.

Example 2.8 *Coupled mode equations*

$$\begin{aligned} \partial_t E_+ \;+\; \partial_x E_+ \;+\; \kappa E_- \;&=\; 0 \\ \partial_t E_- \;+\; \partial_x E_- \;+\; \kappa E_+ \;&=\; 0 \end{aligned} \tag{2.11}$$

Dispersion relation: $\omega_\pm(k) = \pm\sqrt{\kappa^2 + k^2}$. *Note that the first order system (2.11) has the same dispersion relation as the Klein-Gordon equation (2.6), with* $\kappa = m$.

3 Introduction to the Schrödinger equation

In this section we introduce the Schrödinger equation in two ways. First, we mention how it arises in the fundamental description of quantum atomic phenomena. We then show its role in the description of diffraction of classical waves.

3.1 Quantum mechanics

The hydrogen atom: one proton and one electron of mass m and charge e. The state of the atom is given by a function $\psi(x,t)$, complex-valued, defined for all $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. ψ is often called the *wave function*.
Let $\Omega \subset \mathbb{R}^3$. $|\psi(x,t)|^2\ dx$ is a probability measure with the interpretation

$$\int_D |\psi(x,t)|^2\ dx \ = \ \text{Probability (electron } \in \ \Omega \text{ at time } t)$$

Thus, we require

$$\int_{\mathbb{R}^3} |\psi(x,t)|^2\ dx \ = \ \text{Probability}\left(\text{electron } \in \mathbb{R}^3 \text{ at time } t\right) \ = \ 1$$

Given an initial wave function, ψ_0

$$\begin{aligned} i\hbar\, \partial_t \psi \ &= \ H\ \psi \\ H \ &= \ -\frac{\hbar^2}{2m}\, \Delta \ + \ V(x) \end{aligned} \tag{3.1}$$

Here, $\hbar$ denotes Planck's constant divided by 2π. The operator H is called a Schrödinger operator with potential V, a real-valued function determined by the nucleus. For the special case of the hydrogen atom

$$H = \ -\frac{\hbar^2}{2m}\, \Delta \ - \ \frac{e^2}{r}, \quad r = |x| \tag{3.2}$$

The *free electron* (unbound to any nucleus) is governed by the *free Schrödinger equation* ($V \equiv 0$):

$$i\hbar\, \partial_t \psi \ = \ -\frac{\hbar^2}{2m}\, \Delta\psi \tag{3.3}$$

3.2 Free Schrödinger - initial value problem

Initial Value Problem

$$i\partial_t u \ = \ -\Delta\ u, \quad u(x,0) \ = \ f(x) \tag{3.4}$$

Unlike the heat equation, $\partial_t u = \Delta u$, which has an exponentially decaying Gaussian fundamental solution, the fundamental solution of the Schrödinger equation is an oscillatory Gaussian with no spatial decay. For this reason, the derivation of the solution to the initial value problem is more subtle. One approach is to *regularize* the Schrödinger equation by adding a small ($\varepsilon > 0$) diffusive term, which we then take to zero ($\varepsilon \to 0$).

Regularized initial value problem Take $\varepsilon > 0$.

$$i\partial_t u^\varepsilon \;=\; -(1-i\varepsilon)\Delta\; u^\varepsilon, \quad u^\varepsilon(x,0) \;=\; f(x) \tag{3.5}$$

Lemma 3.1

$$f(x) = e^{-\pi a|x|^2}, \quad \Re a \geq 0 \quad \Longrightarrow \quad \hat{f}(\xi) = a^{-\frac{n}{2}} e^{-\frac{\pi}{a}|\xi|^2} \tag{3.6}$$

Solution of regularized IVP

$$\begin{aligned} i\partial_t \hat{u}^\varepsilon &= 4\pi^2(1-i\varepsilon)|\xi|^2\hat{u} \\ \hat{u}^\varepsilon(\xi,t) &= e^{-4\pi^2(i+\varepsilon)|\xi|^2 t}\; \hat{f}(\xi) \\ u^\varepsilon(x,t) &= \int K_t^\varepsilon(x-y)\; f(y)\; dy, \end{aligned} \tag{3.7}$$

where

$$K_t^\varepsilon(x) \;=\; \int e^{-2\pi i x\cdot\xi}\; e^{-4\pi^2(i+\varepsilon)|\xi|^2 t}\; d\xi. \tag{3.8}$$

Here, we have used that for $\varepsilon > 0$, $e^{2\pi i\xi\cdot(x-y)}\; e^{-4\pi^2(i+\varepsilon)|\xi|^2 t}\; f(y) \;\in\; L^1(d\xi dy)$, so we can interchange integrals by Fubini's Theorem.

We now apply Lemma 3.1 with $a = (4\pi(i+\varepsilon)^{-1}$ and obtain

$$K_t^\varepsilon(x) \;=\; (4\pi(i+\varepsilon)t)^{-n/2}\; e^{-\frac{|x|^2}{4t(i+\varepsilon)}}$$

For any $t \neq 0$, if $f \in L^1$, we can pass to the limit $\varepsilon \to 0^+$ by the Lebesgue dominated convergence theorem. We define the free Schrödinger evolution by

$$\begin{aligned} u(x,t) &= e^{i\Delta t} f \;=\; \int K_t(x-y)\; f(y)\; dy \\ K_t(x) &= (4\pi i t)^{-n/2}\; e^{i\frac{|x|^2}{4t}}, \end{aligned} \tag{3.9}$$

where (3.9) is understood as $\lim_{\varepsilon\to 0^+} K_t^\varepsilon \star f$.

For $f \in L^2$, we can use that the Fourier transform is defined (and unitary) on all L^2 to define, by (3.7),

$$u(x,t) \;=\; \Big(e^{-4\pi^2 i|\xi|^2 t}\; \hat{f}(\xi) \Big)^{\check{}}(x,t) \;=\; e^{i\Delta t} f \tag{3.10}$$

3.3 Free Schrödinger in L^p

$e^{i\Delta t}f$ for $f \in L^1(\mathbb{R}^n)$: In this case,

$$|u(x,t)| \;=\; \left| \int K_t(x-y)\; f(y)\; dy \right| \;\leq\; \int |f|\; dy$$

Therefore, if $f \in L^1(\mathbb{R}^n)$, then $e^{i\Delta t} f \in L^\infty(\mathbb{R}^n)$ for $t \neq 0$ and

$$\| \, e^{i\Delta t} f \, \|_{L^\infty} \;\le\; |4\pi t|^{-\frac{n}{2}}| \; \|f\|_{L^1} \tag{3.11}$$

<u>$e^{i\Delta t} f$ **for** $f \in L^2(\mathbb{R}^n)$**:**</u> In this case

$$\int |u(x,t)|^2 dx \;=\; \int |\hat{u}(\xi,t)|^2 d\xi \;=\; \int |e^{-4\pi^2 i|\xi|^2 t} \; \hat{f}(\xi)|^2 d\xi \;=\; \int |f(\xi)|^2 d\xi$$

Thus, if $f \in L^2(\mathbb{R}^n)$, then $e^{i\Delta t} f \in L^2(\mathbb{R}^n)$ and

$$\|e^{i\Delta t} f\|_{L^2} \;=\; \|f\|_{L^2} \tag{3.12}$$

<u>Extension to L^p:</u> Suppose $f \in L^p(\mathbb{R}^n)$ with $1 \le p \le 2$. Using a theorem of M. Riesz on interpolation of linear operators, Theorem 9.1, one can show:

Theorem 3.1 *Let $1 \le p \le 2$ and $2 \le q \le \infty$, where $p^{-1} + q^{-1} = 1$. If $f \in L^p(\mathbb{R}^n)$, then for $t \neq 0$ $e^{i\Delta t} f \in L^q(\mathbb{R}^n)$ and*

$$\|e^{i\Delta t} f\|_{L^q} \;\le\; |4\pi t|^{-(\frac{n}{2} - \frac{n}{q})} \; \|f\|_{L^p} \tag{3.13}$$

3.4 Structural properties of the Schrödinger equation

Invariances properties: The following transformations map solutions of the free Schrödinger equation to solutions of free Schrödinger equation.

(1) Spatial translation: $u(x,t) \mapsto u(x + x_0, t), \; x_0 \in \mathbb{R}^n$

(2) Time translation: $u(x,t) \mapsto u(x, t + t_0), \; t_0 \in \mathbb{R}$

(3) Complex conjugation: $u(x,t) \mapsto u(\bar{x}, t)$

(4) Time-reversal / conjugation: $u(x,t) \mapsto \overline{u(x,-t)}$

(5) Galilean invariance:

$$u(x,t) \mapsto \mathcal{G}_\eta[u](x,t) = e^{i\eta\cdot(x - \eta t)} \; u(x - 2\eta t, t). \tag{3.14}$$

Remark 3.1 *Note that (3.14) contains the two velocities: $v_{phase}(\eta) = \omega(\eta)/\eta = \eta$ and $v_{group}(\eta) = 2\eta$, the group velocity. Energy propagates with the group velocity; see Remark 3.2.*

3.5 Free evolution of a Gaussian wave packet

Consider the evolution of a Gaussian wave-packet in one-space dimension. Let $\eta_0 = 2\pi\xi_0$.

$$\begin{aligned} i\partial_t u &= -\partial_x^2 u \\ u(x,0) &= e^{i\eta_0 x}\ e^{-\frac{x^2}{2L^2}} = g_{L,\eta_0}(x) \end{aligned} \tag{3.15}$$

Thus $u(x,0)$ is an oscillatory and localized initial condition with *carrier oscillation* period ξ_0^{-1} or frequency ξ_0. Its evolution has an elegant and illustrative form:

Theorem 3.2

$$u(x,t) = e^{i\Delta t} g_{L,\eta_0} = \frac{e^{i\eta_0(x-\eta_0 t)}}{\left(1+\frac{2it}{L^2}\right)^{\frac{1}{2}}}\ e^{-\frac{(x-2\eta_0 t)^2}{2L^2(1+\frac{2it}{L^2})}} \tag{3.16}$$

Proof of Theorem 3.2 : The Fourier representation of the solution is:

$$u(x,t) = \int e^{2\pi i\xi x} \hat{g}_{L,2\pi\xi_0}(\xi)\ d\xi \tag{3.17}$$

Note: $\hat{g}_{L,2\pi\xi_0}(\xi) = \hat{g}_{L,0}(\xi-\xi_0) = (2\pi)^{\frac{1}{2}}\ L\ e^{-2\pi^2 L^2(\xi-\xi_0)^2}$. Substitution into (3.17) and grinding away with such tools as completing the square yields the result.

A "better" proof of Theorem 3.2: We prove the result in two steps. Step 1: Treat the case where $\eta_0 = 0$, $u(x,0) = g_{L,0}$. Step 2: Apply the Galilean transformation, $\mathcal{G}_{\eta_0}$, to obtain Theorem for general η_0.
Step 1: $u(t) = e^{i\Delta t} g_{L,0}$ is the convolution of Gaussians. Thus, it is useful to have

Lemma 3.2 *Let $G_a(x) = e^{-a\pi|x|^2}$. Let a and b be such that $\Re a \ge 0$, $\Re b \ge 0$, $a \ne 0$ and $b \ne 0$. Then,*

$$G_a \star G_b = \frac{1}{(a+b)^{\frac{n}{2}}}\ G_{\frac{ab}{a+b}} \tag{3.18}$$

Proof of Lemma 3.2: The Fourier transform of the right hand side of (3.18) is the product of the Fourier transforms, computed using Theorem 1.3. Rewriting this product as a single Gaussian and computing the inverse transform, using Theorem 1.3, gives the result.
Step 2: Note that the Galilean boost: $U(x,t) = \mathcal{G}_{\eta_0}[e^{i\Delta t} g_{L,0}]$ solves the initial value problem for the free Schrödinger equation with initial data $U(x,0) = e^{i\eta_0 x} g_{L,0}$, as desired. This gives the formula (3.16).

Remark 3.2

- *Phase propagates with velocity η_0, the phase velocity*
- *Energy $\sim |u(x,t)|^2$ propagates with velocity $2\eta_0$, the group velocity*
- *Solution disperses (spreads and decays) to zero as $t \uparrow$. This is seen from the general estimate (3.11) as well as the explicit solution (3.16).*
- *However, solution does not decay in L^2. The Schrödinger evolution is unitary in L^2; see (3.12).*
- *Concentrated (sharp) initial conditions (L small) disperse more quickly than spread out initial conditions (L large). The time scale of spreading is $t \sim L^2$.*

3.6 Observables

Recall that $|\psi(x,t)|^2$ has the interpretation of a probability density for a quantum particle to be at position x at time t. $|\hat{\psi}(\xi,t)|^2$ has the interpretation of a probability density for a quantum particle to be at momentum ξ at time t.

The expected value of an *observable or operator* , A, is formally given by[1]

$$\langle A \rangle \;=\; (\psi, A\psi) \;=\; \int \overline{\psi} A \psi \tag{3.19}$$

Theorem 3.3

$$\frac{d}{dt}\langle A \rangle \;=\; i\langle [H,A] \rangle, \tag{3.20}$$

where $[B,A] = BA - AB$.

Examples

(i) $\langle X \rangle$, the average position $= \int x|\psi(x,t)|^2 dx$.

(ii) $\langle \Xi \rangle \;=\; \int \xi |\hat{\psi}(\xi,t)|^2 d\xi$; Let $P_k = -i\partial_{x_k}$, the momentum operator. Then, $\langle P_k \rangle = 2\pi \langle \Xi \rangle$ is the average momentum.

(iii) $\langle |X|^2 \rangle$, the variance or uncertainty in position $= \int |x|^2 |\psi(\xi,t)|^2 dx$.

(iv) $\langle |\Xi|^2 \rangle$, the variance or uncertainty in momentum $= \int |\xi|^2 |\hat{\psi}(\xi,t)|^2 d\xi$.

Exercise 3.1 *Show that* $\langle X_k \rangle(t) = \langle X_k \rangle(0) + 2\langle P_k \rangle(0)\ t$.

3.7 The Uncertainty Principle

Theorem 3.4 *(Uncertainty Inequality) Suppose* xf *and* ∇f *are in* $L^2(\mathbb{R}^n)$. *Then,*

$$\int |f|^2 \;\le\; \frac{2}{n} \left(\int |xf|^2\right)^{\frac{1}{2}} \left(\int |\nabla f|^2\right)^{\frac{1}{2}}$$

or equivalently

$$\int |f|^2 \;\le\; \frac{4\pi}{n} \left(\int |xf|^2\right)^{\frac{1}{2}} \left(\int |\xi \hat{f}|^2\right)^{\frac{1}{2}} \tag{3.21}$$

Exercise 3.2 *(a) Prove the uncertainty inequality, using the pointwise identity*

$$x \cdot \nabla |f|^2 \;=\; \nabla \cdot (x|f|^2) \;-\; n\,|f|^2,$$

(b) Prove that the inequality (3.21) is sharp in the sense that equality is attained for the Gaussian $f(x) = \exp(-|x|^2/2)$.

[1] We proceed formally, without any serious attention to operator domains *etc.* For a fully rigorous treatment, see [3].

Applying the uncertainty inequality (3.21) to a solution of the Schrödinger equation, with initial condition $\|\psi(\cdot,0)\|_{L^2} = 1$ and we have,

$$1 \ = \int |\psi(x,0)|^2 dx = \int |\psi(x,t)|^2 dx \le \frac{4\pi}{n} \left(\int |x\psi(x,t)|^2 \right)^{\frac{1}{2}} \left(\int |\xi\hat{\psi}(\xi,t)|^2 \right)^{\frac{1}{2}} \tag{3.22}$$

The latter, can be written as

$$\frac{n}{4\pi} \ \le \ \sqrt{\langle |X|^2 \rangle (t)} \ \sqrt{\langle |\Xi|^2 \rangle (t)} \tag{3.23}$$

and is called *Heisenberg's uncertainty principle.*

4 Oscillatory integrals and Dispersive PDEs

Consider a scalar constant coefficient partial differential equation

$$\partial_t u \ = \ P(D)\ u, \quad D = (\partial_{x_1}, \ldots, \partial_{x_n}) \tag{4.1}$$

Assuming a real-valued dispersion relation $\omega = \omega(k)$, we have that the solution to the initial value problem with initial data $u(x,0) = g(x)$ has the form

$$u(x,t) \ = \int e^{i(x\cdot\xi - \omega(\xi)t)}\ \hat{g}(\xi)\ d\xi \tag{4.2}$$

Question: What is the large time ($t \to \infty$) behavior of $u(x,t)$?
This question is now considered in the context of the following more general question in the asymptotic analysis of *oscillatory integrals* of the form

$$I(\lambda) \ = \int e^{i\lambda\phi(\xi)}\ h(\xi)\ d\xi, \quad \text{for} \quad \lambda \to \infty \tag{4.3}$$

where $\phi(\xi)$ is a smooth and real-valued a *phase-function.*

Our goal in this section is to make precise the notion that: *as $\lambda \to \infty$, rapid oscillations of $e^{i\lambda\phi(\xi)}$ tend to cancel each other out and the dominant contribution from $I(\lambda)$ comes from a neighborhood of points, ξ, for which $\nabla\phi(\xi) = 0$.*
Here's the idea; when in doubt, integrate by parts. Let's consider the integral: $\int_a^b e^{i\lambda\phi(\xi)} d\xi$. If $\phi' \neq 0$ on $[a,b]$, then

$$\begin{aligned}
\int_a^b e^{i\lambda\phi(\xi)} d\xi \ &= \int_a^b \frac{1}{i\lambda\phi'(\xi)}\ \frac{d}{d\xi} e^{i\lambda\phi(\xi)} d\xi \\
&= \frac{1}{i\lambda\phi'(\xi)}\ e^{i\lambda\phi(\xi)}\ \Big|_{\xi=a}^{\xi=b} \ - \int_a^b e^{i\lambda\phi(s)}\ \frac{d}{ds}\frac{1}{i\lambda\phi'(s)}\ ds
\end{aligned}$$

Therefore, $|\int_a^b e^{i\lambda\phi(\xi)}d\xi| \le C|\lambda|^{-1}\max_{a\le\xi\le b}|\phi'(\xi)|^{-1}$. If there is an interior point ξ_0, $a < \xi_0 < b$, at which $\phi'(\xi_0) = 0$ and $\phi''(\xi_0) \ne 0$. Then, we break the integral into an integral over a small neighborhood of ξ_0 (or, more generally, neighborhoods of any finite set of non-degenerate critical points of ϕ,) and an integral over the complement of this neighborhood. On the complement, $\phi' \ne 0$ so the previous estimate holds. On the small neighborhood of ξ_0, we have $\phi(\xi) \sim \phi(0) + \frac{1}{2}\phi''(\xi_0)(\xi-\xi_0)^2$. Therefore, the contribution from a neighborhood of ξ_0 comes from a Gaussian integral, which can be estimated as $\mathcal{O}\left((|\phi''(\xi_0)|\ |\lambda|)^{-\frac{1}{2}} \right)$. We now proceed with a rigorous treatment.

4.1 Non-stationary phase

Theorem 4.1 *(Non-stationary phase) Let K be a compact subset of $\mathbb{R}^n$. Suppose ϕ is real-valued and defined on a neighborhood, O, of K, such that $\phi \in C^{r+1}(O)$ and such that $\nabla\phi \ne 0$ on K. Let $h \in C_0^r(int(K))$. Then,*

$$| \ I(\lambda) \ | \ \le C\langle\lambda\rangle^{-r} \ \|h\|_{r,\infty}, \tag{4.4}$$

where

$$\|u\|_{r,\infty} \ = \ \sum_{|\alpha|\le r} \|\partial^\alpha u\|_\infty, \tag{4.5}$$

$C = C\left(\max_K |\nabla_\xi\phi|^{-1}, \|\partial\phi\|_{r,\infty}\right)$ and $C(s,s') \to \infty$ as s or s' tend to infinity. In particular, if ϕ and h are C^∞ then $I(\lambda) = \mathcal{O}(\langle\lambda\rangle^{-r})$ for any $r > 0$.

Proof of Theorem 4.1: First note that for $|\lambda| \le 1$ we can use the bound $| \ I(\lambda) \ | \le \|h\|_{L^1}$. Therefore, it suffices to consider $|\lambda| \ge 1$.

We seek an operator $L = \sum_{j=1}^n b^j(\xi)\partial_{\xi_j}$ such that

$$L \ e^{i\lambda\phi(\xi)} \ = \ e^{i\lambda\phi(\xi)}$$

In order to see how to choose b^j, compute:

$$L \ e^{i\lambda\phi(\xi)} \ = \ e^{i\lambda\phi(\xi)} \ \left(\sum_{j=1}^n \ i\lambda \ \partial_{\xi_j}\phi(\xi) \ b^j(\xi) \right)$$

implying the choice

$$b^j(\xi) \ = \ \frac{1}{i\lambda} \ |\nabla\phi(\xi)|^{-2} \ \partial_{\xi_j}\phi(\xi) \tag{4.6}$$

Note that if h has compact support then,

$$\begin{aligned} I(\lambda) \ &= \ \int e^{i\lambda\phi(\xi)} \ h(\xi) \ d\xi \\ &= \ \int L^r(e^{i\lambda\phi(\xi)}) \ h(\xi) \ d\xi \\ &= \ \int e^{i\lambda\phi(\xi)} \ (L^t)^r \ h(\xi) \ d\xi, \end{aligned} \tag{4.7}$$

where

$$L^t \;=\; -\frac{1}{i\lambda}\sum_{j=1}^{n} \partial_{\xi_j}\; \left(\;|\nabla\phi(\xi)|^{-2}\cdot\;\right) \tag{4.8}$$

Since $|e^{i\lambda\phi}| = 1$, we have

$$|\; I(\lambda)\;| \;\le\; \int \;|\;(L^t)^r\; h(\xi)\;|\; d\xi.$$

Note that $(L^t)^r$ contains a factor λ^{-r}. The proof now follows by expanding and estimating the right hand side.

Corollary 4.1 *Let $\hat{g}(\xi)$, appearing in the expression for $u(x,t)$ in (4.2), be smooth and have compact support. Denote by*

$$\Lambda \;=\; \{\nabla_\xi\; \omega(\xi)\;:\;\xi\in\; supp(\hat{g})\;\} \tag{4.9}$$

Let $\mathcal{G}\subset \mathbb{R}^n$ be an open subset which contains Λ. For any $m = 1,2,\ldots,$ there exists a constant $c = c(m,g,\mathcal{G})$, such that for any (x,t) with $x/t\notin\mathcal{G}$:

$$|u(x,t)| \;\le\; c\langle |t|+|x|\rangle^{-m} \tag{4.10}$$

4.2 Stationary Phase

Theorem 4.2 *(Stationary Phase) Let ϕ, defined in a neighborhood of the origin $0\in\mathbb{R}^n$, be a C^∞ and real-valued function. Assume that 0 is a non-degenerate critical point of ϕ, i.e. $\nabla\phi(0) = 0$ and $H_\phi(0) = (\;\phi_{\xi_i\xi_j}(0)\;)$ is invertible. Then, there exist neighborhoods O and O' of the origin and a constant $C>0$, such that for all $h\in C_0^\infty(O)$*

$$|\; I(\lambda)\;| \;\le\; C\;\;|\det H_\phi(0)\;|^{-\frac{n}{2}}\;\;\langle\lambda\rangle^{-\frac{n}{2}}\;\|h\|_{H^s},\quad s>\frac{n}{2} \tag{4.11}$$

Corollary 4.2 *Consider $u(x,t)$, given by (4.2), where $\hat{g}\in C_0^\infty$ and*

$$supp(\hat{g})\;\cap\;\{\xi : \det(\phi_{\xi_i\xi_j})(0) = 0\}\;\;\text{is empty.}$$

Then, for all x,t

$$|u(x,t)| \;\le\; C\;|\det H_\phi(0)|^{-\frac{n}{2}}\;\langle t\rangle^{-\frac{n}{2}} \tag{4.12}$$

The proof of the stationary phase theorem uses the following

Lemma 4.1 *(Morse Lemma)*

Let ϕ satisfy the hypotheses of Theorem 4.2. In particular, $\nabla\phi(0) = 0$ and $H_\phi(0) = (\;\phi_{x_j,x_k}(0)\;)$ is non-singular. There exist open neighborhoods O and O' of the origin and a C^∞ invertible mapping $X : O\to O'$, such that

$$\begin{aligned} X(k) \;&=\; k \;+\; \mathcal{O}(|k|^2)\\ \phi(k) \;&=\; \phi(0) \;+\; \frac{1}{2}\,(X(k), AX(k))\,. \end{aligned} \tag{4.13}$$

for $k\in O$, where here $(a,b) = a^T b$.

Proof of the Morse Lemma: As in the proof of Taylor's Theorem, we have

$$\begin{aligned}
\phi(k) - \phi(0) &= \int_0^1 \frac{d}{dt}\phi(tk)\ dt \\
&= (t-1)\frac{d}{dt}\phi(tk)\ \Big|_0^t \quad - \int_0^t (t-1)\frac{d^2}{dt^2}\phi(tk)\ dt \\
&= \left(k,\ \int_0^1 (1-t)\ H_\phi(tk)\ dt\ k \right) \\
&\equiv \frac{1}{2}\ (k, B(k)k), \quad \text{where} \qquad (4.14)
\end{aligned}$$

$$B_{ij}(k) \ = \ 2 \int_0^1 \ \phi_{k_i k_j}(sk)\ (1-s)\ ds \qquad (4.15)$$

We seek a C^∞ n by n matrix $R(k)$, such that $R^*(k)AR(k) = B(k)$. If we can find such a matrix, $R(k)$, with $R(k) = I + \mathcal{O}(|k|)$, then

$$(k, B(k)k) \ = \ (k, R^*(k)AR(k)k) \ = ([R(k)k], A[R(k)k])$$

and defining $X(k) = R(k)k$ does the trick.

We construct $R(k)$ by applying the implicit function theorem to the matrix equation

$$F(R,B) \ \equiv \ R^*(k)AR(k) \ - \ B(k) \ = \ 0, \qquad (4.16)$$

in a neighborhood of the solution $R = I,\ B = A$.

Here's the setup. Let

(i) $M \ = \$ the vector space of all n by n matrices.

(ii) $M_s \ = \$ the vector space of all n by n symmetric matrices.

Then, the mapping $F : (R,B) \mapsto F(R,B)$ maps $M \times M_s \to M_s$, by the symmetry of A. We also have $F(I,A) = 0$. To apply the implicit function theorem in a neighborhood of (I,A), we first compute the Jacobian $F_R(R,B)$ evaluated at $(R,B) = (I,A)$.

<u>Computation of $F_R(R,B)$</u>:

$$\begin{aligned}
F(R+\epsilon C, B) \ - \ F(R,B) &= (R^* + \epsilon C^*)A(R+\epsilon C) \ - \ R^*AR \\
&= R^*AR \ + \ \epsilon(C^*AR \ + \ R^*AC) \ + \ \mathcal{O}(\epsilon^2) \qquad (4.17)
\end{aligned}$$

Thus,

$$\begin{aligned}
F_R(R,B)\ C &= C^*AR \ + \ R^*AC, \quad \text{and therefore} \\
F_R(I,A)\ C &\equiv T(C) \ = \ C^*A \ + \ AC \qquad (4.18)
\end{aligned}$$

To apply the implicit function theorem, we need to check that $T : M \to M_s$ is one to one and onto.

(a) T is one to one. Proof: Exercise

(b) T is onto: Let $D \in M_s$. We need to show that $C^*A + AC = D$ has a solution. Let $C = \frac{1}{2}A^{-1}D$, which exists because A is assumed invertible. Then one can easily check that $T(C) = D$.

By the implicit function theorem, there exists an neighborhood $\mathcal{A} \subset M_s$ of A and a C^∞ map $R : \mathcal{A} \to M$, such that $F(R(A), A) = 0$, $R(A) = I$. Now choose O, an open neighborhood of $k = 0$, so that $k \in \mathcal{O}$ implies that $B(k) \in \mathcal{A}$, and take $X(k) = R(B(k))k$. This completes the proof.

Proof of Stationary Phase Theorem: Consider the integral

$$I(\lambda) \ = \ \int e^{i\lambda\phi(\xi)} \ h(\xi) \ d\xi.$$

By the Morse Lemma, in a sufficiently small neighborhood, O, of $\xi = 0$ we have

$$\phi(\xi) \ = \ \phi(0) \ + \ \frac{1}{2} \ \langle X(\xi), AX(\xi)\rangle, \ \text{where } A = H_\phi(0) \tag{4.19}$$

is the Hessian matrix of $\phi(\xi)$ at the critical point $\xi = 0$. Assume that h is supported within O. Then,

$$I(\lambda) \ = \ e^{i\lambda\phi(0)} \ \int e^{i\lambda\langle X(\xi),AX(\xi)\rangle/2} \ h(\xi) \ d\xi$$

Set $y = X(\xi)$. Then,

$$\begin{aligned} I(\lambda) \ &= \ e^{i\lambda\phi(0)} \ \int e^{i\lambda\langle y,Ay\rangle/2} \ (h \circ X^{-1})(y) \ \left|\det D_\xi X(X^{-1}(y))\right|^{-1} dy \\ &= \ e^{i\lambda\phi(0)} \ \int v(y) \ \overline{e^{-i\lambda\langle y,Ay\rangle/2}} \ dy, \ \ \text{where} \\ v(y) \ &= \ (h \circ X^{-1})(y) \ \left|\det D_\xi X(X^{-1}(y))\right|^{-1} \end{aligned}$$

By the Plancherel Theorem 1.6,

$$\begin{aligned} |\ I(\lambda)\ | \ &= \ \left| \int \mathcal{F}v(\eta) \ \overline{\mathcal{F}e^{-i\frac{\lambda\langle y,Ay\rangle}{2}}(\eta)} \ d\eta \right| \\ &\le \ C \ |\det(A)\lambda|^{-\frac{n}{2}} \ \int |\mathcal{F}v(\eta)| \ d\eta \\ &\le \ C' \ |\det(A)\lambda|^{-\frac{n}{2}} \ \|v\|_{H^s} \\ &\le \ C'' \ |\det(H_\phi(0)) \ \lambda|^{-\frac{n}{2}} \ \|h\|_{H^s}, \ \ \ s > n/2 \end{aligned} \tag{4.20}$$

4.3 Degenerate dispersion and Van der Corput's Lemma

It may happen that the oscillatory integral, $I(\lambda)$, has points of stationary phase, which are *degenerate*. We consider an example:

Airy / linearized KdV:

For concreteness, consider the initial value problem for the linearized KdV (Airy equation); see Example 2.6:

$$\partial_t u \ + \ \partial_x^3 u \ = \ 0, \ \ u(x,0) = g(x) \tag{4.21}$$

The dispersion relation, as noted in section 2, is $\omega(k) = -k^3$. For a large class of initial conditions, the solution can be represented, via the Fourier transform (see section 1) as a superposition of plane waves:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int e^{i(kx-\omega(k)t)}\hat{g}\left(\frac{k}{2\pi}\right)\ dk \\ &= \int K_t(x-y)\ g(y)\ dy, \quad \text{where} \\ K_t(x) &= \frac{1}{2\pi}\int\ e^{i(kx-\omega(k)t)}\ dk \end{aligned} \tag{4.22}$$

The large time, $t \to \infty$, asymptotics of $u(x,t)$ are governed by those of $K_t(x)$. By sections 4.1 and 4.2, the large time behavior of K_t is governed by a neighborhood of points, where the phase $\phi(k;x,t) = kx/t + k^3$, is stationary. Thus we consider points, k_0, for which

$$\frac{x}{t} - \omega'(k) \ = \ \frac{x}{t} \ + \ 3k^2 \ = \ 0, \quad i.e.\ k_0^2(x,t) \ = \ -\frac{x}{3t}$$

Note however that $\omega''(k_0(x,t))$ vanishes at $x = 0$ and therefore, our basic theorem on stationary phase does not apply as $t \to \infty$ for all x. Thus, we require a version of the stationary phase theorem that can handle more degenerate situations.

We consider the case where $I(\lambda)$ is a one-dimensional integral. The key tool is *Van der Corput's Lemma* [4].

Theorem 4.3 *Let $\phi(\xi)$ be real-valued and smooth on (a,b). Assume that*

$$\left|\ \frac{d^k}{dx^k}\ \phi(x)\ \right| \ \geq\ 1, \quad x \in (a,b)$$

In the case, $k = 1$, assume additionally that $\phi'(x)$ is monotone on the interval (a,b). Then,

$$\left|\ \int_a^b e^{i\lambda\phi(x)}dx\ \right| \ \leq\ c_k\lambda^{-\frac{1}{k}}. \tag{4.23}$$

The inequality (4.23) holds with the choice of constant $c_k = 5\ 2^k - 2$.

Proof of Van der Corput's Lemma: We begin with the case $k = 1$.

$$\begin{aligned} \int_a^b e^{i\lambda\phi(x)}dx &= \int_a^b \frac{1}{i\lambda\phi'(x)}\frac{d}{dx}e^{i\lambda\phi(x)}dx \\ &= \frac{1}{i\lambda\phi'(x)}e^{i\lambda\phi(x)}\ \Big|_{x=a}^{x=b} \ -\ \int_a^b e^{i\lambda\phi(x)}\ \frac{d}{dx}\left(\frac{1}{i\lambda\phi'(x)}\right)\ dx \end{aligned}$$

Therefore, using that ϕ' is monotone,

$$\begin{aligned} |\ I(\lambda)\ | &\leq \frac{2}{\lambda} \ +\ \frac{1}{\lambda}\int_a^b \left|\ \frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)\ \right|\ dx \\ &= \frac{2}{\lambda} \ +\ \frac{1}{\lambda}\left|\ \int_a^b \frac{d}{dx}\left(\frac{1}{\phi'(x)}\right)\ dx\ \right| \\ &\leq \frac{2}{\lambda} \ +\ \left|\ \frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\ \right| \ \leq\ \frac{3}{\lambda} \end{aligned}$$

This settled the case: $k = 1$.

We now turn to the case $k \geq 2$. We proceed by induction on k. Let's assume that the proposition holds for the case, k, and prove that it holds for $k+1$. Since ϕ is smooth, $|\phi^k(x)|$ attains its minimum on $[a, b]$. Let $x = c$ denote the location of this minimum. By hypothesis $\phi^{(k)}(x)$ is monotone on (a, b) and therefore, either $\phi^{(k)}(c) = 0$ or c is an endpoint a or b.

If c is an interior point at which $\phi^{(k)}(c) = 0$, then we write $I(\lambda)$ as

$$I(\lambda) \ = \ \left(\int_a^{c-\delta} \ + \ \int_{c-\delta}^{c+\delta} \ + \ \int_{c-\delta}^{b} \right) e^{i\lambda\phi(\xi)} \ d\xi$$

Note that $\phi^{(k)}(x) = \phi^{(k)}(c) + \phi^{(k+1)}(\eta)(x-c) = \phi^{(k+1)}(\eta)(x-c)$ and therefore if $x \in [a, c-\delta]$ or $x \in [c+\delta, b]$ we have $\phi^{(k+1)}(x) \geq \delta$. Therefore, by the induction hypothesis on the case k, we get

$$\left| \int_a^{c-\delta} \ + \ \int_{c-\delta}^{b} \right| \ \leq \ \frac{2c_k}{(\lambda\delta)^{\frac{1}{k}}}$$

Clearly, the remaining contribution to $I(\lambda)$ can be bounded above by 2δ. Therefore, for any δ small and positive

$$|\ I(\lambda)\ | \ \leq \ \frac{2c_k}{(\lambda\delta)^{\frac{1}{k}}} \ + \ 2\delta \tag{4.24}$$

Choose $\delta = \lambda^{-\frac{1}{k}}$. Then, we have

$$|\ I(\lambda)\ | \ \leq \ \frac{2c_k + 2}{\lambda^{\frac{1}{k}}} \tag{4.25}$$

Therefore, c_k satisfies the first order difference equation $c_{k+1} = 2c_k + 2$ with initial data $c_1 = 3$, with solution displayed in the statement of the theorem.

Note that if $\phi^{(k)}(c) = 0$ and c is an endpoint of $[a, b]$, then the above argument gives δ instead of 2δ on the right hand side of (4.24). Therefore, the previous bound applies. Finally, a similar argument can be given if $\min_{\xi\in[a,b]} |\phi^{(k)}(\xi)| = |\phi^{(k)}(c)| \neq 0$ and therefore $c = a$ or $c = b$.

The following corollary of Van der Corput's Lemma is also useful.

Corollary 4.3 *Assume ϕ is as in Theorem 4.3 and assume $\psi'(x)$ is defined and integrable on $[a, b]$. Then,*

$$\left| \int_a^b e^{i\lambda\phi(x)} \ \psi(x) \ dx \right| \ \leq \ \frac{c_k}{\lambda^{\frac{1}{k}}} \ \left(\ |\psi(b)| \ + \ \|\psi'\|_{L^1(a,b)} \ \right) \tag{4.26}$$

Proof of Corollary 4.3:

$$\begin{aligned}
\int_a^b e^{i\lambda\phi(x)} \ \psi(x) \ dx \ &= \ \int_a^b \frac{d}{dx} \int_a^x e^{i\lambda\phi(t)} dt \ \psi(x) \ dx \\
&= \ \psi(b) \int_a^b e^{i\lambda\phi(x)} dx \ - \ \int_a^b \int_a^x e^{i\lambda\phi(t)} dt \ \psi'(x) \ dx
\end{aligned}$$

Therefore, by Theorem 4.3

$$\left| \int_a^b e^{i\lambda\phi(x)} \ \psi(x) \ dx \right| \ \le \ \left(\ |\psi(b)| \ + \ \|\psi'\|_{L^1(a,b)} \ \right) \ \sup_{x\in[a,b]} \left| \int_a^x e^{i\lambda\phi(t)} dt \ \right|$$
$$\le \ \frac{c_k}{\lambda^{\frac{1}{k}}} \ \left(\ |\psi(b)| \ + \ \|\psi'\|_{L^1(a,b)} \ \right)$$

4.4 PDE asymptotics via localization in Fourier space

In this subsection we continue our discussion of the Airy / linearized KdV equation initiated at the beginning of section 4.3. In particular, we now show how one uses the above results on oscillatory integrals to study the large time asymptotics of solutions.

We introduce a smooth *cutoff function* $\chi(k)$, defined as follows. Let $\chi(k)$ be a C^∞ function which is identically equal to one for $|k| \le 1$ and identically equal to zero for $|k| \ge 2$.

4.5 Notes and references for Section 4

References on stationary and non-stationary phase - see [3, 4]. Reference on Van der Corput's Lemma - see [4] Chapter 8.

5 The Nonlinear Schrödinger / Gross-Pitaevskii Equation

Nonlinear Schrödinger equations

$$i\partial_t\Phi \ = \ -\Delta\Phi \ + \ f(x, |\Phi|^2)\Phi \tag{5.1}$$

form a class of nonlinear, conservative (Hamiltonian) dispersive PDEs, which arise in many fields of application. Here, Δ denotes the $n-$ dimensional Laplace operator. We list several application areas:

(1) **Nonlinear optics:** Propagation of laser beams through a nonlinear medium, (gas, water) with the refractive index exhibits a dependence on the the local field intensity, if the latter is suffficiently large [**?**]. In this case,

$$i\partial_z\Phi \ = \ -\left(\ \partial_x^2 + \partial_y^2 \ \right) \Phi \ - \ |\Phi|^2\Phi \tag{5.2}$$

Thus, $n = 2$ and $f(x, |\Phi|^2) = -|\Phi|^2$. Here, Φ denotes the slowly varying envelope of the highly oscillatory electric field, which is nearly-monochromatic. If the propagation is in a waveguide, with transverse refractive index profile, then we have [**?**, **?**]

$$i\partial_z\Phi \ = \ -\left(\ \partial_x^2 + \partial_y^2 \ \right) \Phi \ + \ V(x, y)\Phi \ - \ |\Phi|^2\Phi. \tag{5.3}$$

(2) **Macroscopic quantum systems:** The effective dynamics of a quantum system consisting of $N-$ Bosons, where N is large, in the mean-field limit [**?**]. Here, $n = 3$, $f(x, |\Phi|^2) = |\Phi|^2$ or, more generally, $f(x, |\Phi|^2) = \int K(x-y)|\Phi(y)|^2\, dy$, where $0 < K \in L^1$ and $K(x) = K(-x)$. In this case, NLS is sometimes referred to as the *Gross-Pitaevskii* equation.

(3) **Dynamics of waves in a nearly-collisionless plasma** Here, $n = 2$ or $n = 3$ and $f(x, |\Phi|^2) = |\Phi|^2$. See [**?**].

(4) **Hydrodynamics:** The motion of a vortex filament governed by the Euler equations of fluid mechanics [**?**]. Here, $n = 1$ and $f(x, |\Phi|^2) = |\Phi|^2$.

5.1 "Universality" of NLS

Why the ubiquity of NLS? In this section we show the manner in which NLS naturally arises as an envelop equation governing the evolution of small amplitude *wave-packets* in a weakly nonlinear and strongly-dispersive system. The calculation presented is quite general; although implemented for the nonlinear Klein-Gordon equation, it can be carried out for a very general system of the above type.

We begin with a prototypical nonlinear dispersive wave equation, the *nonlinear Klein-Gordon equation*:

$$\partial_t^2 v \ - \ \partial_x^2 v \ + \ m^2 v \ = \ \lambda |v|^2 v \tag{5.4}$$

Here, $v = v(x,t)$ is a complex valued function. λ is a "nonlinear coupling parameter" which we take to be of order one; more on the role of lambda later.

We shall consider the case of *weakly nonlinear solutions*. Thus we introduce a small parameter ε and define

$$v(x,t) \ = \ \varepsilon u^\varepsilon(x,t) \tag{5.5}$$

Thus,

$$\partial_t^2 u^\varepsilon \ - \ \partial_x^2 u^\varepsilon \ + \ m^2 u^\varepsilon \ = \ \lambda \varepsilon^2 |u^\varepsilon|^2 u^\varepsilon \tag{5.6}$$

is an equation in which the small parameter, ε, is explicit. We consider (5.6) with order one initial data which are of "nearly plane wave type" or *nearly mono-chromatic*:

$$u^\varepsilon(x,0) \ \sim \ A(\varepsilon x)\, e^{ikx}, \ \ \partial_t u^\varepsilon(x,0) \ \sim \ -i\omega(k)\ A(\varepsilon x)\, e^{ikx} \tag{5.7}$$

where $A(X)$ is a localized function, say A smooth and rapidly decaying.

5.2 Multiple scales

We view u^ε as a function of fast and slow variables:

$$\begin{aligned} u^\varepsilon(x,t) \ &= \ u^\varepsilon(x, X; t, T_1, T_2, \dots) \\ X \ = \ \varepsilon x, \quad T_j \ &= \ \varepsilon^j t \end{aligned} \tag{5.8}$$

Thus, we take for initial conditions

$$\begin{aligned} u^\varepsilon(x,X;0,0,\dots) &= A(X)\ e^{ikx} \\ \partial_t u^\varepsilon(x,X;0,0,\dots) &= -i\omega(k)\ A(X)\ e^{ikx} \end{aligned} \tag{5.9}$$

Multiscale Expansion We expand the solution in a formal series in powers of ε and treat the slow and fast variables as *independent* variables. We must then also rewrite the PDE as a PDE with respect to this extended list of independent variables:

$$\begin{aligned} u^\varepsilon &= u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots \\ \partial_t &\to \partial_t + \varepsilon\partial_{T_1} + \varepsilon^2\partial_{T_2} + \dots \\ \partial_x &\to \partial_x + \varepsilon\partial_X \end{aligned}$$

Thus,

$$\begin{aligned} \partial_t^2 &\to \partial_t^2 + 2\varepsilon\partial_t\partial_{T_1} + \varepsilon^2\left(\partial_{T_1}^2 + 2\partial_t\partial_{T_2}\right) + \varepsilon^3\left(2\partial_{T_1}\partial_{T_2} + 2\partial_t\partial_{T_3}\right) + \dots \\ \partial_x^2 &\to \partial_x^2 + 2\varepsilon\partial_x\partial_X + \varepsilon^2\partial_X^2 \end{aligned}$$

The nonlinear Klein-Gordon becomes

$$L_\varepsilon u^\varepsilon = \lambda\,\varepsilon^2\,|u^\varepsilon|^2 u^\varepsilon, \text{ where} \tag{5.10}$$

$$L_\varepsilon = \left[\partial_t^2 - \partial_x^2 + m^2\right] + 2\varepsilon\left[\partial_t\partial_{T_1} - \partial_x\partial_X\right] + \varepsilon^2\left[\partial_{T_1}^2 + 2\partial_t\partial_{T_2} - \partial_X^2\right] + \mathcal{O}(\varepsilon^3) \tag{5.11}$$

Substitution of the expansion for u^ε, using that

$$|u^\varepsilon|^2 u^\varepsilon = |u^0|^2 u^0 + 2\varepsilon u^0 u^1 (u^0)^* + \varepsilon (u^0)^2 (u^1)^* + \dots$$

and equation of like orders of ε, we obtain the following

5.3 Hierarchy of equations

$$\begin{aligned} \mathcal{O}(\varepsilon^0) &: \ \left(\partial_t^2 - \partial_x^2 + m^2\right) u^0 = 0 \\ \mathcal{O}(\varepsilon^1) &: \ \left(\partial_t^2 - \partial_x^2 + m^2\right) u^1 = -2\left[\partial_t\partial_{T_1} - \partial_x\partial_X\right] u^0 \\ \mathcal{O}(\varepsilon^2) &: \ \left(\partial_t^2 - \partial_x^2 + m^2\right) u^2 = -2\left[\partial_t\partial_{T_1} - \partial_x\partial_X\right] u^1 - \left[\partial_{T_1}^2 - \partial_X^2 + 2\partial_t\partial_{T_2}\right] u^0 \\ &\qquad + \lambda|u^0|^2 u^0 \\ &\qquad \cdots\cdots\cdots \\ \mathcal{O}(\varepsilon^j) &: \ \left(\partial_t^2 - \partial_x^2 + m^2\right) u^j = \mathcal{S}(u^0,\dots,u^{j-1}), \quad j \ge 1. \end{aligned}$$

5.4 Solution of equation hierarchy

$\mathcal{O}(\varepsilon^0)$: We take u^0 to be a plane wave solution:

$$u^0(x, X; t, T_1, T_2, \dots) \;=\; A(X; T_1, T_2, \dots)\, e^{i(kx-\omega(k)t)} \tag{5.12}$$

The slowly varying amplitude, A, is to be determined at higher order in the perturbation scheme.

$\mathcal{O}(\varepsilon^1)$: Using the expression (5.12) for u^0, we find that u^1 satisfies:

$$\begin{aligned}\left(\, \partial_t^2 \;-\; \partial_x^2 \;+\; m^2 \,\right) u^0 \;&=\; -2\,[\, \partial_t \partial_{T_1} \;-\; \partial_x \partial_X \,]\, u^0 \\ &=\; 2i\,[\, \omega(k)\partial_{T_1} A \;+\; k\ \partial_X A \,]\ e^{i(kx-\omega(k)t)} \\ &= 2i\omega(k)\ \left[\, \partial_{T_1} A \;+\; \frac{k}{\omega(k)} \partial_X A \,\right] e^{i(kx-\omega(k)t)} \\ &= 2i\omega(k)\ \left[\, \partial_{T_1} A \;+\; \omega'(k)\partial_X A \,\right] e^{i(kx-\omega(k)t)},\end{aligned} \tag{5.13}$$

where we have used that $\omega^2 = m^2 + k^2$ and therefore $\omega'(k) = k/\omega(k)$.

Exercise 5.1 *Prove the following*

Proposition 5.1 *If $\alpha \neq 0$, then the PDE*

$$\left(\, \partial_t^2 \;-\; \partial_x^2 \;+\; m^2 \,\right) U \;=\; \alpha\ e^{i(kx-\omega(k)t}$$

has resonant forcing and its general solution grows linearly in time.

It follows that the expansion $u^\varepsilon = u^0 + \varepsilon u^1 + \dots$ will break down on times of order ε^{-1} in the sense that εu^1 will become comparable in size with u^0 unless A is constrained so that the right hand side of (5.13) vanishes. This gives the equation:

$$\partial_{T_1}\ A \;+\; \omega'(k)\ \partial_X\ A \;=\; 0 \tag{5.14}$$

Equation (5.14) implies that the amplitude propagates with the *group velocity*, $\omega'(k)$. We have

$$A \;=\; A(Y, T_2, \dots), \quad \text{where} \quad Y = X - \omega'(k) T_1$$

Furthermore, u^1 is now seen to satisfy a homogeneous equation and we take $u^1 \equiv 0$. The solution we have thus far constructed is

$$u^\varepsilon \;=\; A(X - \omega'(k)T_1, T_2, \dots)\ e^{i(kx-\omega(k)t)} \;+\; \varepsilon^2 u_2 \;+\; \dots$$

Using that $u^1 = 0$ as well as the expression for u^0 we obtain the following equation for u^2:

$$\left(\, \partial_t^2 \;-\; \partial_x^2 \;+\; m^2 \,\right) u^2 \;=\; \left[\, 2i\omega(k)\partial_{T_2} A \;-\; \left(\, \partial_{T_1}^2 - \partial_X^2 \,\right) A \;+\; \lambda |A|^2 A \,\right]\ e^{i(kx-\omega(k)t)} \tag{5.15}$$

By Proposition 5.1 we require

$$2i\omega(k)\partial_{T_2} A \;-\; \left(\, \partial_{T_1}^2 - \partial_X^2 \,\right) A \;+\; \lambda |A|^2 A \;=\; 0 \tag{5.16}$$

We can simplify (5.16) by making the following observations:

$$\partial_{T_1} = -\omega' \partial_Y, \quad \partial_Y = \partial_X, \quad \partial_{T_1}^2 - \partial_X^2 = \left(\, (\omega')^2 - 1 \, \right) \partial_Y^2$$

This gives

$$2i\omega(k)\partial_{T_2} A \; + \; \left(\, 1 - (\omega')^2 \, \right) \partial_Y^2 A \; + \; \lambda |A|^2 A \; = \; 0 \tag{5.17}$$

Finally, note that

$$\omega^2 = m^2 + k^2 \quad \Longrightarrow \quad \omega\omega' = k \quad \Longrightarrow \quad \omega\omega'' + (\omega')^2 = 1$$

Thus,

$$2i\omega(k)\partial_{T_2} A \; + \; \omega(k)\omega''(k) \; \partial_Y^2 A \; + \; \lambda |A|^2 A \; = \; 0 \tag{5.18}$$

5.5 Conclusion and Theorem

Definition We call a wave equation strongly dispersive at wave number k_0 if its dispersion relation satisfies $\omega''(k_0) \neq 0$.

Conclusion: A nearly monochromatic (about wave number k) wave packet in a strongly dispersive system will translate with the group velocity, $\omega'(k)$, and modulate in accordance with the nonlinear Schrödinger equation (5.18).

The following result can be proved:

Theorem 5.1 *Let $A(Y, T_2)$ satisfy the nonlinear Schrödinger equation (5.18). There exists a small constant $\varepsilon_0 > 0$, such that for any $\tau > 0$ and $\varepsilon < \varepsilon_0$, the nonlinear Klein-Gordon equation (5.4) has solutions*

$$v(x,t) \; = \; \varepsilon \; \left[\; A(\varepsilon(x - \omega'(k)t), \varepsilon^2 t) \; e^{i(kx - \omega(k)t)} \; + \; \varepsilon \; w^\varepsilon(x,t) \; \right],$$

where w^ε satisfies the following bound:

$$\| \; w^\varepsilon(\cdot, t) \; \|_{H^1(\mathbb{R})} \; \leq \; C_1, \quad |w^\varepsilon(x,t)| \; \leq \; C_2, \quad 0 \leq t \; \leq \tau\varepsilon^{-2}$$

6 Structural Properties of NLS

6.1 Hamiltonian structure

6.2 Symmetries and conserved integrals

7 Formulation in H^1 and the basic well-posedness theorem

8 Special solutions of NLS - nonlinear plane waves and nonlinear bound states

9 Appendices

9.1 The M. Riesz Convexity Theorem

A linear operator, T, is of type (p,q), where $p^{-1}+q^{-1}=1$, if there exists a positive constant k such that for all $f \in L^p$

$$\|Tf\|_q \;\leq k\; \|f\|_p \tag{9.1}$$

Theorem 9.1 *[5] Let T be of type (p_i,q_i) with norm k_i, $i=0,1$. Then, T is of type (p_θ,q_θ) with (p_θ,q_θ) norm*

$$k_\theta \;\leq\; k_0^{1-\theta}\; k_1^{\theta}, \tag{9.2}$$

provided

$$(p_\theta^{-1},q_\theta^{-1}) \;=\; (1-\theta)(p_0^{-1},q_0^{-1}) \;+\; \theta(p_1^{-1},q_1^{-1}) \tag{9.3}$$

9.2 The Implicit Function Theorem

Definition 9.1 *A mapping between Banach spaces X and Z, $F: X \to Z$, is (Frechét) differentiable if there is a bounded linear transformation $\mathcal{L}: X \to Z$, such that*

$$\|F(x+\xi)-F(x)-\mathcal{L}\xi\|_Z \;=\; o(\|\xi\|_X)$$

as $|\xi\|_X \to 0$.

Theorem 9.2 *[2] Let X, Y and Z denote Banach spaces. Let F denote a continuous mapping*

$$F: U \subset X \times Y \to Z, \quad (x,y) \mapsto F(x,y),$$

where U is open. Assume F is (Frechét) differentiable with respect to x and that $F_x(x,y)$ is continuous in U. Let $(x_0,y_0) \in U$ and $F(x_0,y_0)=0$. If the linear operator $\eta \mapsto \mathcal{L}\eta \equiv F_x(x_0,y_0)\eta$ is one to one and onto Z (an isomorphism from X to Z), then

(i) There exists a ball $\{y : \|y-y_0\|_Y < r\} = B_r(y_0)$ and a unique continuous mapping $u: B_r(y_0) \to X$ such that $u(y_0)=x_0$ and $F(u(y),y)=0$.

(ii) $u_y \in C^p$ if $F \in C^p$, $1 < p \leq \infty$.

Remark 9.1 *If $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ and $Z = \mathbb{R}^m$, this reduces to the usual finite dimensional case. The m by m matrix, $F_x(x_0,y_0)$, is assumed to be invertible.*

9.3 The Contraction Mapping Principle

Theorem 9.3 *Let $T: X \to X$ denote a mapping from a complete metric space, X, into itself. Assume that T is a contraction on X, i.e. there exists k with $0<k<1$, such that for any α and β in X*

$$\rho(T\alpha,T\beta) \;\leq k\; \rho(\alpha,\beta) \tag{9.4}$$

Then, there exists a unique element of X, α_*, for which $T\alpha_* = \alpha_*$. Moreover, starting with an arbitrary element $\alpha_0 \in X$ and defining $\alpha_j = T\alpha_{j-1}$, $j=1,2,3,\ldots$, we have $\alpha_* = \lim_{j\to\infty}\alpha_j$.

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