Radiation Induced Instability in Interconnected Systems

Patrick Hagerty[1]
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
hagerty@umich.edu

Anthony M. Bloch[2]
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
abloch@math.lsa.umich.edu

Michael I. Weinstein[3]
Bell Labs - Lucent Technologies and
Department of Mathematics
University of Michigan
600 Mountain Avenue
Murray Hill, NJ 07974
miw@research.bell-labs.com

Abstract

In this paper we discuss the stability and instability properties of two classes of conservative dynamical systems which have two interconnected components: a finite dimensional and an infinite-dimensional subsystem. The finite dimensional component is a linear mechanical system with gyroscopic terms and it is coupled to a wave equation defined on an infinite spatial domain via two different types of coupling – integral and point coupling. In particular we analyze the conditions under which connection to a wave system induces instability in the finite dimensional system.

1 Introduction

In this paper, we analyze the stability of a class of interconnected finite- and infinite-dimensional systems. We consider linear finite-dimensional mechanical systems with gyroscopic terms coupled to a wave equation via two different types of coupling – integral and point coupling. Dissipative perturbations are known to induce instability in certain Hamiltonian systems; see Bloch, Krishnaprasad, Marsden and Ratiu [1994]. Since the origins of dissipation (*e.g.* friction, viscosity...) lie in the transfer of energy from one form (energy of one subsystem) to another form (that of a second subsystem) of a larger conservative system, it is natural to expect the analogue of the above destabilization phenomenon to be present within the more fundamental context of conservative systems which exhibit internal energy transfer. In this paper, we explore this in the context of a gyroscopic oscillating mechanical system coupled to an extended wave system (infinite string). Due to the coupling, motion within the mechanical system generates waves which can be carried off to infinity. Such *radiation damping* has been studied in models arising in the theory of quantum resonances, ionization type problems and nonlinear waves; see Soffer and Weinstein [1998a, 1998b, 1999]. We derive an exact *reduced* equation for the mechanical system, which has an additional term, capturing the effects induced by the radiating wave system. We then analyze the stability properties of the reduced system in order to deduce the effects of the coupling.

We discuss the stability of the overall system under various assumptions on the eigenvalues of the finite-dimensional part of the system. In particular, we consider the case of gyroscopic stability (i.e. stability induced by the gyroscopic terms in the system equations) and analyze when the wave equation destabilizes the system. Since gyroscopic systems may be shown to be the normal form for a large class of physical systems of interest, this provides a fairly widely applicable model for studying the effect of wave coupling on finite-dimensional systems. Stability of interconnected systems is of interest for many different applications including the study of satellite or space station motion. Such systems include a mixture of rigid and flexible components. The Lagrangian or Hamiltonian structure of such systems is also of interest (see Maschke and van der Schaft [1997] or Bloch and Crouch [1998] for example).

We show how the systems may be viewed as a finite-dimensional n-port connected to a wave equation.

In the integral coupling case, we show that the reduced mechanical system is a finite-dimensional linear

[1]Research partially supported by National Science Foundation grant DMS–9803181 and AFOSR grant F49620-96-1-0100.
[2]Research partially supported by National Science Foundation grant DMS–9803181 and AFOSR grant F49620-96-1-0100.
[3]Research partially supported by National Science Foundation Grant DMS-9500997

system with integral feedback. Furthermore, we show integral coupling destabilizes all gyroscopic systems, not just gyroscopically stabilized ones. This turns out to be in contrast to the case of point coupling of the gyroscopic finite-dimensional system to the wave equation. Using a method of analysis from Komech [1995b], a dissipative term of Rayleigh type in the gyroscopic system explicitly arises from the point coupling. Applying results from Bloch, et. al [1994], we show that radiation induces instability in a gyroscopically stabilized system.

2 Two degree of freedom Chetaev system

Gyroscopically stabilized systems exhibit interesting instability when perturbed by dissipative forces. Via coupling the gyroscopic system to a wave equation, we desire to induce similar destabilizing effects from radiation. The general form of a gyroscopic system is

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = 0, \tag{2.1}$$

where $\mathbf{q} \in \mathbb{R}^n$, M is a positive-definite symmetric $n \times n$ matrix, S is skew, and V is symmetric.

We prove our integral coupling results here for a two degree of freedom gyroscopic system both for exposition purposes and because this is an interesting physical model. As in Bloch et. al. [1994] we shall call this the *Chetaev system* (see Chetaev [1961]). The general case of integral coupling can be analyzed in an essentially similar fashion and will be discussed in a forthcoming paper. (In the point coupling case discussed below we do analyze the general n-dimensional system.)

Consider firstly then a two degree of freedom gyroscopic system with $\mathbf{q} = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 0 & -g \\ g & 0 \end{bmatrix}$, and $V = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$,

$$\begin{aligned} \ddot{x} - g\dot{y} + \alpha x &= 0, \\ \ddot{y} + g\dot{x} + \beta y &= 0, \end{aligned} \tag{2.2}$$

where α, β and g are fixed real constants of arbitrary sign. The above system of equations models a bead at the center of a rotating disc subject to a central restoring force. The equations are also the linearized equations of motion of a rotating spherical pendulum in a gravitational field. (See Bloch et. al. [1994], Baillieul and Levi [1991], Chetaev [1961] for physical discussions.) It has been shown that if the system is unstable for $g = 0$, then the Chetaev system is gyroscopically stabilized if $g^2 + \alpha + \beta \geq 2\sqrt{\alpha\beta}$, (i.e. the eigenvalues are purely imaginary).

We also present the uncoupled wave equation as

$$\ddot{w} - c^2 \frac{\partial^2 w}{\partial \xi^2} = 0. \tag{2.3}$$

Loosely, we physically interpret the wave equation as modeling the motion of a string. Furthermore, the interaction between the mechanical system and the wave equation can be interpreted as an interaction between a bead on a rotating disc and a string.

3 Integral Coupling

We first investigate a local field coupling of the mechanical gyroscopic system to the wave equation. We construct the coupling so that the interaction is local and so that the system as a whole remains Hamiltonian. Coupling of this type is important in various physical models – see Soffer and Weinstein [1998a, 1998b, 1999]. For a coupling parameter κ, we have

$$\begin{aligned} \ddot{x} - g\dot{y} + \alpha x &= \kappa \int_{\mathbb{R}} \chi(\xi) w(\xi, t) d\xi, \\ \ddot{y} + g\dot{x} + \beta y &= \kappa \int_{\mathbb{R}} \chi(\xi) w(\xi, t) d\xi, \\ \ddot{w} - c^2 \frac{\partial^2 w}{\partial \xi^2} &= \kappa \chi(\xi), \end{aligned} \tag{3.1}$$

where $\chi(\xi)$ is a distribution. Desiring a local interaction, we choose $\chi(\xi)$ to be the Dirac- δ distribution. (So in fact, even though we will continue to use the terminology "integral coupling' for the type of system discussed in this section, the special case we consider here is in fact also a type of point coupling.)

The above coupled system remains Hamiltonian with H defined as

$$\begin{aligned} H =& \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \alpha x^2 + \beta y^2) - \kappa(x+y)w(\xi, 0) \\ &+ \frac{1}{2}\int_{\mathbb{R}} \dot{w}^2 + c^2 w_\xi^2 d\xi. \end{aligned} \tag{3.2}$$

Notice that the Hamiltonian is indefinite; hence the energy of the system does not uniformly bound the motion of the bead and the motion of the string.

Even though the bead and the string seem intimately coupled, we can decouple the motion of the bead from that of the string as follows. Taking the Fourier transform with respect to ξ in the wave equation yields

$$\hat{w}(k,t) = \int_{\mathbb{R}} e^{-ik\xi} w(\xi) d\xi, \tag{3.3}$$

$$\hat{w}_{tt}(k,t) + c^2 k^2 \hat{w}(k,t) = \kappa(x(t) + y(t)). \tag{3.4}$$

Solving the o.d.e. for $\hat{w}$, we have

$$\begin{aligned} \hat{w}(k,t) =& \kappa \int_0^t \frac{\sin(ck(t-s))}{ck}(x(s) + y(s))ds \\ &+ \hat{w}_{free}(k,t). \end{aligned} \tag{3.5}$$

The above equation is the retarded Green's function for the wave equation and w_{free} is the homogeneous solution satisfying initial conditions. Neglecting w_{free}, we have

$$w(0,t) = \frac{\kappa}{2c}\int_0^t \mathrm{sgn}(t-s)(x(s)+y(s))ds \quad (3.6)$$

$$= \frac{\kappa}{2c}\int_0^t x(s)+y(s)ds \quad (3.7)$$

Thus the mechanical system decouples as

$$\begin{aligned}\ddot{x} - g\dot{y} + \alpha x &= \gamma \int_0^t x(s)+y(s)\ ds,\\ \ddot{y} + g\dot{x} + \beta y &= \gamma \int_0^t x(s)+y(s)\ ds,\end{aligned} \quad (3.8)$$

where $\gamma = \frac{\kappa^2}{2c}$. The field coupling of the wave equation to the gyroscopic system reduces to integral coupling of the finite-dimensional system.

4 Differentiated System

We now analyze the stability of the system (3.8).

Suppose that initially the coupling field is identically zero, hence $w_{free}(0,t) = 0$ for all time $t > 0$.

We write 3.8 in matrix form, setting $Q = [x\ y\ \dot{x}\ \dot{y}\ \ddot{x}\ \ddot{y}]^T$.

Differentiating 3.8, we have

$$\dot{Q} = BQ, \quad (4.1)$$

where

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ \gamma & \gamma & -\alpha & 0 & 0 & g\\ \gamma & \gamma & 0 & -\beta & -g & 0 \end{bmatrix}. \quad (4.2)$$

Computing the characteristic equation, $p(\mu)$, of the matrix B, we have

$$p = \mu(\mu^5 + (g^2+\alpha+\beta)\mu^3 - 2\gamma\mu^2 + \alpha\beta\mu - \gamma(\alpha+\beta)). \quad (4.3)$$

Let μ_1, ... ,μ_5 be the non-zero eigenvalues of A. Then we have

$$\prod_{i=1}^{5}\mu_i = \gamma(\alpha+\beta). \quad (4.4)$$

Recall the system is gyroscopically stabilized when $g^2 + \alpha + \beta \geq 2\sqrt{\alpha\beta}$. Also note the $trace(B) = 0$. Hence if $\gamma(\alpha+\beta) \neq 0$, then B has an eigenvalue with positive real part. In particular if $\alpha, \beta < 0$ and $\gamma > 0$, then there is a real negative eigenvalue. Thus, there exists a conjugate pair of eigenvalues with positive real part. Hence the differentiated system is unstable.

If an addition we have $\alpha \neq \beta$, we can calculate the speed at which the eigenvalues leave the imaginary axis (see the analogous calculation in McKay [1991], Bloch et. al. [1994] and references therein)

$$\mu' = \left.\frac{d\mu}{d\gamma}\right|_{\gamma=0}. \quad (4.5)$$

Computing μ', we have

$$\mu' = \frac{2\mu^2 + (\alpha+\beta)}{5\mu^4 + 3(w_0^2+w_1^2)\mu^2 + w_0^2 w_1^2}, \quad (4.6)$$

where the eigenvalues of the unperturbed system are $\pm iw_0, \pm iw_1$. Simplifying in terms of w_0 and w_1, we have

$$(\pm iw_j)' = \frac{-2w_j^2 + (\alpha+\beta)}{2w_j^2(w_j^2 - w_{1-j}^2)}. \quad (4.7)$$

Integral coupling thus induces movement of the eigenvalues in a fashion similar to to the manner in which dissipation induces movement of the eigenvalues: the eigenvalues closer to (further from) the origin move to the right (left) half plane.

Now a C^3 solution to the undifferentiated system is a solution to the differentiated system. Conversely, a solution to the differentiated system satisfies the original system if in the initial conditions are satisfied; that is

$$\begin{aligned}\ddot{x}(0) - g\dot{y}(0) + \alpha x(0) = 0,\\ \ddot{y}(0) + g\dot{x}(0) + \beta y(0) = 0,\end{aligned} \quad (4.8)$$

which is only a linear constraint on the initial conditions. This constraint can be satisfied while exhibiting a negative eigendirection.

Summarizing we have

Theorem 4.1 *Coupling the finite-dimensional system (2.2) via integral coupling to a wave equation as in system (3.1) where $\chi(\xi)$ is a delta function yields the reduced finite-dimensional system (3.8). All smooth solutions of this system are unstable.*

Note that this is unlike the effect of Rayleigh dissipation on the gyroscopic system, i.e. dissipation arising from a term consisting of a positive definite symmetric matrix multiplying velocities. Rayleigh dissipation induces instability only if the original system is gyroscopically stable (i.e. it is unstable for $g = 0$ but stable for nontrivial g). Here instability is always induced. This is contrast to the case of point coupling analyzed in the next section where we see the same behavior as in the Rayleigh dissipation case. (See Chetaev [1961] and Bloch et. al. [1994] for a discussion of Rayleigh dissipation induced instability).

We now discuss two examples.

Example. For $\alpha = \beta = -1, c = 1$, and $g = 10$, the origin of the uncoupled system is gyroscopically stabilized. Letting $\kappa = 0.1$, we show by numerically computing the eigenvalues how the origin becomes unstable. With initial conditions of

$$\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}_0 \approx \epsilon \begin{bmatrix} -5.25 \times 10^{-3} \\ -1.25 \times 10^{2} \\ -1.25 \times 10^{3} \\ 6.71 \times 10^{-4} \end{bmatrix} \tag{4.9}$$

where $\epsilon > 0$, we see exponential growth of the coupled system. Hence, integral coupling destabilizes the origin in a gyroscopically stabilized system.

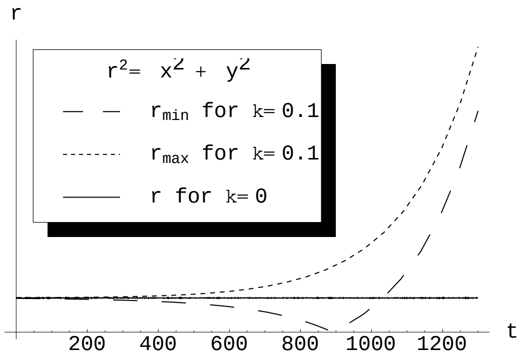

Figure 4.1: Radial distance of gyroscopically stabilized bead from the origin with integral coupling $\kappa = 0.1$ and without coupling $\kappa = 0$.

In the case of integral coupling, figure 4.1 shows the bounds, r_{min} and r_{max}, of the highly oscillatory radially position of the bead. Since the initial conditions have a component in an unstable eigendirection, we see exponential growth compared to the stable motion of the uncoupled system.

Example. As discussed above we have shown that integral coupling has a broader destabilizing effect than dissipation. In particular, the integral coupling can destabilize a system regardless of whether or not it is gyroscopically stabilized, while Rayleigh dissipative perturbations destabilize a Chetaev system only if it is initially gyroscopically stabilized. We now numerically compute an example which remains stable under dissipative perturbations, but in which integral coupling induces instability. Let $\alpha = \beta = 1, c = 1, \kappa = 0.1$, and $g = 10$. We numerically compute the eigenvalues. With initial conditions of

$$\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}_0 \approx \epsilon \begin{bmatrix} -7.79 \times 10^{-2} \\ 1.38 \\ -1.37 \times 10^{-1} \\ -7.67 \times 10^{-3} \end{bmatrix} \tag{4.10}$$

where $\epsilon > 0$, we again see exponential growth of the coupled system.

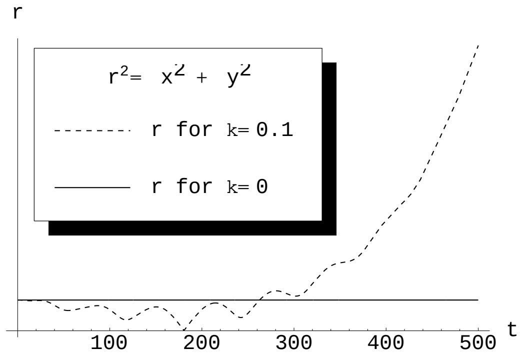

Figure 4.2: Radial distance of stabilized bead from the origin with integral coupling $\kappa = 0.1$ and without coupling $\kappa = 0$.

5 Network theory and integral feedback

In this section, we discuss the linked Chetaev system and the wave equation from the point of view of network theory and linear systems theory. As have seen, the wave coupling induces instability. It is of interest (see e.g. Baras et. al. [1974]) to analyze infinite-dimensional systems from the point of view of systems theory and in particular to understand asymptotic stability or instability. One problem of interest is the Darlington synthesis problem (see e.g. e.g. Brockett [1966], Krishnaprasad [1980] and references therein) where one shows that one can realize any positive real tranfer functions by terminating a lossless 2-port by a 1-ohm resistor. One can then extend this to the infinite-dimensional Hamiltonian setting by terminating by an infinite transmission line.

The situation here is somewhat different but related:

In particular, we view the Chetaev system as a two port connected to a wave system with integral coupling. The previous calculations show that this system can be reduced to a two port with integral feedback.

Consider the two degree of freedom Chetaev system with inputs:

$$\begin{aligned} \ddot{x} - g\dot{y} + \alpha x &= u_x, \\ \ddot{y} + g\dot{x} + \beta y &= u_y, \end{aligned} \tag{5.1}$$

Without coupling, we can think of this system as a linear four dimensional first order system with 2 inputs u_x and u_y (to be determined) and two outputs x and y. This makes the Chetaev system a classical two port.

We can also think of the wave equation as a first order system:

$$\frac{\partial}{\partial t}\mathbf{w} = A\mathbf{w}, \tag{5.2}$$

where $\mathbf{w} = [w, \dot{w}]^T$ and $A = \begin{bmatrix} 0 & 1 \\ c^2 \frac{\partial^2}{\partial \xi^2} & 0 \end{bmatrix}$. We think of the input in the second variable $\dot{w}$.

With the integral coupling, we connect the finite and the infinite systems through their inputs. Equivalently, the Chetaev system is given dynamical feedback through the wave equation. In this context, the decoupled system (3.8) becomes a first order system with integral feedback. Moreover, the integral feedback destabilizes the system in a not-necessarily Hamiltonian fashion: as we have seen we may only have one unstable eigenvalue.

We remark also that the instead of the mechanical systems discussed here we can physically realize the Chetaev system via coupled LC-circuits. The discussion also extends essentially without change to the n-port case.

6 Point coupling

Here we investigate a different model for the coupling of the gyroscopic system coupled to wave equation: we consider a model of a string coupled to the bead, in which the position of the string is constrained to pass through the bead. We show that this point coupling perturbation destabilizes mechanical systems which, when isolated, is gyroscopically stabilized. The system 2.1 with $S = 0$ is unstable if V has a negative eigendirection. The system is gyroscopically stabilized if for $S \neq 0$ the system is spectrally stable.

We begin with a model of a string in $\mathbb{R}^{2n+1}$, whose transverse vibrations are independent. Suppose the string lies initially along the $x_{2n+1}-$ axis. For simplicity, we denote the x_{2n+1} dimension as the z dimension. Coupling the transverse motion to a $2n-$dimensional gyroscopic system, we generalize the method described in Komech [1995b] to solve the following

$$\begin{aligned}\frac{\partial^2 \mathbf{w}}{\partial t^2}(z,t) =& c^2 \frac{\partial^2 \mathbf{w}}{\partial z^2}(z,t), \quad z \in \mathbb{R} - \{0\}, t \in \mathbb{R},\\ \mathbf{w}(0+,t) =& \mathbf{w}(0-,t) = \mathbf{q}(t),\\ M\ddot{\mathbf{q}}(t) =& - S\dot{\mathbf{q}}(t) - V\mathbf{q}(t)\\ &+ T[\frac{\partial \mathbf{w}}{\partial z}(0+,t) - \frac{\partial \mathbf{w}}{\partial z}(0-,t)],\end{aligned} \tag{6.1}$$

where c is the speed of transverse waves in the string, T is the tension of the string, and $\mathbf{w} = \begin{bmatrix} w_1(z,t) & \cdots & w_n(z,t)\end{bmatrix}^T$ is the displacement of the string in the first n dimensions. A model of this type with $S = 0$ was introduced by Lamb [1900].

Since our simplified model decomposes into $2n$ independent 1-dimensional wave equations, we can use d'Alembert decomposition to two traveling waves

$$\mathbf{w}(z,t) = \mathbf{f}_\pm(z - ct) + \mathbf{g}_\pm(z + ct), \qquad \pm z > 0, \tag{6.2}$$

where $\mathbf{f}_\pm, \mathbf{g}_\pm$ are distributions on $\mathbb{R}^{2n}$. In components, we have

$$\mathbf{f}_\pm = \begin{bmatrix} f_{\pm 1} & \cdots & f_{\pm 2n}\end{bmatrix}^T, \tag{6.3}$$
$$\mathbf{g}_\pm = \begin{bmatrix} g_{\pm 1} & \cdots & g_{\pm 2n}\end{bmatrix}^T, \tag{6.4}$$

where $f_i, g_i, i = \pm 1, \cdots, \pm 2n$ are distributions on $\mathbb{R}$. By the d'Alembert method of solving the wave equation, we have the formulas for $\pm z > 0$

$$\mathbf{f}_\pm(z) = \frac{1}{2}\mathbf{w}_0(z) - \frac{1}{2c}\int_0^z \mathbf{w}_1(v)dv, \tag{6.5}$$
$$\mathbf{g}_\pm(z) = \frac{1}{2}\mathbf{w}_0(z) + \frac{1}{2c}\int_0^z \mathbf{w}_1(v)dv, \tag{6.6}$$

where $\mathbf{w}_1(z) = \frac{\partial \mathbf{w}}{\partial t}(z,0)$, and $\mathbf{w}_0(z) = \mathbf{w}(z,0)$. Since the d'Alembert formula for $|z| \geq c|t|$, is defined in terms of known functions, the method remains valid and we have

$$\begin{aligned}\mathbf{w}(z,t) =& \frac{\mathbf{w}_0(z-ct) + \mathbf{w}_0(z+ct)}{2}\\ &+ \frac{1}{2c}\int_{z-ct}^{z+ct} \mathbf{w}_1(v)dv.\end{aligned} \tag{6.7}$$

To use the d'Alembert formula for $|z| < c|t|$, we need to define $\mathbf{f}_+(z)$ for $z < 0$ and to define $\mathbf{g}_-(z)$ for $z > 0$. from the coupling condition and from continuity of the string, we have for $t > 0$

$$\mathbf{f}_+(-ct) + \mathbf{g}_+(ct) = \mathbf{f}_-(-ct) + \mathbf{g}_-(ct) = \mathbf{q}(t), \tag{6.8}$$

$$\begin{aligned}M\ddot{\mathbf{q}}(t) =& - S\dot{\mathbf{q}}(t) - V\mathbf{q}(t) + T[\frac{\partial \mathbf{f}_+}{\partial z}(-ct)\\ &+ \frac{\partial \mathbf{g}_+}{\partial z}(ct) - \frac{\partial \mathbf{f}_-}{\partial z}(-ct) - \frac{\partial \mathbf{g}_-}{\partial z}(ct)].\end{aligned} \tag{6.9}$$

Differentiating and solving for $\mathbf{f}'_+$ and $\mathbf{g}'_-$, where the $'$ denotes $\frac{\partial}{\partial z}$, we have for positive t

$$-c\mathbf{f}'_+(-ct) = \dot{\mathbf{q}}(t) - c\mathbf{g}'_+(ct), \tag{6.10}$$
$$c\mathbf{g}'_-(ct) = \dot{\mathbf{q}}(t) + c\mathbf{f}'_-(-ct). \tag{6.11}$$

Substituting into equation (6.9), we arrive at a differential equation for $\mathbf{q}$ in terms of known functions for positive t

$$\begin{aligned}M\ddot{\mathbf{q}}(t) =& - S\dot{\mathbf{q}}(t) - V\mathbf{q}(t) - \frac{2T}{c}\dot{\mathbf{q}}(t)\\ &+ T[\mathbf{w}'_0(ct) - \mathbf{w}'_0(-ct)\\ &+ \frac{1}{c}(\mathbf{w}_1(ct) + \mathbf{w}_1(-ct))].\end{aligned} \tag{6.12}$$

Note that the time derivatives are considered to be in the distributional sense and thus equation (6.12) holds for almost every $t > 0$.

The effect of coupling to a wave equation (6.12) is intuitively manifested by the presence of an explicit dissipative term. If we assume that the initial conditions of the string are such that $\mathbf{w_1}$ and $\mathbf{w_0}$ both have compact support, then for large t we have $\pm ct$ outside the support of $|\mathbf{w_0}| + |\mathbf{w_1}|$ reducing equation 6.12 to

$$\begin{aligned}M\ddot{\mathbf{q}}(t) =& - S\dot{\mathbf{q}}(t) - V\mathbf{q}(t) - \frac{2T}{c}\dot{\mathbf{q}}(t),\\ &\pm ct \notin \operatorname{supp}(|\mathbf{w_0}| + |\mathbf{w_1}|).\end{aligned} \tag{6.13}$$

Constructing a Lyapunov function and invoking the Lyapunov instability theorem, Bloch et. al. [1994] explicitly show that the above system is spectrally unstable if V has a negative eigendirection. If the unperturbed system is gyroscopically stabilized then the dissipative term induces instability.

Thus we have

Theorem 6.1 *If the finite-dimensional gyroscopic system in (6.1) is gyroscopically stable, the point coupling to a wave equation induces instability for initial data* $\mathbf{w}_0$ *and* $\mathbf{w}_1$ *with compact support.*

This *radiation induced* instability the analog of dissipation induced instability in gyroscopically stabilized systems. This is consistent with Krein signature results on instability (see Arnold and Avez [1968] and Bloch et. al. [1994]), in constrast to the stronger instability one sees in the integral coupling case.

7 Results

We have shown that both integral coupling and point coupling of a finite-dimensional system to an infinite dimensional system allows for radiation induced instability. We derived the equivalence between integral coupling to a local wave field and integral feedback in the reduced description. Also, point coupling to a wave field is shown to be equivalent to Rayeigh dissipation in the reduced description. In the examples discussed, integral coupling destabilizes arbitrary gyroscopic systems, while point coupling destabilizes gyroscopically stabilized systems analogously to dissipation induced instability. In both cases, local field and point, the coupling allows for energy transfer between the finite dimensional gyroscopic system and the wave equation. Further investigations include exploring the destabilizing effect of different field coupling, such as the Klein-Gordon equation.

Acknowledgement: We would like to thank Roger Brockett, P.S. Krishnaprasad, and J.E. Marsden for valuable conversations on this material.

References

Arnold, A. and A. Avez, *Ergodic Problems of Classical Mechanics*, Addison Wesley, 1968.

Baillieul, J. and M. Levi [1991], Constrained relative motions in rotational mechanics *Arch. Rat. Mech. An.* **115**, 101-135.

Baras, J.S., R.W. Brockett and P.A. Fuhrmann [1974] State-space models for infinite-dimensional systems. *IEEE Trans. Aut. Control* **19**, 693-700.

Bloch, A.M. and P.E. Crouch [1998] Representation of Dirac structures on vector spaces and L-C circuits *Proc. Symp. in Pure Math* **64** (Ferreyra et. al. eds.), 103-117.

Bloch, A.M., Krishnaprasad, P.S., Marsden, J.E., and Ratiu, T.S. [1994] Dissipation induced instabilities. *Analyse Nonlineaire, Annales Institute H. Poincaré* **11**, 37-90.

Brockett, R.W [1966] Path integrals, Lyapunov functions and quadratic minimization. *Proc. 4th Allerton Conference*, 665-697.

Chetaev, N. G. [1961] *The Stability of Motion*, Pergamon Press, New York.

Folland, G.B. [1992] *Fourier Analysis and its Applications*, Wadsworth & Brooks/Cole.

Komech, A. I. [1995a] On the stabilization of string-oscillator interaction. *Russian Journal of Mathematical Physics* **3**, 2, 227-247.

Komech, A.I. [1995b] On stabilization of string-nonlinear oscillator interaction. *Journal of Mathematical Analysis and Applications* **196**, 384-409.

Krishnaprasad, P.S. [1980] On the geometry of linear passive systems. *Lectures in Applied Mathematics*, Algebraic Geometric Methods in Systems Theory (C.I. Byrnes and C.F. Martin eds.), 253-275.

Lamb, H. [1900] On the peculiarity of the wave-System due to the free vibrations of a nucleus in an extended medium, *Proceeding of the London Math. Society*, **32**, 208-211.

MacKay, R. [1991] Movement of eigenvalues of Hamiltonian equilibria under non-Hamiltonian perturbation, *Phys. Lett.* **A155**, 266-268.

Maschke, B.M. and and A.J. van der Schaft [1997] Interconnected mechanical systems, in *Modelling and Control of Mechanical Systems* (A. Astolfi et. al. eds.) Imperial College Press, London.

Routh, E. J. [1905] *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies*, MacMillan and Company.

Soffer, A. and Weinstein, M.I. [1998a] Time Dependent Resonance Theory, *Geom. func. anal.* **8** 1086–1128.

Soffer, A. and Weinstein, M.I. [1998b] Nonautonomous Hamiltonians, *J. Stat. Phys.* **93** 359–391.

Soffer, A. and Weinstein, M.I. [1999] Resonances, Radiation Damping and Instability in Hamiltonian Nonlinear Wave Equations, *Invent. Math.* **136** 9–74.