SIAM J. MATH. ANAL.
Vol. 33, No. 1, pp. 16–52
© 2001 Society for Industrial and Applied Mathematics

PARAMETRICALLY EXCITED HAMILTONIAN PARTIAL DIFFERENTIAL EQUATIONS*

E. KIRR† AND M. I. WEINSTEIN‡

Abstract. Consider a linear autonomous Hamiltonian system with a time-periodic bound state solution. In this paper we study the structural instability of this bound state relative to time almost periodic perturbations which are small, localized, and Hamiltonian. This class of perturbations includes those whose time dependence is periodic but encompasses a large class of those with finite (quasi-periodic) or infinitely many noncommensurate frequencies. Problems of the type considered arise in many areas of applications including ionization physics and the propagation of light in optical fibers in the presence of defects. The mechanism of instability is radiation damping due to resonant coupling of the bound state to the continuum modes by the time-dependent perturbation. This results in a transfer of energy from the discrete modes to the continuum. The rate of decay of solutions is slow and hence the decaying bound states can be viewed as metastable. These results generalize those of A. Soffer and M. I. Weinstein, who treated localized time-periodic perturbations of a particular form. In the present work, new analytical issues need to be addressed in view of (i) the presence of infinitely many frequencies which may resonate with the continuum as well as (ii) the possible accumulation of such resonances in the continuous spectrum. The theory is applied to a general class of Schrödinger operators.

Key words. Hamiltonian partial differential equations, parametric resonance, time-dependent perturbation theory, Fermi golden rule, energy transfer, metastable states

AMS subject classifications. 37L50, 35B34, 35B40, 37K55, 35B35, 35P25, 11K70

PII. S0036141099363456

1. Introduction.

1.1. Overview.

Consider a dynamical system of the form

$$i\partial_t \phi \;=\; H_0\ \phi, \tag{1.1}$$

where H_0 denotes a self-adjoint operator on a Hilbert space $\mathcal{H}$. We further assume that H_0 has only one eigenstate $\psi_0 \in \mathcal{H}$ with corresponding simple eigenvalue λ_0. Thus,

$$b_*(t) \;=\; e^{-i\lambda_0 t}\psi_0 \tag{1.2}$$

is a time-periodic *bound state* solution of the dynamical system (1.1). We next introduce the perturbed dynamical system

$$i\partial_t \phi \;=\; (H_0 \;+\; \varepsilon W(t))\,\phi. \tag{1.3}$$

In this paper we prove that if the perturbation, $\varepsilon W(t)$, is small, "generic," and almost periodic in time,[1] then solutions of the perturbed dynamical system (1.3) tend to zero

*Received by the editors October 26, 1999; accepted for publication (in revised form) October 26, 2000; published electronically April 12, 2001. This research was supported in part by National Science Foundation grant DMS-950097.

http://www.siam.org/journals/sima/33-1/36345.html

†Department of Mathematics, University of Michigan, Ann Arbor, MI and Department of Applied Mathematics, University Babeş-Bolyai, Cluj, Romania (ekirr@math.lsa.umich.edu). Part of this work was done while this author participated in the Bell Labs/Lucent Student Intern Program.

‡Mathematical Sciences Research, Bell Laboratories - Lucent Technologies, Murray Hill, NJ 07974 (miw@research.bell-labs.com).

[1]See the appendix in section 9 as well as [2, 9] for definitions and results on almost periodic functions.

as $t \to \pm\infty$. It follows that the state, $b_*(t)$, does not continue or deform to a time-periodic or even time-almost-periodic state. Thus, $b_*(t)$ is structurally unstable with respect to this class of perturbations. Our methods yield a detailed description of the transient (t large but finite) and long time ($t \to \pm\infty$) behavior solutions to the initial value problem. Theorems 2.1–2.3 contain precise statements of our main results. The following picture emerges concerning time evolution (1.3) for initial data given by the bound state, ψ_0, of the unperturbed problem. Let

$$P(t) = |(\ \psi_0, \phi(t)\ )|^2 \tag{1.4}$$

be the modulus square of the projection of the solution at time t onto the state ψ_0.[2] Then,

(i) $P(t) \ \sim \ 1 - C_W\ |t|^2$ for $|t|$ small,[3]

(ii) $P(t) \ \sim \ \exp(-2\varepsilon^2\Gamma t)$ for $t \leq \mathcal{O}((\varepsilon^2\Gamma)^{-1}),\ \Gamma \ = \mathcal{O}(W^2)$, and

(iii) $P(t) \ \sim \ \langle t\rangle^{-\alpha}$ for $|t| >> (\varepsilon^2\Gamma)^{-1}$ for some $\alpha > 0$.

The time $\tau = (\varepsilon^2\Gamma)^{-1}$ is called the *lifetime* of the state $b_*(t)$, which can be thought of as being *metastable* due to its slow decay. The mechanism for large time decay is resonant coupling of the bound state with continuous spectrum due to the time-dependent perturbation. Our analysis makes explicit the slow transfer of energy from the discrete to continuum modes and the accompanying radiation of energy out of any compact set.

Phenomena of the type considered here are of importance in many areas of theoretical physics and applications. Examples include ionization physics [3, 4, 10] and the propagation of light in optical fibers in the presence of defects [13]; see the discussion below.

The results of this article generalize those of Soffer and Weinstein [22], where the case

$$W(t) \ = \ \cos(\mu t)\ \beta, \quad \beta = \beta^* \tag{1.5}$$

was considered. The method used is a time-dependent/dynamical systems approach introduced in [21], [23] in a perturbation theory of operators with embedded eigenvalues in their continuous spectrum. These ideas were also used in a study of resonant radiation damping of nonlinear systems [24], as well as in a class of parametric resonance problems [22]; see also [12]. New analytical questions must be addressed in view of (i) the presence of infinitely many frequencies which may resonate with the continuum as well as (ii) the possible accumulation of such resonances in the continuous spectrum. This leads to a careful use of almost periodic properties of the perturbation (Theorems 2.1 and 2.2) and hypothesis **(H6)** (Theorem 2.3), which is easily seen to hold when the perturbation, $W(t)$, consists of a sum over a finite number of frequencies, μ_j.

A special case for which the hypotheses of our theorems are verified is the case of the Schrödinger operator $H_0 \ = \ -\Delta \ + \ V(x)$. Here, $V(x)$ is a real-valued function of $x \in \mathbb{R}^3$ which decays sufficiently rapidly as $|x| \to \infty$. In this setting Soffer and Weinstein [22] studied in detail the structural instability of $b_*(t)$ by considering the perturbed dynamical system (1.3) with $W(x,t) \ = \ \beta(x)\ \cos(\mu t)$. Here, we consider

[2] (f, g) denotes the inner product of f and g. If ψ_0 is normalized, then $P(t)$ has the quantum mechanical interpretation of the probability that the system at time t is in the state ψ_0.

[3] We do not discuss the short time behavior in this article; see [12]. This small time behavior is related to the "watched pot" effect in quantum measurement theory [15].

a class of perturbations of the form $W(x,t) = \sum_j \beta_j(x) \cos \mu_j t$, where the sum may be finite or infinite and where the frequencies μ_j need not be commensurate, e.g., $W(x,t) = \beta_1(x) \cos t + \beta_2(x) \cos \sqrt{2}t$, where $\beta_i(x)$, $i = 1, 2$, is rapidly decaying as $x \to \infty$.

In addition to the problem of ionization by general time-varying fields, we mention other motivations for considering the class of time-dependent perturbations sketched above and defined in detail in section 2.

(a) An area of application to which our analysis applies is the propagation of light through an optical fiber [13]. In the regime where backscattering can be neglected, the propagation of waves down the length of the fiber is governed by a Schrödinger equation:

$$i\partial_z \phi = (-\Delta_\perp + V(x_\perp)) \phi + W(x_\perp, z)\phi. \tag{1.6}$$

Here, ϕ denotes the slowly varying envelope of the highly oscillatory electric field, a function of z, the direction of propagation along the fiber, and $x_\perp \in \mathbb{R}^2$, the transverse variables. $V(x_\perp)$ denotes an unperturbed index of refraction profile and $W(x_\perp, z)$ the small fluctuations in refractive index along the fiber. These can arise due to defects introduced either accidentally or by design. The models considered allow for distributions of defects which are far more general than periodic. Our analysis addresses the simple situation of energy in a single transverse mode propagating and being radiated away due to coupling by defects to continuum modes. The bound state channel sees an effective damping. In particular the results of this paper have been applied to a study of structural instability of so-called breather modes of planar "soliton wave guides" [12]. The case of multiple transverse modes is of great interest [13]. Here, one has the phenomena of coupling among discrete modes as well as the coupling of discrete to continuum/radiation modes [7]. There is extensive interesting work on this problem in the case where $W(x_\perp, z)$ is a stochastic process in z and radiation is neglected [8].

(b) Nonlinear problems can be viewed as linear time-dependent potential problems where the time-dependent potential is given by the solution. *A priori* one knows little about the time dependence of the solution of a nonlinear problem. Nonlinearity is expected, in general, to excite infinitely many frequencies. Therefore results of a general nature for potentials with very general time dependence are of interest. This point of view is adopted by Sigal [19, 20], who considers the case where the nonlinear term defines a time-periodic perturbation and then proceeds to study the resonance problem via time-independent Floquet analysis applied to the so-called Floquet Hamiltonian. The dilation analytic techniques used were first applied in the context of time-periodic Hamiltonians by Yajima [26, 27, 28]. Floquet-type methods were also used in the time-periodic context by Vainberg [25]. The general class of perturbations we consider are not treatable by Floquet analysis and time-dependent analysis appears necessary.

1.2. Outline of the method. We now give a brief outline of our approach. For simplicity consider the initial value problem

$$i\partial_t \phi(t,x) = H_0 \phi(t,x) + \varepsilon W(t,x) \phi(t,x), \tag{1.7}$$

$$\phi|_{t=0} = \phi(0), \tag{1.8}$$

where

$$H_0 = -\Delta + V(x),\ W(t,x) = g(t)\ \beta(x),\ g(t) = \sum_j g_j\ e^{-i\mu_j t} \tag{1.9}$$

is a real-valued almost periodic function of t, and $\beta(x)$ is a real-valued and rapidly decaying function of x as $|x| \to \infty$. The unperturbed problem ($\varepsilon = 0$) can be trivally written as two decoupled equations governing the bound state amplitude, $a(t)$, and dispersive components, $\phi_d(t)$, of the solution. Specifically, let

$$\phi(t) \;=\; a(t)\;\psi_0(x) \;+\; \phi_d(t,x),\;(\psi_0,\phi_d(t)) \;=\; 0. \tag{1.10}$$

Then,

$$\begin{aligned} i\partial_t a(t) &= \lambda_0 a(t), \\ i\partial_t\;\phi_d(t,x) &= H_0\;\phi_d(t,x) \end{aligned} \tag{1.11}$$

with initial conditions

$$\begin{aligned} a(0) &= (\psi_0,\phi(0))\,, \\ \phi_d(0) &= \mathbf{P_c}\phi(0), \end{aligned} \tag{1.12}$$

where

$$\mathbf{P_c} f \equiv\; f - (\psi_0, f)\,\psi_0$$

defines the projection onto the continuous spectral part of H_0.

For initial data $a(0) = 1$, $\phi_d(0) = 0$, we have $a(t) = e^{-i\lambda_0 t}$, $\phi_d(t) \equiv 0$, corresponding to the bound state, $b_*(t)$.

We now ask the following:

(a) *Under the small perturbation $\varepsilon W(t,x)$, does the bound state deform or continue to a nearby periodic or even almost periodic solution?*

(b) *How do solutions to the perturbed initial value problem behave as $|t| \to \infty$?*

For small perturbations $\varepsilon W(t,x)$ it is natural to use the decomposition (1.10). Substitution of (1.10) into (1.3) yields a weakly coupled system for $a(t)$ and $\phi_d(t)$. This system is derived and analyzed in detail in sections 4–6.

In order to illustrate the main idea, we introduce a simplified system having the same general character:

$$\begin{aligned} i\partial_t a(t) \;&=\; \lambda_0\;a(t) \;+\; \varepsilon g(t)\;\;(\beta\psi_0, \phi_d(t)) \\ i\partial_t \phi_d(t,x) \;&=\; -\Delta\phi_d(t,x) \;+\; \varepsilon a(t) g(t)\beta(x)\psi_0(x). \end{aligned} \tag{1.13}$$

Here, we have replaced H_0 on its continuous spectral part by $-\Delta$.

If $\varepsilon\beta$ is small, then $A(t) \equiv e^{i\lambda_0 t}a(t)$ is slowly varying ($\partial_t A(t) \;=\; \mathcal{O}(\varepsilon\beta)$). In particular, we have

$$\begin{aligned} i\partial_t A(t) \;&=\; \varepsilon e^{i\lambda_0 t}\;g(t)\;\;(\beta\psi_0, \phi_d(t)) \\ i\partial_t \phi_d(t,x) \;&=\; -\Delta\phi_d(t,x) \;+ A(t)e^{-i\lambda_0 t}\;\varepsilon g(t)\beta(x)\psi_0(x). \end{aligned} \tag{1.14}$$

Viewing $A(t)$ as nearly constant, we see that the inhomogeneous source term in (1.14) has frequencies $\lambda_0 + \mu_j$; see (1.9). Therefore, if $\lambda_0 + \mu_j > 0$ for some j, then $\lambda_0 + \mu_j$ lies in the continuous spectrum of $-\Delta$ (H_0) and therefore ϕ_d satisfies a resonantly forced wave equation. A careful expansion and analysis to second order in the perturbation $\varepsilon W(t)$ (see the proof of Proposition 4.1) reveals the system for $A(t)$ and $\phi_d(t)$ can be rewritten in the following form, in which the effect of this resonance is made explicit:

$$\partial_t A(t) \;=\; (-\varepsilon^2\Gamma \;+\; \rho(t))\;A(t) \;+\; E(t; A(t), \phi_d(t)), \tag{1.15}$$

$$i\partial_t \phi_d(t,x) \;=\; H_0\;\phi_d(t,x) \;+\; \mathbf{P_c}\;F(t,x; A(t), \phi_d(t)). \tag{1.16}$$

The terms $E(t)$ and $F(t,x)$ formally tend to zero if $A(t)$ tends to zero and if the "local energy" of $\phi_d(t)$ tends to zero as $t \to \infty$. The strategy of sections 5 and 6 is to derive coupled estimates for $A(t)$ and a measure of the local energy of ϕ_d from which one can conclude, for $\varepsilon W(t)$ small, that solutions to (1.15)–(1.16) decay in an appropriate sense. The key to the decay of solutions is the constant Γ, given by

$$\Gamma \equiv \frac{\pi}{4} \sum_{\{j\,:\,\lambda_0+\mu_j>0\}} |g_j|^2 \ (\mathbf{P_c}\beta\psi_0,\ \delta(H_0-\lambda_0-\mu_j)\mathbf{P_c}\beta\psi_0)\,; \tag{1.17}$$

see also hypothesis **(H5)** of section 2. The quantity Γ is a generalization of the well-known Fermi golden rule arising in the theory of radiative transitions in quantum mechanics [3, 4, 10]. For the example at hand, (1.9), the sum in (1.17) is over all j for which $\mu_j+\lambda_0$ is strictly positive, i.e., lies in the continuous spectrum of H_0. Thinking of H_0 as having a spectral decomposition in terms of eigenfunctions and generalized eigenfunctions, let $e(\lambda)$ denote a generalized eigenfunction associated with the energy λ. Then each term in the sum (1.17) is of the form

$$|(\ e(\lambda_0+\mu_j), \beta\psi_0\ )|^2\,. \tag{1.18}$$

Thus, clearly $\Gamma > 0$, generically.

Neglecting for the moment the oscillatory function $\rho(t)$ in (1.15), we see that coupling of the bound state by the time dependent perturbation to the continuum-radiation modes, at the frequencies $\mu_j+\lambda_0 > 0$, leads to decay of the bound state. The leading order of (1.15)–(1.16) is a normal form in which this internal damping effect is made explicit; energy is transferred from the discrete to the continuous spectral components of the solution while the total energy remains independent of time:

$$\begin{aligned} \|\ \phi(t)\ \|_2^2 &= |a(t)|^2 \ +\ \|\ \phi_d(t)\ \|_2^2 \\ &= |a(0)|^2 \ +\ \|\ \phi_d(0)\ \|_2^2. \end{aligned} \tag{1.19}$$

1.3. Energy flow; contrast with the analysis of [22]. The goal is to show that energy flows out of the bound state channel into dispersive spectral components. The normal form above is the system in which this energy flow is made explicit. Once the normal form (1.15)–(1.16) has been derived, it is natural to seek coupled estimates for $A(t)$ and $\phi_d(t)$ from which their decay can be deduced. This is implemented in section 6. A natural first step is to introduce the auxiliary function

$$\tilde{A}(t) \equiv e^{\int_0^t \rho(s)\ ds}\ A(t) \tag{1.20}$$

for then $\tilde{A}(t)$ satisfies simplified equation of the form

$$\partial_t \tilde{A}(t) = -\varepsilon^2\Gamma\ \tilde{A}(t) \ +\ \tilde{E}(t;\tilde{A}(t),\phi_d(t)). \tag{1.21}$$

If $\Re\int_0^t \rho(s)\ ds$ is uniformly bounded, then modulo time-decay estimates on $\tilde{E}(t;\tilde{A},\phi_d)$ and $F(t;\tilde{A},\phi_d)$, the decay of $\tilde{A}(t)$ and therefore of $A(t)$ follows. For the class of perturbations considered in [22], $\rho(t)$ is a periodic function, having only a finite number of commensurate frequencies, none of them zero. Therefore, in this case $\Re\int_0^t \rho(s)\ ds$ is uniformly bounded. However, in the present case $\rho(t)$ is almost periodic with mean $M(\Re\rho) = 0$ (see section 9); $\rho(t)$ is displayed in (4.12). $\Re\rho(t)$ has, in general, infinitely many frequencies, $\mu_k-\mu_j,\ k\neq j$ which may accumulate at zero. Most delicate is the case where, along some subsequence, $\mu_k-\mu_j \to 0$. It is well known that the integral

of an almost periodic function of mean zero is not necessarily bounded [2], so we are in need of a strategy for estimating the effects of $\Re \int_0^t \rho(s)\, ds$. We address the estimation of $\Re \int_0^t \rho(s)\, ds$ in two different ways corresponding to Theorem 2.3 (see also section 5.1) and Theorems 2.1–2.2 (see also section 5.2). In section 5.1 $\Re \int_0^t \rho(s)\, ds$ is estimated under the hypothesis **(H6)** which requires that the rate of accumulation of a subset of frequencies $\{\mu_j\}_{j\in I}$ is balanced by the decay of the Fourier coefficients g_j as $j \to \infty,\ j \in I$. This leads to a bound on $\Re \int_0^t \rho(s)\, ds$ (Proposition 5.2). In section 5.2 the estimates are based on a more refined analysis; the almost periodic function $\rho(t)$ is decomposed into a part with bounded integral and a part which has mean zero. The latter is controlled using results on the rate at which an almost periodic function approaches its mean.

1.4. Fermi golden rule and obstructions to Poincaré continuation. In the theory of ordinary differential equations it is a standard procedure, given a periodic solution of an unperturbed problem, to seek a periodic or almost periodic solution of a slightly perturbed dynamical system. We now investigate this procedure in the context of (1.7) and its solution $b_*(t)$ for $\varepsilon = 0$. Seek a solution of the form

$$\phi(t) \;=\; b_*(t) \;+\; \phi_1(t) \;+\; \mathcal{O}(\varepsilon^2\beta^2). \tag{1.22}$$

Here, $\phi_1 \;=\; \mathcal{O}(\varepsilon\beta)$.[4] Substitution of (1.22) into (1.7) yields the equation

$$i\partial_t\phi_1 \;=\; H_0\phi_1 \;+\; \varepsilon\beta\; g(t)\; b_*(t). \tag{1.23}$$

This equation has a solution in the class of almost periodic solutions of t with values in the Hilbert space $\mathcal{H}$ only if $\beta\; g(t)\; b_*(t)$ is "orthogonal" to the null space of $i\partial_t - H_0$.

We now derive this condition. Let $e(\zeta)$ be a solution of $H_0 e(\zeta) = \zeta e(\zeta)$. Then, taking the scalar product of (1.23) with $e^{-i\zeta t}\; e(\zeta)$ and applying the operator $\lim_{T\uparrow\infty} T^{-1}\int_0^T \cdot\, dt$ to the resulting equation gives

$$0 \;=\; \lim_{T\uparrow\infty}\; T^{-1} \int_0^T \; e^{i\zeta t}\; e^{-i\lambda_0 t}\; g(t)\; dt\;\; (e(\zeta), \beta\psi_0)\,. \tag{1.24}$$

Substitution of the expansion for $g(t)$ yields

$$\sum_{j\in\mathbb{Z}} \; g_j\; \delta(\zeta, \lambda_0 + \mu_j)\;\; (e(\zeta), \beta\psi_0) \;=\; 0, \tag{1.25}$$

where $\delta(a,b) = 0$ if $a \neq b$ and $\delta(a,a) = 1$. If ζ, which lies in the spectrum of H_0, satisfies $\zeta = \lambda_0 + \mu_k$ for some $k \in \mathbb{Z}$ (which will be the case in our example if $\lambda_0 + \mu_k > 0$), then we have that

$$(e(\lambda_0 + \mu_k), \beta\psi_0) \;=\; 0 \tag{1.26}$$

is a necessary condition for the existence of a family of solutions of (1.7) which converges to $b_*(t)$ as the perturbation $W(t)$ tends to zero. We immediately recognize the inner product in (1.26) as the projection of $\beta\psi_0$ onto the generalized eigenmode at the resonant frequency $\lambda_0 + \mu_k$ which arises in (1.17); see also (1.18). Therefore the obstruction to continuation of $b_*(t)$ to a nearby almost periodic state of the system can be identified with the damping mechanism.

[4] This argument is heuristic so we do not specify the norm with which the size of β is measured.

1.5. Outline. The paper is structured as follows. In section 2 we give a general formulation of the problem. The hypotheses on H_0, the unperturbed Hamiltonian, and $W(t)$, the perturbation, are introduced and discussed. There are two types of theorems: Theorems 2.1 and 2.2 and Theorem 2.3. Although the conclusions of these are quite similar, as discussed above, they differ in a key hypothesis on the perturbation $W(t)$, which is relevant in the case where $W(t)$ has infinitely many frequencies which may resonate with the continuous spectrum. In section 3 we apply the results of section 2 to the case of Schrödinger operators $H_0 = -\Delta + V(x)$ defined on $L^2(\mathbb{R}^3)$. To check the key local energy decay hypotheses we use results of Jensen and Kato [5] on expansions of the resolvent of H_0 near zero energy, the edge of the continuous spectrum. In section 4 the dynamical system (1.3) is reformulated as a system governing the interaction of the bound state and dispersive part of the solution. This section contains an important computation in which the key resonance is made explicit and a perturbed "normal form" for the bound state evolution is derived (Proposition 4.1). Sections 5 and 6 contain estimates for the bound state and dispersive parts of the solution for intermediate and large time scales. In section 7 we discuss extensions of our Theorems 2.1–2.3 to a more general class of perturbations. We shall frequently make use of some singular operators which are rigorously defined in section 8, an appendix, and of elements of the theory of almost periodic functions [2, 9], which are assembled in section 9, the second appendix.

Notations and terminology. Throughout this paper we will use the following notations:

$\mathbb{N} = \{1,\ 2,\ 3, \ldots\}$;

$\mathbb{N}_0 = \{0,\ 1,\ 2,\ 3, \ldots\}$;

$\mathbb{Z} = \{\ldots, -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \ldots\}$;

for z a complex number, $\Re z$ and $\Im z$ denote, respectively, its real and imaginary parts; a generic constant will be denoted by C, D, etc;

$\langle x \rangle = \left(1 + |x|^2\right)^{\frac{1}{2}}$;

$\mathcal{L}(\mathcal{A}, \mathcal{B}) =$ the space of bounded linear operators from $\mathcal{A}$ to $\mathcal{B}$; $\mathcal{L}(\mathcal{A}, \mathcal{A}) \equiv \mathcal{L}(\mathcal{A})$.

Functions of self-adjoint operators are defined via the spectral theorem; see, for example, [17]. The operators containing boundary value of resolvents or singular distributions applied to self-adjoint operators are defined in section 8.

2. General formulation and main results. Consider the general system

$$\begin{aligned} i\partial_t \phi(t) &= (H_0 + W(t))\, \phi(t), \\ \phi|_{t=0} &= \phi(0). \end{aligned} \tag{2.1}$$

Here, $\phi(t)$ denotes a function of time, t, with values in a complex Hilbert space $\mathcal{H}$.

Hypotheses on H_0.

(H1) H_0 is self-adjoint on $\mathcal{H}$ and both H_0 and $W(t)$, $t \in \mathbb{R}^1$, are densely defined on a subspace $\mathcal{D}$ of $\mathcal{H}$.

The norm on $\mathcal{H}$ is denoted by $\|\cdot\|$ and the inner product of $f, g \in \mathcal{H}$, by (f, g).

(H2) The spectrum of H_0 is assumed to consist of an absolutely continuous part, $\sigma_{\rm cont}(H_0)$, with associated spectral projection $\mathbf{P_c}$ and a single isolated eigenvalue λ_0 with corresponding normalized eigenstate, ψ_0, i.e.,

$$H_0\psi_0 = \lambda_0\psi_0,\ \|\psi_0\| = 1. \tag{2.2}$$

The manner in which we shall measure the decay of solutions is typically in a local decay sense, e.g., for the scalar Schrödinger equation governing a function defined on

$\mathbb{R}^n$ we measure local decay using the norms $f \mapsto \|\langle x\rangle^{-s} f\|_{L^2}$, where $s > 0$. So that our theory applies to a class of general systems (involving, for example, vector equations with matrix operators), we assume the existence of self-adjoint "weights" w_- and w_+ such that

(i) w_+ is defined on a dense subspace of $\mathcal{H}$ and on which $w_+ \geq cI,\ c > 0$.

(ii) $w_- \in \mathcal{L}(\mathcal{H})$ such that Range$(w_-) \subseteq$ *Domain*(w_+).

(iii) $w_+\ w_-\ \mathbf{P_c}\ =\ \mathbf{P_c}$ on $\mathcal{H}$ and $\mathbf{P_c}\ =\ \mathbf{P_c}\ w_-\ w_+$ on the domain of w_+.

In the scalar case, w_+ and w_- correspond to multiplication by $\langle x\rangle^s$ and $\langle x\rangle^{-s}$, respectively; see section 3.

The following hypothesis ensures that the unperturbed dynamics satisfies sufficiently strong dispersive time-decay estimates. Let $\{\mu_j\}_{j\in\mathbb{Z}}$ denote the set of Fourier exponents associated with the perturbation W (see hypothesis **(H4)** below).

(H3) *Local decay estimates on* e^{-iH_0t}.

Let $r_1 > 1$. There exist w_+ and w_-, as above, and a constant $\mathcal{C}$ such that for all $f \in \mathcal{H}$ satisfying $w_+ f \in \mathcal{H}$ we have

$$\text{(2.3)}\quad \textbf{(a)}\quad \|w_- e^{-iH_0t}\mathbf{P_c} f\| \leq \mathcal{C}\ \langle t\rangle^{-r_1} \|w_+ f\| \text{ for } t \in \mathbb{R};$$

$$\text{(2.4)}\quad \textbf{(b)}\quad \|w_- e^{-iH_0t}(H_0 - \lambda_0 - \mu_j - i0)^{-1}\ \mathbf{P_c} f\| \leq\ \mathcal{C}\ \langle t\rangle^{-r_1}\ \|w_+ f\| \text{ for } t \geq 0$$

and for all $j \in \mathbb{Z}$. For $t < 0$ estimate (2.4) is assumed to hold with $-i0$ replaced by $+i0$. See section 8 for the definition of the singular operator in (2.4).

Remark 2.1. There is a good deal of literature on local energy decay estimates of the form (2.3) for $e^{-iH_0t}\mathbf{P_c}$ in the case $H_0 = -\Delta + V(x)$ on $L^2(\mathbb{R}^n)$. These results require sufficient regularity and decay of the potential $V(x)$. We refer the reader to [5, 6] and [14]; see also [16, 18].

Remark 2.2. Estimates of the type **(H3b)** are obtained in [22], [23, Appendix A]. A key point here is that we require that one can choose the constant, $\mathcal{C}$, in (2.4) to hold for all μ_j. It appears difficult to deduce this uniformity of the constant by the general arguments used in [22] and [23]. However, in section 3, where we apply our results to a class of Schrödinger operators, we can verify (**H3b**) using known results on the spectral measure.

(H4) Hypotheses on the perturbation $W(t)$.

We consider time-dependent symmetric perturbations of the form

$$\text{(2.5)}\qquad W(t) = \frac{1}{2}\beta_0 + \sum_{j\in\mathbb{N}} \cos(\mu_j t)\ \beta_j \text{ with } \beta_j^* = \beta_j \text{ and } \sum_{j\in\mathbb{N}_0} \|\beta_j\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

In many applications, β_j are spatially localized scalar or matrix functions. Note that formula (2.5) can be rewritten in the form

$$\text{(2.6)}\qquad W(t) = \frac{1}{2}\ \sum_{j\in\mathbb{Z}} \exp(-i\mu_j t)\beta_j,$$

where $\mu_0 = 0$ and for $j < 0$, $\mu_j = -\mu_{-j},\ \beta_j = \beta_{-j}$. Thus, $W(t)$ is an almost periodic function with values in the Banach space $\mathcal{L}(\mathcal{H})$ with the Fourier exponents $\{\mu_j\}_{j\in\mathbb{Z}}$ and corresponding Fourier coefficients $\{\beta_j\}_{j\in\mathbb{Z}}$; see, for example, [9].

To measure the size of the perturbation W, we introduce the norm

$$\text{(2.7)}\qquad |||W||| \ \equiv\ \frac{1}{2}\sum_{j\in\mathbb{Z}} \|w_+\ \beta_j\|_{\mathcal{L}(\mathcal{H})}\ +\ \frac{1}{2}\sum_{j\in\mathbb{Z}} \|\ \beta_j\ \|_{\mathcal{L}(\mathcal{H}_-,\mathcal{H}_+)},$$

which is assumed to be finite. Here $\mathcal{H}_+$, respectively, $\mathcal{H}_-$, denote the closure of the domain of w_+, respectively, the range of $\mathbf{P_c}$, with norm $f \to \|w_+ f\|$, respectively, $f \to \|w_- f\|$.

Remark 2.3. A special case which arises in various models is

$$W(t) \;=\; g(t)\beta, \tag{2.8}$$

where

$$g(t) \;=\; \sum_j \; g_j \cos \mu_j t, \tag{2.9}$$

$\|w_+\beta\|_{\mathcal{L}(\mathcal{H})} + \; \|\beta\|_{\mathcal{L}(\mathcal{H}_-,\mathcal{H}_+)} \; < \; \infty$ and the sequence $\{g_j\}$ is absolutely summable.

Remark 2.4. Our results are valid in the more general case

$$W(t) = \frac{1}{2}\beta_0 + \sum_{j\in\mathbb{N}} \cos(\mu_j t + \delta_j)\; \beta_j,$$

where β_j are self-adjoint such that expression (2.7) is finite. This follows because the proofs use only the self-adjointness of W and the expansion

$$W(t) = \frac{1}{2} \sum_{j\in\mathbb{Z}} \exp(-i\mu_j t)\tilde{\beta}_j,$$

where $\tilde{\beta}_j = e^{-i\mathrm{sgn}(j)\delta_j}\beta_j$ and $\mu_{-j} = -\mu_j,\ \mu_0 = 0$.

We will impose a resonance condition which says that $\{\lambda_0 + \mu_j\}_{j\in\mathbb{Z}} \cap \sigma_{\mathrm{cont}}(H_0)$ is nonempty and that there is nontrivial coupling; see section 1.4. Let us first denote by I_{res} the following set:

$$I_{res} \;=\; \{j \in \mathbb{Z} \;:\; \lambda_0 + \mu_j \in \sigma_{\mathrm{cont}}(H_0)\}. \tag{2.10}$$

(H5) *Resonance condition.* Fermi golden rule.

I_{res} is nonempty and furthermore, there exists $\theta_0 > 0$, independent of W, such that

$$\Gamma \;\equiv\; \frac{\pi}{4} \sum_{j\in I_{res}} (\mathbf{P_c}\beta_j\psi_0,\; \delta(H_0 - \lambda_0 - \mu_j)\mathbf{P_c}\beta_j\psi_0.) \geq \theta_0 |||W|||^2 > 0. \tag{2.11}$$

Remark 2.5. For the exact definition of the Dirac-type operator in (2.11), see section 8. That Γ is finite is a consequence of the estimate (8.8) and

$$\Gamma \leq \frac{C_0}{\pi} \sum_j \|w_+\beta_j\|^2 \leq \frac{C_0}{\pi} |||W|||^2; \tag{2.12}$$

see also [1].

We now state our main results.

THEOREM 2.1. *Let us fix H_0 and $W(t)$ satisfying hypotheses* **(H1)**–**(H5)**. *Consider the initial value problem*

$$\begin{aligned} i\partial_t \phi(t) &= (H_0 + \varepsilon W(t))\, \phi(t), \\ \phi|_{t=0} &= \phi(0) \end{aligned} \tag{2.13}$$

with $w_+\phi(0) \in \mathcal{H}$. Then, there exists an $\varepsilon_0 > 0$ (depending on $\mathcal{C}$, r_1, and θ_0) such that whenever $|\varepsilon| < \varepsilon_0$, the solution, $\phi(t)$, of (2.13) *satisfies the local decay estimate*

$$\|w_- \ \phi(t)\| \leq C\langle t\rangle^{-r_1}\|w_+ \ \phi(0)\|, \quad t \in \mathbb{R}. \tag{2.14}$$

Under the same hypotheses as Theorem 2.1, we obtain more detailed information on the behavior of $\phi(t)$.

THEOREM 2.2. *Assume the hypotheses of Theorem* 2.1. *For any $0 < \gamma < \Gamma$ there exist the constants C and D (depending on $\mathcal{C}$, r_1, θ_0, and γ) such that any solution of* (2.13)*, for $|\varepsilon| < \varepsilon_0$ and $w_+\phi(0) \in \mathcal{H}$, satisfies*

$$\begin{aligned}
\phi(x,t) &= a(t)\psi_0 + \phi_d(t), \quad (\psi_0 \ , \ \phi_d(t),) \ = \ 0,\\
a(t) &= \ a(0) \ e^{-\varepsilon^2(\Gamma-\gamma)|t|}e^{i\omega(t)} \ + \ R_a(t),\\
P(t) &= \ P(0) \ e^{-2\varepsilon^2(\Gamma-\gamma)|t|} \ + \ R_a'(t),\\
\phi_d(t) &= e^{-iH_0t} \ \mathbf{P_c}\phi(0) \ + \ \tilde{\phi}(t),
\end{aligned} \tag{2.15}$$

where Γ is given by (2.11) *and $\omega(t)$ is a real-valued phase given by*

$$\begin{aligned}
\omega(t) \ &= \ -\lambda_0 t - \varepsilon\left(\psi_0, \ \int_0^t W(s)ds \ \psi_0\right)\\
&\quad + \frac{1}{4}\varepsilon^2 t\sum_{j\in\mathbb{Z}}\left(\beta_j\psi_0, \mathrm{P.V.}(H_0-\lambda_0-\mu_j)^{-1}\mathbf{P_c}\beta_j\psi_0\right)\\
&\quad + \ \frac{1}{4}\varepsilon^2 \ \Re\int_0^t \sum_{j,k\in\mathbb{Z}, j\neq k} e^{i(\mu_k-\mu_j)t}\left(\beta_k\psi_0, (H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right).
\end{aligned} \tag{2.16}$$

$P(t)$ is defined in (1.4) *and for any fixed $T_0 > 0$ we have*

$$|R_a(t)| \ \leq \ C \ |\varepsilon| \ |||W|||, \ |t| \leq \frac{T_0}{\varepsilon^2\Gamma}, \tag{2.17}$$

$$|R_a'(t)| \ \leq \ D \ |\varepsilon| \ |||W|||, \ |t| \leq \frac{T_0}{\varepsilon^2\Gamma}. \tag{2.18}$$

Moreover,

$$|R_a(t)| = \mathcal{O}(\langle t\rangle^{-r_1}), \ |R_a'(t)| = \mathcal{O}(\langle t\rangle^{-r_1}), \ |t| \to \infty.$$

Finally, $\tilde{\phi} = \phi_1 + \phi_2$ is given in (4.9) *with $\|w_-\tilde{\phi}(t)\| = \mathcal{O}(\langle t\rangle^{-r_1})$ as $|t| \to \infty$. Therefore, by* **(H3)** *$\|w_-\phi_d(t)\| = \mathcal{O}(\langle t\rangle^{-r_1})$ as $|t| \to \infty$.*

Remark 2.6. Suppose the initial data is given by the bound state of the unperturbed problem, i.e., $\phi(x,0) = \psi_0(x)$, $a(0) = 1$, $\phi_d(0) = 0$. Then, from the expansion of the solution we have that for $0 \leq t \leq \varepsilon^{-2}\Gamma^{-1}$ that $P(t)$ (see (1.4)) is of order $e^{-2\varepsilon^2(\Gamma-\gamma)t}$ with an error of order ε. Hence it is natural to view the state $\psi_0 e^{-i\lambda_0 t}$ as a metastable state with lifetime $\tau = \varepsilon^{-2}(\Gamma-\gamma)^{-1} \sim \varepsilon^{-2}|||W|||^{-2}$. Although $\gamma > 0$ is arbitrary we have not inferred that the actual lifetime is $\tau = \varepsilon^{-2}\Gamma^{-1}$ under hypotheses **(H1)**–**(H5)**. The reason is that the constants C and D in the estimates (2.17) and (2.18) blow up as $\gamma \searrow 0$. In order to remedy this we need an additional hypothesis.

(H6) *Control of small denominators.*

There exists $\xi > 0$, independent of W, such that

$$\sum_{j\in I_{res}, \ k\in\mathbb{Z}, \ j\neq k}\left|\frac{1}{\mu_j-\mu_k}(\mathbf{P_c}\beta_k\psi_0, \delta(H_0-\lambda_0-\mu_j)\mathbf{P_c}\beta_j\psi_0)\right| \leq \xi \ |||W|||^2. \tag{2.19}$$

Remark 2.7. By (8.8) we have that

$$\sum_{j\in I_{res},\ k\in\mathbb{Z},\ j\neq k} |(\beta_k\psi_0, \delta(H_0-\lambda_0-\mu_j)\beta_j\psi_0)| \leq C\ \pi^{-1}\ |||W|||^2 \tag{2.20}$$

is finite (see also Remark 2.5). Thus, **(H6)** is important only if

$$\inf\{|\mu_j-\mu_k|:\ j,k\in\mathbb{Z},\ j\neq k \text{ and } \lambda_0+\mu_j\in\sigma_{\rm cont}(H_0)\}=0, \tag{2.21}$$

i.e., the Fourier exponents $\{\mu_j\}$ are such that $\lambda_0+\mu_j$ accumulate in σ_c. In particular, if the perturbation $W(t)$ consists of a trigonometric polynomial

$$W(t)\ =\ \sum_{j=1}^{N}\ \cos\mu_j t\ \beta_j, \tag{2.22}$$

then **(H6)** is trivially satisfied.

Remark 2.8. Hypothesis **(H6)** can be imposed by balancing the clustering of the frequencies $\lambda_0+\mu_j$ in the continuous spectrum of H_0 with rapid decay of $(\beta_k\psi_0\ ,\ \delta(H_0-\lambda_0-\mu_j)\ \beta_j\psi_0)$ as $j,k\to\infty$. Let $\beta_j(x)=g_j\ \beta(x)$. Then, $W(t,x)=\sum_j\ g_j\cos(\mu_j t)\ \beta(x)$. Using Remark 2.5 we find that the left-hand side of (2.19) is bounded by $\sum_{j,k\in\mathbb{Z};j\neq k}|g_jg_k|\ |\mu_j-\mu_k|^{-1}\ |||W|||^2$. The constant ξ in (2.19) is finite if, for example, $\mu_j=2|\lambda_0|+|j|^{-1},\ g_j=|j|^{-2-\tau},\ \tau>0$.

In case **(H6)** is satisfied we have the following improvement of Theorem 2.2.

THEOREM 2.3. *Assume the hypotheses* **(H1)**–**(H6)** *hold. Then there exist* ε_0 *and the constants* C, D *(depending on* $\mathcal{C}$, r_1, θ_0 *and* ξ*) such that any solution of* (2.13)*, for* $|\varepsilon|<\varepsilon_0$ *and* $w_+\phi(0)\in\mathcal{H}$*, satisfies*

$$\begin{aligned}
\phi(x,t) &= a(t)\psi_0+\phi_d(t),\quad (\psi_0\ ,\ \phi_d(t))\ =\ 0,\\
a(t) &= a(0)\ e^{-\varepsilon^2\Gamma|t|}\ e^{i\omega(t)}\ +\ R_a(t),\\
P(t) &= P(0)\ e^{-2\varepsilon^2\Gamma|t|}\ +\ R_a'(t),\\
\phi_d(t) &= e^{-iH_0t}\ \mathbf{P_c}\phi(0)\ +\ \tilde{\phi}(t).
\end{aligned} \tag{2.23}$$

Here, $\omega(t)$ *is given by* (2.16) *and* $R_a(t)$, $R_a'(t)$, $w_-\phi_d(t)$ *satisfy the estimates of Theorem* 2.2*.*

3. An application: The Schrödinger equation. In this section we verify hypotheses **(H1)**–**(H4)** in the particular case of the Schrödinger equation on the three-dimensional space with a time almost periodic and spatially localized perturbing potential:

$$i\partial_t\phi\ =\ (-\Delta+V(x))\,\phi\ +\ \varepsilon W(x,t)\phi \tag{3.1}$$

with $\phi:\mathbb{R}^3\times\mathbb{R}\to\mathbb{C},\ (x,t)\to\phi(x,t)$, and

$$W(x,t)=\frac{1}{2}\beta_0(x)+\sum_{j\in\mathbb{N}}\cos(\mu_j t)\beta_j(x),$$

where $\mu_j\in\mathbb{R},\ j\in\mathbb{N}_0$, and $\beta_j:\mathbb{R}^3\to\mathbb{R},\ j\in\mathbb{N}$, are localized functions. Models of the sort considered in this example occur in the study of ionization of an atom by a time-varying electric field; see [10, 4].

We take $\mathcal{H} = L^2(\mathbb{R}^3)$ and $H_0 \equiv -\Delta + V(x)$, where $V(x)$ is real-valued with moderately short range. More precisely, we suppose that there exists $\sigma > 4$ and a constant D such that

$$|V(x)| \leq D(1+|x|)^{-\sigma}. \tag{3.2}$$

Thus, H_0 is self-adjoint and densely defined in L^2. In what follows we assume that H_0 has exactly one eigenvalue which is strictly negative and that the remainder of the spectrum is absolutely continuous and equal to the positive half-line. Our results can be extended to operators with strictly negative, multiple eigenvalues [7].

We first discuss the local decay hypothesis **(H3)**. As weights used to measure local energy decay we take $w_\pm \equiv \langle x\rangle^{\pm s}$, where $s > 7/2$ and fix $r_1 = 3/2$. Our aim is to obtain the estimates

$$\textbf{(H3a)} \quad \|w_- e^{-iH_0 t}\mathbf{P_c} f\| \leq \mathcal{C}\ \langle t\rangle^{-3/2}\|w_+ f\|, \tag{3.3}$$

$$\textbf{(H3b)} \quad \|w_- e^{-iH_0 t}(H_0 - \lambda_0 - \mu_j - i0)^{-1}\ \mathbf{P_c} f\| \leq\ \mathcal{C}\ \langle t\rangle^{-3/2}\ \|w_+ f\| \tag{3.4}$$

for all $\mu_j \in \mathbb{Z}$ with $\mathcal{C}$ independent of j.

We shall assume that the frequencies $\{\lambda_0 + \mu_j\}$ do not accumulate at zero, the edge of the continuous spectrum of H_0:

$$m_* \ \equiv\ \min\{\ |\lambda_0 + \mu_j|\ :\ j \in \mathbb{Z}\ \} >\ 0. \tag{3.5}$$

To prove (3.3) and (3.4) we use the spectral representation for the operators $e^{-iH_0 t}\ \mathbf{P_c}$ and $e^{-iH_0 t}\ (H_0 - \lambda_0 - \mu_j - i0)^{-1}\mathbf{P_c}$, namely,

$$e^{-iH_0 t}\mathbf{P_c} = \int_0^\infty e^{-i\lambda t}E'(\lambda)d\lambda, \tag{3.6}$$

$$e^{-iH_0 t}(H_0 - \lambda_0 - \mu_j - i0)^{-1}\mathbf{P_c} = \int_0^\infty e^{-i\lambda t}(\lambda - \lambda_0 - \mu_j - i0)^{-1}E'(\lambda)d\lambda, \tag{3.7}$$

where $E'(\lambda) = \pi^{-1}\ \Im(H_0 - \lambda - i0)^{-1}$ is the spectral density induced by H_0, [5].

The technique of getting **(H3a)** from (3.6) is presented in [5, section 10] and it can be summarized in the following way. We decompose the integral in (3.6) in two parts, corresponding to low energies (λ near zero) and high energies (λ away from zero) by writing

$$\begin{aligned} E' &= \chi E' + (1-\chi)E', \\ w_- e^{-iH_0 t}\mathbf{P_c} w_- &= \int_0^\infty e^{-i\lambda t}\chi(\lambda)w_- E'(\lambda)w_- d\lambda + \int_0^\infty e^{-i\lambda t}(1-\chi(\lambda))w_- E'(\lambda)w_- d\lambda \\ &=\ S_1\ +\ S_2. \end{aligned} \tag{3.8}$$

Here, $\chi(\lambda)$ is a smoothed characteristic function of a neighborhood of origin, chosen so that

$$\begin{aligned} \chi(\lambda)\ &\equiv\ 1,\ |\lambda|\ \leq\ \frac{1}{2}m_*, \\ \chi(\lambda)\ &\equiv\ 0,\ |\lambda|\ \geq\ \frac{3}{4}m_*. \end{aligned}$$

To estimate the two integrals in (3.8) we make use of the detailed results of [5] on the family of operators $\{E'(\lambda)\}$. First, by Theorem 8.1 and Corollary 8.2 of [5], $w_-\partial_\lambda^k E'(\lambda)w_-$ is bounded on L^2 and satisfies

$$\|\ w_-\partial_\lambda^k E'(\lambda)w_-\ \|_{\mathcal{B}(L^2)} = O(\lambda^{-(k+1)/2}) \text{ as } \lambda \to \infty \tag{3.9}$$

for $k \in \{0,1,2,3\}$. Integration by parts twice in the second integral in (3.8) and use of the estimate (3.9) with $k=2$ yields the estimate

$$\| S_2 \|_{\mathcal{B}(L^2)} = o(t^{-2}) \text{ as } t \to \infty. \tag{3.10}$$

Next, by Theorem 6.3 of [5] we have the low energy asymptotic expansion

$$w_- E'(\lambda) w_- = -\lambda^{-1/2} B_{-1} + \lambda^{1/2} B_1 + o(\lambda^{1/2}) \text{ as } \lambda \to 0, \tag{3.11}$$

where $B_{-1},\ B_1$ are bounded linear operators on L^2. Use of this expansion in the first integral of (3.8) yields the expansion in $\mathcal{B}(L^2)$:

$$S_1 = (\pi i)^{-1/2} t^{-1/2} B_{-1} - (4\pi i)^{-1/2} t^{-3/2} B_1 + o(t^{-3/2}) \text{ as } t \to \infty. \tag{3.12}$$

Thus, **(H3a)** is satisfied provided that B_{-1} is the null operator or equivalently $H_0\psi = 0$ has no solution with the property $w_-\psi \in L^2(\mathbb{R}^3)$. The last condition holds for generic potentials $V(x)$ and when it is violated one says that H_0 has *zero energy resonance*; see [5] for details.

In the same way one can prove **(H3b)** from the spectral representation (3.7) provided that the integral is nonsingular, i.e., $\lambda_0+\mu_j < 0$. In the case $\lambda_0+\mu_j \geq m_* > 0$ we first decompose the singular integral in two parts, one away from singularity point, $\lambda_0 + \mu_j$, and the other in a neighborhood of it by using the smoothed characteristic function

$$\chi_j(\lambda) = \chi(\lambda - \lambda_0 - \mu_j), \tag{3.13}$$

which is supported in a neighborhood of $\lambda_0 + \mu_j$, which does not include $\lambda = 0$:

$$\begin{aligned} e^{-iH_0 t}(H_0 - \lambda_0 - \mu_j - i0)^{-1}\mathbf{P_c} = &\int_0^\infty e^{-i\lambda t}(\lambda - \lambda_0 - \mu_j)^{-1}(1 - \chi_j(\lambda))E'(\lambda)d\lambda \\ &+ \int_0^\infty e^{-i\lambda t}(\lambda - \lambda_0 - \mu_j - i0)^{-1}\chi_j(\lambda)E'(\lambda)d\lambda. \end{aligned} \tag{3.14}$$

The nonsingular integral may be treated as above while the singular one defines the singular operator

$$T_j = e^{-iH_0 t}(H_0 - \lambda_0 - \mu_j - i0)^{-1}\chi_j(H_0)\mathbf{P_c}$$

via the spectral theorem. Here, $T_j = \lim_{\eta \searrow 0} T_j^\eta$, where

$$T_j^\eta \equiv e^{-iH_0 t}(H_0 - \lambda_0 - \mu_j - i\eta)^{-1}\chi_j(H_0)\mathbf{P_c}.$$

To estimate its L^2 operator norm we use the integral representation

$$w_- T_j^\eta w_- = \frac{1}{i}\int_t^\infty e^{i(\lambda_0+\mu_j+i\eta)(s-t)} w_- e^{-iH_0 s}\chi_j(H_0)\mathbf{P_c} w_- ds. \tag{3.15}$$

But this reduces to the evaluation of

$$w_- e^{-iH_0 s}\chi_j(H_0)\mathbf{P_c} w_- = \int_0^\infty e^{-i\lambda s}\chi_j(\lambda) w_- E'(\lambda) w_- d\lambda,\ s \geq t, \tag{3.16}$$

where we used again the spectral representation theorem. Integration by parts three times in (3.16) and use of the estimate (3.9) with $k=3$ implies

$$\|w_- e^{-iH_0 s}\chi_j(H_0)\mathbf{P_c} w_-\|_{\mathcal{B}(L^2)} = o(s^{-3}) \text{ as } t \to \infty.$$

Replacing this in (3.15), integrating and passing to the limit as $\eta \searrow 0$ we obtain an $o(t^{-2})$ estimate for T_j which is even better than we need to satisfy **(H3b)**.

Moving now towards hypothesis **(H4)**, we may choose the time-dependent perturbation to be of the form

$$W(x,t) \ = \frac{1}{2}\beta_0 \ + \ \sum_{j\in\mathbb{N}} \cos\mu_j t \ \beta_j(x) \tag{3.17}$$

with β_j rapidly decaying in x, e.g., $\langle x\rangle^{2s}\|\beta_j(x)\| \leq C_j$ for all $x \in \mathbb{R}^3$, $j \in \mathbb{N}_0$, where $\sum_{j\in\mathbb{N}_0} C_j < \infty$. Thus, **(H4)** is satisfied as well.

Therefore, our main results Theorems 2.1–2.2 on the structural instability of the unperturbed bound state and large time behavior for systems of the form (3.1) apply provided **(H5)**, the Fermi golden rule resonance condition, holds. For results concerning more general perturbations than the ones in (3.1) see section 7.

4. Decomposition and derivation of the dispersive normal form. The results of this section rely on hypothesis **(H1)** through **(H4)** only, so they may and will be used in proving Theorems 2.1–2.3.

As in [21, 22] and [23], we begin by deriving a decomposition of the solution, $\phi(t)$, which will facilitate the study of its large time behavior. Let

$$\phi(t) = a(t)\psi_0 + \phi_d(t) \tag{4.1}$$

with the orthogonality condition

$$(\psi_0, \phi_d(t)) = 0 \ \text{ for all } \ t. \tag{4.2}$$

Note therefore that $\phi_d \ = \ \mathbf{P_c}\phi_d$.

We proceed by first inserting (4.1) into (2.13), which yields the equation

$$\begin{aligned} i\partial_t a(t)\psi_0 \ + \ i\partial_t\phi_d(t) \ &= \ \lambda_0 a(t)\psi_0 \ + H_0\phi_d(t) \\ &\quad + \ \varepsilon a(t)W(t)\psi_0 \ + \ \varepsilon W(t)\phi_d(t). \end{aligned} \tag{4.3}$$

Taking the inner product of (4.3) with ψ_0 we get the following equation for $a(t)$:

$$\begin{aligned} i\partial_t a \ &= \ \lambda_0 a(t) \ + \ \varepsilon\,(\psi_0, W(t)\psi_0)\,a(t) \ + \ \varepsilon\,(\psi_0, W(t)\phi_d)\,, \\ a(0) \ &= \ (\psi_0, \phi(0))\,. \end{aligned} \tag{4.4}$$

In deriving (4.4) we have used that ψ_0 is normalized and the relation

$$(\psi_0, \partial_t\phi_d) = 0, \tag{4.5}$$

a consequence of (4.2).

Applying $\mathbf{P_c}$ to (4.3), we obtain an equation for ϕ_d:

$$\begin{aligned} i\partial_t\phi_d(t) \ &= \ H_0\phi_d(t) \ + \ \varepsilon\mathbf{P_c}W(t)\phi_d(t) \ + \ \varepsilon a(t)\mathbf{P_c}W(t)\psi_0, \\ \phi_d(0) \ &= \ \mathbf{P_c}\phi(0). \end{aligned} \tag{4.6}$$

Since we are after a slow resonant decay phenomenon, it will prove advantageous to extract the fast oscillatory behavior of $a(t)$. We therefore define

$$A(t) \equiv e^{i\lambda_0 t}a(t). \tag{4.7}$$

Then, (4.4) reads

$$\partial_t A \;=\; -i\varepsilon A\,(\psi_0, W(t)\psi_0) \;-\; i\varepsilon e^{i\lambda_0 t}\;(\psi_0, W(t)\phi_d(t))\,. \tag{4.8}$$

Solving (4.6) by Duhamel's formula we have

$$\begin{aligned}\phi_d(t) &= e^{-iH_0 t}\phi_d(0) \;-\; i\varepsilon\int_0^t e^{-iH_0(t-s)}\mathbf{P_c}W(s)a(s)\psi_0 ds\\ &\quad -i\varepsilon\int_0^t e^{-iH_0(t-s)}\mathbf{P_c}W(s)\phi_d(s)\;ds\\ &\equiv \phi_0(t)\;+\;\phi_1(t)\;+\;\phi_2(t).\end{aligned} \tag{4.9}$$

By standard methods, the system (4.8)–(4.9) for $A(t)$ and $\phi_d(t) = \phi(t) - e^{-i\lambda_0 t}\,A(t)\,\psi_0$ has a global solution in t with

$$A \in C^1(\mathbb{R}),\;\; \|\phi_d(t)\| \in C^0(\mathbb{R}),\;\; \|w_-\phi_d(t)\| \in C^0(\mathbb{R}).$$

Our analysis of the $|t| \to \infty$ behavior is based on a study of this system.

By inserting (4.9) into (4.8) we get

$$\partial_t A(t) \;=\; -i\varepsilon A(t)\,(\psi_0, W(t)\psi_0) \;-\; i\varepsilon e^{i\lambda_0 t}\sum_{j=0}^{2}(\psi_0, W(t)\phi_j)\,. \tag{4.10}$$

We next give a detailed expansion of the sum in (4.10). It is in the $j = 1$ term that the key resonance is found. This makes it possible to find a normal form for (4.10) in which *internal damping* in the system is made explicit. This damping reflects the transfer of energy from the discrete to continuum modes of the system and the associated radiative decay of solutions.

PROPOSITION 4.1. *For $t > 0$,*

$$\partial_t A(t) \;=\; \left(-\varepsilon^2\Gamma \;+\; \rho(t)\right)\;A(t) \;+\; E(t), \tag{4.11}$$

where Γ is defined in (2.11),

$$\begin{aligned}\rho(t) \;&=\; -i\varepsilon\,(\psi_0,\; W(t)\psi_0)\\ &\quad +\;\frac{i}{4}\varepsilon^2\sum_{j\in\mathbb{Z}}\left(\beta_j\psi_0, \mathrm{P.V.}.(H_0-\lambda_0-\mu_j)^{-1}\mathbf{P_c}\beta_j\psi_0\right)\\ &\quad +\;\frac{i}{4}\varepsilon^2\sum_{j,k\in\mathbb{Z},j\neq k} e^{i(\mu_k-\mu_j)t}\left(\beta_k\psi_0, (H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right)\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}E(t) \;&=\; -\frac{i}{4}\varepsilon^2 A(0)e^{i\lambda_0 t}\sum_{j,k\in\mathbb{Z}} e^{i\mu_k t}\;\left(\beta_k\psi_0,\; e^{-iH_0 t}\;(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right)\\ &\quad -\frac{i}{4}\varepsilon^2 e^{i\lambda_0 t}\sum_{j,k\in\mathbb{Z}} e^{i\mu_k t}\\ &\quad \left(\beta_k\psi_0, \int_0^t e^{-iH_0(t-s)}(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}e^{-i(\lambda_0+\mu_j)s}\partial_s A(s)\beta_j\psi_0\right)ds\\ &\quad -i\varepsilon e^{i\lambda_0 t}\;(\psi_0,\; W(t)\phi_0(t))\\ &\quad -i\varepsilon e^{i\lambda_0 t}\;(\psi_0,\; W(t)\phi_2(t))\,.\end{aligned} \tag{4.13}$$

Here, ϕ_0 and ϕ_2 are given in (4.9).

Although the proposition is stated for $t > 0$, an analogous proposition with $-\varepsilon^2\Gamma$ replaced by $\varepsilon^2\Gamma$ holds for $t < 0$. The modification required to treat $t < 0$ is indicated in the proof.

Remark 4.1. (1) The point of (4.11) is that the source of damping, $\Gamma > 0$, which arises due to the coupling of the discrete bound state to the continuum modes by the almost periodic perturbation is made explicit. Note that $\Re\rho(t)$ is of order $\varepsilon^2|||W|||^2$ as the first two terms of $\rho(t)$ are purely imaginary inducing only a phase shift in the solution, $A(t)$. The last term is of the same order as the damping and may compete with it. A key point of our analysis is to assess the contribution of this last term in (4.12).

(2) The leading order part of (4.11) is the analogue of the dispersive normal form derived in [24] for a class of nonlinear dispersive wave equations.

Proof of Proposition 4.1. Using the expression for $W(t)$ in (2.6), which is a uniform convergent series with respect to $t \in \mathbb{R}$, and the definition $A(t) = e^{i\lambda_0 t}a(t)$, we get from (4.9)

$$\begin{aligned}\phi_1(t) = & -\frac{i\varepsilon}{2}\int_0^t e^{-iH_0(t-s)}e^{-i\lambda_0 s}A(s)\mathbf{P_c}\sum_{j\in\mathbb{Z}}e^{-i\mu_j s}\ \beta_j\psi_0\ ds \\ = & -\frac{i\varepsilon}{2}\sum_{j\in\mathbb{Z}}\int_0^t e^{-iH_0(t-s)}e^{-i(\lambda_0+\mu_j)s}A(s)\mathbf{P_c}\beta_j\psi_0\ ds. \qquad (4.14)\end{aligned}$$

We would like to integrate by parts each of the integrals in the above sum. We cannot proceed directly since the resolvents of H_0 in $\lambda_0 + \mu_j,\ j \in \mathbb{Z}$, would appear and hypothesis **(H5)** implies that some of the $\lambda_0 + \mu_j,\ j \in \mathbb{Z}$, are in the spectrum of H_0. Instead we regularize ϕ_1 by defining

$$\phi_1^\eta(t) = -\frac{i}{2}\varepsilon\sum_{j\in\mathbb{Z}}\int_0^t e^{-iH_0(t-s)}e^{-i(\lambda_0+\mu_j+i\eta)s}A(s)\mathbf{P_c}\beta_j\psi_0\ ds \qquad (4.15)$$

for η positive and arbitrary and $t > 0$. Note that $\phi_1(t) = \lim_{\eta\searrow 0}\phi_1^\eta(t)$ uniformly with respect to t on compact intervals.

Now, integration by parts for each integral in expression (4.15) and letting η tend to zero from above gives the following expansion of $(\psi_0, W(t)\phi_1(t))$:

$$\begin{aligned}(\psi_0, W(t)\phi_1(t)) = & \left(W(t)\psi_0,\ -\frac{\varepsilon}{2}\ e^{-i\lambda_0 t}\sum_{j\in\mathbb{Z}}e^{-i\mu_j t}A(t)(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right) \\ & +\left(W(t)\psi_0,\ \frac{\varepsilon}{2}A(0)\sum_{j\in\mathbb{Z}}e^{-iH_0 t}(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right) \\ & +\left(W(t)\psi_0,\ \frac{\varepsilon}{2}\sum_{j\in\mathbb{Z}}\int_0^t e^{-iH_0(t-s)}(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}e^{-i(\lambda_0+\mu_j)s}\partial_s A(s)\beta_j\psi_0 ds\right).\end{aligned} \qquad (4.16)$$

The definition of the singular operators in the above computation is given in section 8. The choice of regularization, $+i\eta$, in (4.15) ensures that the latter two terms in the expansion of ϕ_1, (4.16), decay dispersively as $t \to +\infty$; see hypothesis **(H3)** and section 6. For $t < 0$, we replace $+i\eta$ with $-i\eta$ in (4.15).

To further expand the first series in (4.16) we use the identities (8.5). The proof of Proposition 4.1 is now completed by substitution of (8.5) in the expansion (4.16) for ϕ_1 and of the result into the second term of the sum in (4.10). □

In the next sections we estimate the remainder terms in (4.9) and (4.11).

5. Estimates on the bound state amplitude. Our strategy is as follows. Equations (4.9) and (4.11) comprise a dynamical system governing $\phi_d(t)$ and $a(t) = A(t)e^{-i\lambda_0 t}$, the solution of which is equivalent to the original equation (1.1). In this and in the following section we derive a coupled system of estimates for $A(t)$ and $\phi_d(t)$. This section is focused on obtaining estimates for the bound state amplitude $A(t)$ in terms of $\phi_d(t)$, while the following section is focused on obtaining dispersive estimates for $\phi_d(t)$ in terms of $A(t)$. We treat only the case $t > 0$ since the modifications for the case $t < 0$ are obvious. The coupled system of estimates shows that $A(t)$ decays in time, provided $\phi_d(t)$ is dispersively decaying and vice-versa. We exploit the assumed smallness of the perturbation εW to "close" the resulting inequalities and prove the decay of both $A(t)$ and $\phi_d(t)$.

The main difference from the strategy employed in [22] for the estimation of the bound state amplitude is related to the presence of *infinitely many* frequencies in the perturbation $W(t)$. In particular, one can have an *accumulation of resonances* in the continuous spectrum of H_0. We have two strategies for obtaining estimates for $A(t)$ which correspond to the use of hypotheses **(H1)**–**(H5)** (Theorems 2.1 and 2.2) or hypotheses **(H1)**–**(H6)** (Theorem 2.3). These strategies revolve around estimation of $\Re \int_0^t \rho(s)\, ds$, where ρ is given by (4.12). Hypothesis **(H6)**, which controls certain "small divisors" which arise from the clustering of frequencies, ensures that

$$\Re \int_0^t \rho(s)\, ds \;\leq\; C\, \varepsilon^2 |||W|||^2. \tag{5.1}$$

This, in turn, implies that the contribution of $\rho(t)$ in the size of $A(t)$ is of order $\varepsilon^2|||W|||^2$. Without hypothesis **(H6)** we carefully decompose $\rho(t)$ as

$$\rho(t) \;=\; \varepsilon^2\sigma(t) \;+\; \eta(t),$$

where $\sigma(t)$ is a real almost periodic function with mean, $M(\sigma)$, zero and $\Re \int_0^t \eta(s)\, ds \leq C\varepsilon^2||W|||^2$. As in the previous case, the contribution of the $\eta(t)$ in the size of $A(t)$ is of order $\varepsilon^2|||W|||^2$. On the other hand, $\sigma(t)$ competes with the damping term $\varepsilon^2\Gamma$ in (4.11), but being oscillatory (i.e., of mean zero) and of the same size as the damping it allows the latter to eventually dominate.

As the above discussion suggests it is simplest to start by assuming **(H6)** to get sharper estimates on $A(t)$ (Theorem 2.3) and then to relax this assumption (Theorem 2.2). We begin with a simple lemma which we shall use in a number of places in this and in the next section.

LEMMA 5.1. *Let* $\alpha > 1$.

$$\int_0^t \langle t-s\rangle^{-\alpha}\, \langle s\rangle^{-\beta}\, ds \;\leq C_{\alpha,\beta}\, \langle t\rangle^{-\min(\alpha,\beta)}. \tag{5.2}$$

Proof. The bound is obtained by viewing the integral as decomposed into a part over $[0, t/2]$ and the part over $[t/2, t]$. We estimate the integral over $[0, t/2]$ by bounding $\langle t-s\rangle^{-\alpha}$ by its value at $t/2$ and explicitly computing the remaining integral. The integral over $[t/2, t]$ is computed by bounding $\langle s\rangle^{-\beta}$ by its value at $t/2$ and again

computing explicitly the remaining integral. Putting the two estimates together yields the lemma.

We now turn to the estimate for $A(t)$ in terms of the dispersive norm of $\phi_d(t)$ and local decay estimates for $e^{-iH_0t}\mathbf{P_c}(H_0)$.

5.1. Estimates for $A(t)$ under the hypotheses of Theorem 2.3.

PROPOSITION 5.1. *Suppose* **(H1)–(H6)** *hold. Then $A(t)$, the solution of* (4.11)*, can be expanded as*

$$A(t) \;=\; e^{\int_0^t \rho(s)ds}\left(e^{-\varepsilon^2\Gamma t}A(0) + R_A(t)\right), \tag{5.3}$$

$$R_A(t) \;=\; \int_0^t \; e^{-\varepsilon^2\Gamma(t-\tau)}\; \tilde{E}(\tau)\; d\tau, \tag{5.4}$$

where $\tilde{E}(t)$ is given in (4.13) *and* (5.9)*. For any $\alpha > 1$, there exists a $\delta > 0$ such that $R_A(t)$ satisfies the estimates for $T > 2(\varepsilon^2\Gamma)^{-\alpha}$,*

$$\begin{aligned}\sup_{2(\varepsilon^2\Gamma)^{-\alpha}\le t\le T}\langle t\rangle^{r_1}\;|R_A(t)| &\le C_1 e^{-(\varepsilon^2\Gamma)^{-\delta}}\sup_{0\le\tau\le(\varepsilon^2\Gamma)^{-\alpha}}|E(\tau)|\\ &\quad + C\varepsilon^2\Gamma^{-1}\sup_{(\varepsilon^2\Gamma)^{-\alpha}\le\tau\le T}\left(\langle\tau\rangle^{r_1}|E(\tau)|\right),\end{aligned} \tag{5.5}$$

$$\sup_{0\le t\le 2(\varepsilon^2\Gamma)^{-\alpha}}\langle t\rangle^{r_1}\;|R_A(t)| \le\; D\;(\varepsilon^2\Gamma)^{-\alpha(r_1+1)}\sup_{0\le\tau\le 2(\varepsilon^2\Gamma)^{-\alpha}}|E(\tau)|. \tag{5.6}$$

Proof. To prove (5.5) we begin with (4.11). Let

$$\tilde{A}(t) \equiv e^{-\int_0^t \rho(s)ds}A(t). \tag{5.7}$$

Then, $\tilde{A}$ satisfies the equation

$$\partial_t\tilde{A} = -\varepsilon^2\Gamma\tilde{A} + \tilde{E}(t), \tag{5.8}$$

$$\tilde{E}(t) \equiv e^{-\int_0^t \rho(s)ds}E(t). \tag{5.9}$$

Solving (5.8) we get

$$\tilde{A}(t) \;=\; e^{-\varepsilon^2\Gamma t}\tilde{A}(0) \;+\; \int_0^t e^{-\varepsilon^2\Gamma(t-s)}\tilde{E}(s)\; ds \tag{5.10}$$

$$\equiv\; e^{-\varepsilon^2\Gamma t}\tilde{A}(0) \;+\; R_A(t). \tag{5.11}$$

Below, in Proposition 5.2 we show that the real part of the integral of $\rho(t)$ is uniformly bounded and of order $\mathcal{O}(\varepsilon^2|||W|||^2)$ for $t \ge 0$. Therefore, for some $C > 0$, we have by (5.7) and (5.9)

$$C^{-1}|\tilde{A}(t)| \le |A(t)| \le C|\tilde{A}(t)|, \tag{5.12}$$

$$C^{-1}|\tilde{E}(t)| \le |E(t)| \le C|\tilde{E}(t)|. \tag{5.13}$$

Consequently, it is sufficient to estimate $\tilde{A}(t)$, in terms of $\tilde{E}(t)$.

Remark 5.1. Estimates of $R_a(t)$, which appear in the statement of Theorem 2.3, are related to those for $R_A(t)$ via

$$R_a(t) \;=\; e^{-i\lambda_0 t+\int_0^t \rho(s)\; ds}\; R_A(t) - \left(1 - e^{\Re\int_0^t \rho(s)ds}\right)e^{-\varepsilon^2\Gamma t}e^{iw(t)}a(0). \tag{5.14}$$

Hence, by Proposition 5.2,

$$|R_a(t)| \ \leq \ C \ |R_A(t)| + \mathcal{O}(\varepsilon^2 |||W|||^2). \tag{5.15}$$

From (5.10) we have for any $M > 0$

$$\begin{aligned} |\tilde{A}(t)| \ &\leq |A(0)| e^{-\varepsilon^2 \Gamma t} + \int_0^M e^{-\varepsilon^2 \Gamma (t-s)} |\tilde{E}(s)| ds + \int_M^t e^{-\varepsilon^2 \Gamma (t-s)} |\tilde{E}(s)| \ ds \\ &= \ |A(0)| e^{-\varepsilon^2 \Gamma t} \ + \ I_1(t) \ + \ I_2(t). \end{aligned} \tag{5.16}$$

Set

$$M = (\varepsilon^2 \Gamma)^{-\alpha}, \quad \alpha > 1.$$

We now estimate the terms $I_1(t)$ and $I_2(t)$ in (5.16) for $2(\varepsilon^2\Gamma)^{-\alpha} \leq t \leq T$.

$$\begin{aligned} \langle t \rangle^{r_1} \ I_1(t) \ &= \ \langle t \rangle^{r_1} \ \int_0^M e^{-\varepsilon^2 \Gamma (t-s)} |\tilde{E}(s)| ds \\ &\leq \ \langle t \rangle^{r_1} e^{-\frac{1}{2}\varepsilon^2 \Gamma t} \cdot \int_0^M e^{-\varepsilon^2 \Gamma (\frac{1}{2} t - s)} \ ds \ \cdot \sup_{0 \leq \tau \leq (\varepsilon^2 \Gamma)^{-\alpha}} |\tilde{E}(\tau)| \\ &\leq \ \sup_{2(\varepsilon^2\Gamma)^{-\alpha} \leq t \leq T} \left(\langle t \rangle^{r_1} e^{-\frac{1}{2}\varepsilon^2 \Gamma t} \right) \cdot C (\varepsilon^2 \Gamma)^{-1} \cdot \sup_{0 \leq \tau \leq (\varepsilon^2 \Gamma)^{-\alpha}} |\tilde{E}(\tau)| \\ &\leq \ C e^{-(\varepsilon^2 \Gamma)^{-\delta}} \sup_{0 \leq \tau \leq (\varepsilon^2 \Gamma)^{-\alpha}} |\tilde{E}(\tau)| \end{aligned} \tag{5.17}$$

for some $\delta > 0$. Therefore,

$$\sup_{2(\varepsilon^2\Gamma)^{-\alpha} \leq t \leq T} (\langle t \rangle^{r_1} \ I_1(t)) \ \leq \ C e^{-(\varepsilon^2 \Gamma)^{-\delta}} \sup_{0 \leq \tau \leq (\varepsilon^2 \Gamma)^{-\alpha}} |\tilde{E}(\tau)|. \tag{5.18}$$

We estimate $I_2(t)$ on the interval $2(\varepsilon^2\Gamma)^{-\alpha} \leq t \leq T$ as follows:

$$\langle t \rangle^{r_1} \ I_2(t) \ \leq \ \langle t \rangle^{r_1} \ \int_{(\varepsilon^2\Gamma)^{-\alpha}}^t e^{-\varepsilon^2 \Gamma (t-s)} \langle s \rangle^{-r_1} \ ds \sup_{(\varepsilon^2\Gamma)^{-\alpha} \leq \tau \leq T} \left(\langle \tau \rangle^{r_1} \tilde{E}(\tau) \right). \tag{5.19}$$

The integral is now bounded above using the estimate

$$\langle t \rangle^{r_1} \ \int_{(\varepsilon^2\Gamma)^{-\alpha}}^t e^{-\varepsilon^2 \Gamma (t-s)} \ \langle s \rangle^{-r_1} \ ds \ \leq \ C (\varepsilon^2 \Gamma)^{-1}, \ t \geq 2(\varepsilon^2 \Gamma)^{-\alpha}. \tag{5.20}$$

This gives

$$\sup_{2(\varepsilon^2\Gamma)^{-\alpha} \leq t \leq T} \langle t \rangle^{r_1} \ I_2(t) \ \leq \ C (\varepsilon^2 \Gamma)^{-1} \sup_{(\varepsilon^2\Gamma)^{-\alpha} \leq \tau \leq T} \left(\langle \tau \rangle^{r_1} \tilde{E}(\tau) \right). \tag{5.21}$$

Assembling the estimates (5.18) and (5.21) yields estimate (5.5) of Proposition 5.1 provided that (5.12) and (5.13) hold. Estimate (5.6) is a simple consequence of the definition of $R_A(t)$.

Thus it remains to prove (5.12) and (5.13). By (5.7) and (5.9) it is necessary and sufficient to verify the following proposition.

PROPOSITION 5.2. *Assume hypotheses* **(H1)**–**(H6)**. *If ρ is given by* (4.12), *then*

$$\Re \int_0^t \rho(s)\, ds \le C\varepsilon^2 |||W|||^2,\ t \ge 0, \tag{5.22}$$

for some constant C depending on $\mathcal{C}$, r_1, and ξ; see **(H6)**.

Proof of Proposition 5.2. Using the estimates (8.7) and (8.9) we can infer that $\rho(t)$, given by (4.12) is a series which converges uniformly on any compact subset of $\mathbb{R}$. For each fixed t, it can therefore be integrated term-by-term to give

$$\begin{aligned}\Re \int_0^t \rho(s)ds &= \frac{\varepsilon^2}{4}\Re\, i \sum_{j,k\in\mathbb{Z},j\neq k} \int_0^t e^{i(\mu_k-\mu_j)s} \left(\beta_k\psi_0, (H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right) ds \\ &= \frac{\varepsilon^2}{4} \sum_{j,k\in\mathbb{Z},j\neq k} \Re \frac{e^{i(\mu_k-\mu_j)t}-1}{\mu_k-\mu_j} \left(\beta_k\psi_0, (H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right). \end{aligned} \tag{5.23}$$

Define

$$\tilde\rho_{j,k} \equiv \frac{e^{i(\mu_k-\mu_j)t}-1}{\mu_k-\mu_j} \left(\beta_k\psi_0, (H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right).$$

Then (5.23) can be expressed as

$$\Re \int_0^t \rho(s)ds = \frac{\varepsilon^2}{4} \sum_{j,k\in\mathbb{Z},j\neq k} \Re\tilde\rho_{j,k} = \frac{\varepsilon^2}{8} \sum_{j,k\in\mathbb{Z},j\neq k} \Re(\tilde\rho_{j,k}+\tilde\rho_{k,j}). \tag{5.24}$$

Now, since

$$\begin{aligned}\tilde\rho_{k,j} &= -\frac{e^{-i(\mu_k-\mu_j)t}-1}{\mu_k-\mu_j} \left(\beta_j\psi_0, (H_0-\lambda_0-\mu_k-i0)^{-1}\mathbf{P_c}\beta_k\psi_0\right) \\ &= -\left[\frac{e^{i(\mu_k-\mu_j)t}-1(\dots)}{\mu_k-\mu_j}\right]^* \left(\beta_k\psi_0, (H_0-\lambda_0-\mu_k+i0)^{-1}\mathbf{P_c}\beta_j\psi_0,\right)\end{aligned}$$

we have

$$\begin{aligned}&\Re(\tilde\rho_{j,k}+\tilde\rho_{k,j}) \\ &= \Re\frac{e^{i(\mu_k-\mu_j)t}-1}{\mu_k-\mu_j} \left(\beta_k\psi_0, (H_0-\lambda_0-\mu_j-i0)^{-1} - (H_0-\lambda_0-\mu_k+i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right).\end{aligned} \tag{5.25}$$

Moreover, by (8.5) we can infer

$$\Re(\tilde\rho_{j,k}+\tilde\rho_{k,j}) = \Re\left(e^{i(\mu_k-\mu_j)t}-1\right)\rho_{j,k} + 2\Im\left(e^{-i(\mu_k-\mu_j)t}-1\right)\delta_{j,k},$$

where, for $j\neq k\in\mathbb{Z}$,

$$\rho_{j,k} \equiv \frac{1}{\mu_k-\mu_j} \left(\beta_k\psi_0, (H_0-\lambda_0-\mu_j-i0)^{-1} - (H_0-\lambda_0-\mu_k-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right), \tag{5.26}$$

and for $j\neq k,\ j\in\mathbb{Z},\ k\in I_{res}$,

$$\delta_{j,k} \equiv \frac{\pi}{\mu_k-\mu_j} \left(\beta_j\psi_0, \delta(H_0-\lambda_0-\mu_k)\beta_k\psi_0\right). \tag{5.27}$$

Thus, by (5.24) and (5.25)

$$\Re\int_0^t \rho(s)ds = \frac{\varepsilon^2}{8}\sum_{j,k\in\mathbb{Z},j\neq k}\Re\big(e^{i(\mu_k-\mu_j)t}-1\big)\rho_{j,k}+\frac{\varepsilon^2}{4}\sum_{k\in I_{res},k\neq j\in\mathbb{Z}}\Im\big(e^{-i(\mu_k-\mu_j)t}-1\big)\delta_{j,k}. \tag{5.28}$$

We now derive a uniform bound for $\Re\ \int_0^t \rho(s)\ ds$.

Estimating the modulus of the above sum, we have for any t

$$\left|\ \Re\ \int_0^t \rho(s)\ ds\ \right| \ \leq\ \frac{\varepsilon^2}{4}\sum_{j,k\in\mathbb{Z},j\neq k}|\rho_{j,k}|+\frac{\varepsilon^2}{2}\sum_{k\in I_{res},k\neq j\in\mathbb{Z}}|\delta_{j,k}|\,. \tag{5.29}$$

By **(H6)**,

$$\sum_{k\in I_{res},k\neq j\in\mathbb{Z}}|\delta_{j,k}|\ \leq\ \pi\xi\ |||W|||^2. \tag{5.30}$$

We now bound the first term in (5.29). This requires an estimate of

$$|\rho_{j,k}| = \left|\frac{1}{\mu_k-\mu_j}\left(\beta_k\psi_0,(H_0-\lambda_0-\mu_j-i0)^{-1}-(H_0-\lambda_0-\mu_k-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right)\right|$$

for $j\neq k\in\mathbb{Z}$. We rely on the hypothesis **(H3b)** (singular local decay estimate (2.4)), which implies smoothness of the resolvent of H_0 near accumulation points in $\sigma_{\rm cont}(H_0)$ of the set $\{\lambda_0+\mu_j\}_{j\in\mathbb{Z}}$.

In order to treat both $\lambda_0+\mu_j\in\sigma_{\rm cont}(H_0)$ and $\lambda_0+\mu_j\notin\sigma_{\rm cont}(H_0)$ case simultaneously we regularize $\rho_{j,k}$:

$$\rho^\eta_{j,k}\equiv\frac{1}{\mu_k-\mu_j}\left(\beta_k\psi_0,(H_0-\lambda_0-\mu_j-i\eta)^{-1}-(H_0-\lambda_0-\mu_k-i\eta)^{-1}\mathbf{P_c}\beta_j\psi_0\right). \tag{5.31}$$

Clearly $\rho_{j,k}=\lim_{\eta\searrow 0}\rho^\eta_{j,k}$.

Now by the standard resolvent formula we have

$$\rho^\eta_{j,k}=\left(\beta_k\psi_0,(H_0-\lambda_0-\mu_k-i\eta)^{-1}(H_0-\lambda_0-\mu_j-i\eta)^{-1}\mathbf{P_c}\beta_j\psi_0\right).$$

Thus, using the singular local decay estimate **(H3b)**, we get

$$\begin{aligned}
|\rho_{j,k}| &= \left|\lim_{\eta\searrow 0}\int_0^\infty\left(\beta_k\psi_0,e^{-i(H_0-\lambda_0-\mu_k-i\eta)s}(H_0-\lambda_0-\mu_j-i\eta)^{-1}\mathbf{P_c}\beta_j\psi_0\right)ds\right|\\
&\leq \lim_{\eta\searrow 0}\int_0^\infty e^{-\eta s}\left|\left(w_+\beta_k\psi_0,w_-e^{-iH_0s}(H_0-\lambda_0-\mu_j-i\eta)^{-1}\mathbf{P_c}w_-w_+\beta_j\psi_0\right)\right|ds\\
&\leq \|w_+\beta_k\|\|w_+\beta_j\|\int_0^\infty\|w_-e^{-iH_0s}(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}w_-\|ds\\
&\leq \mathcal{C}\|w_+\beta_k\|\|w_+\beta_j\|\int_0^\infty\langle s\rangle^{-r_1}ds\\
&\leq C\|w_+\beta_k\|\|w_+\beta_j\| \qquad (5.32)
\end{aligned}$$

for some constant C depending on $\mathcal{C}$ and r_1. Summing on $j,k\in\mathbb{Z},\ j\neq k$, yields

$$\sum_{j,k\in\mathbb{Z},\ j\neq k}|\rho_{j,k}|\ \leq\ C\ |||W|||^2 \tag{5.33}$$

for some $C > 0$; see (2.7). Use of the bounds (5.30) and (5.33) in (5.29) gives

$$\left| \Re \int_0^t \rho(s)\, ds \right| \le C\, \varepsilon^2 |||W|||^2$$

for some constant C depending on $\mathcal{C}$, r_1, and ξ.

This completes the proof of Proposition 5.2 and therewith Proposition 5.1. ∎

5.2. Estimates for $A(t)$ under the hypotheses of Theorem 2.1. In this subsection we work under the hypotheses of Theorem 2.1. In particular, we drop hypothesis **(H6)**. *We shall reuse the notation $\tilde{A}$ and $\tilde{E}$ for functions which are different from but related to those defined in section* 5.1.

PROPOSITION 5.3. *Suppose* **(H1)**–**(H5)** *hold. Then $A(t)$, the solution of* (4.11)*, can be expanded as*

$$A(t) = e^{\int_0^t \eta(s)ds}\Big(e^{-\varepsilon^2(\Gamma t - \int_0^t \sigma(s)ds)}A(0) + R_A(t)\Big), \tag{5.34}$$

$$R_A(t) = \int_0^t e^{-\varepsilon^2\Gamma(t-\tau)+\varepsilon^2\int_\tau^t \sigma(s)ds}\, \tilde{E}(\tau)\, d\tau, \tag{5.35}$$

where

$$\sigma(t) \equiv -\frac{\pi}{4}\Re \sum_{j\in I_{res}, j\neq k\in\mathbb{Z}} e^{i(\mu_k-\mu_j)t}\, (\ \beta_k\psi_0, \delta(H_0-\lambda_0-\mu_j)\beta_j\psi_0) \tag{5.36}$$

is a real almost periodic function with mean $M(\sigma) = 0$, η in (5.46) *is a function whose real part has a bounded time integral of order $\mathcal{O}(\varepsilon^2|||W|||^2)$ and $\tilde{E}(t)$ is given in* (5.42)*; see also* (4.13)*. For any $\alpha > 1$, there exists $\delta > 0$ such that $R_A(t)$ satisfies the estimates*

$$\begin{aligned}\sup_{2(\varepsilon^2\Gamma/2)^{-\alpha}\le t\le T}\langle t\rangle^{r_1}|R_A(t)| &\le C_1 e^{-(\varepsilon^2\Gamma/2)^{-\delta}} \sup_{0\le\tau\le(\varepsilon^2\Gamma/2)^{-\alpha}}|E(\tau)| \\ &\quad + C(\varepsilon^2\Gamma)^{-1}\sup_{(\varepsilon^2\Gamma/2)^{-\alpha}\le\tau\le T}\left(\langle\tau\rangle^{r_1}|E(\tau)|\right),\end{aligned} \tag{5.37}$$

$$\sup_{0\le t\le 2(\varepsilon^2\Gamma/2)^{-\alpha}}\langle t\rangle^{r_1}\ |R_A(t)| \le\ D\ (\varepsilon^2\Gamma/2)^{-\alpha(r_1+1)} \sup_{0\le\tau\le 2(\varepsilon^2\Gamma/2)^{-\alpha}}\ |E(\tau)|. \tag{5.38}$$

Proof. As in the previous subsection we begin with the equation for $A(t)$:

$$\partial_t A(t) = \big(\rho(t) - \varepsilon^2\Gamma\ \big)\, A(t) + E(t), \tag{5.39}$$

where $\rho(t)$ and $E(t)$ are given by (4.12)–(4.13). In the previous section we transformed away the term $\rho(t)A(t)$ using the "integration factor" $\exp(\int_0^t \rho(s)\ ds)$. Under the current hypotheses, this can't be done because without **(H6)** $\Re\int_0^t \rho(s)\ ds$ may be unbounded as $t\to\infty$, which could cause the estimates (5.12)–(5.13) to break down. Instead, we proceed by a more refined analysis of $\rho(t)$, which we now outline.

We express $\rho(t)$ as $\rho(t) = \varepsilon^2\sigma(t) + \eta(t)$, where $\eta(t)$ has a time integral whose real part can be bounded by the estimates of section 5.1 and a part, $\varepsilon^2\sigma(t)$, which is almost periodic and of mean zero. Using this decomposition of $\rho(t)$ we write (5.39) as

$$\partial_t A(t) = \big[-\varepsilon^2\Gamma + \varepsilon^2\sigma(t) + \eta(t)\big]\, A(t) + E(t).$$

Next introduce the change of variables

$$\tilde{A}(t) \;\equiv\; e^{-\int_0^t \eta(s)\ ds}\ A(t) \tag{5.40}$$

and obtain a reduction to

$$\partial_t \tilde{A} = \left[\ -\varepsilon^2\Gamma \;+\; \varepsilon^2\sigma(t)\ \right]\tilde{A} \;+\; \tilde{E}(t), \tag{5.41}$$

$$\tilde{E}(t) \equiv e^{-\int_0^t \eta(s)ds} E(t). \tag{5.42}$$

With this strategy in mind we now proceed to derive the decomposition of $\rho(t)$. We are mostly interested in its real part, so we start with it.

$$\begin{aligned}
\Re\rho(t) &= \Re\frac{i\varepsilon^2}{4}\sum_{j,k\in\mathbb{Z},j\neq k} e^{i(\mu_k-\mu_j)t}\left(\beta_k\psi_0,(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right)\\
&= -\frac{\varepsilon^2}{4}\Im\sum_{j,k\in\mathbb{Z},j\neq k} e^{i(\mu_k-\mu_j)t}\left(\beta_k\psi_0,(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right)\\
&\equiv -\frac{\varepsilon^2}{4}\sum_{j,k\in\mathbb{Z},j\neq k}\Im\eta_{j,k}\\
&= \frac{\varepsilon^2}{8}\sum_{j,k\in\mathbb{Z},j\neq k}\Im\left(\eta_{j,k}+\eta_{k,j}\right). \qquad (5.43)
\end{aligned}$$

In a manner similar to the derivation of (5.25) from (5.24) we find

$$\Im\eta_{k,j} = \Im e^{i(\mu_k-\mu_j)t}\left(\beta_k\psi_0,(H_0-\lambda_0-\mu_k+i0)^{-1}\mathbf{P_c}\beta_j\psi_0\right). \tag{5.44}$$

Using (8.5) in (5.44) and then replacing it in (5.43) we get

$$\begin{aligned}
\Re\rho(t) \;&=\; \frac{\pi}{4}\varepsilon^2\Re\sum_{k\in I_{res},k\neq j\in\mathbb{Z}} e^{i(\mu_j-\mu_k)t}\left(\beta_j\psi_0,\delta(H_0-\lambda_0-\mu_k)\beta_k\psi_0\right)\\
&-\frac{1}{8}\varepsilon^2\Im\sum_{j,k\in\mathbb{Z},j\neq k} e^{i(\mu_k-\mu_j)t}\big(\beta_k\psi_0,\big[(H_0-\lambda_0-\mu_j-i0)^{-1}\\
&\qquad\qquad -(H_0-\lambda_0-\mu_k-i0)^{-1}\big]\mathbf{P_c}\beta_j\psi_0\big)\\
&=\; \Re\ \eta(t)+\varepsilon^2\sigma(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho(t) \;&=\; \Re\rho(t) \;+\; i\Im\rho(t)\\
&=\; \eta(t) \;+\; \varepsilon^2\sigma(t), \qquad (5.45)
\end{aligned}$$

where

$$\begin{aligned}
\eta(t) &= i\ \Im\ \rho(t) - \frac{1}{8}\varepsilon^2\Im\sum_{j,k\in\mathbb{Z},j\neq k} e^{i(\mu_k-\mu_j)t}\\
&\left(\beta_k\psi_0,\left[(H_0-\lambda_0-\mu_j-i0)^{-1}-(H_0-\lambda_0-\mu_k-i0)^{-1}\mathbf{P_c}\right]\beta_j\psi_0\right), \qquad (5.46)\\
\sigma(t) &= -\frac{\pi}{4}\Re\sum_{j\in I_{res},j\neq k\in\mathbb{Z}} e^{i(\mu_k-\mu_j)t}\left(\ \beta_k\psi_0,\delta(H_0-\lambda_0-\mu_j)\beta_j\psi_0\right);
\end{aligned}$$

see also (5.36).

Note that $\Re \int_0^t \eta(s)ds$ is uniformly bounded in t. To see this, recall the definition of $\rho_{j,k}$ in Lemma 5.2 (see (5.26)):

$$\rho_{j,k} \equiv \frac{1}{\mu_k - \mu_j} \left(\beta_k\psi_0, \left[(H_0 - \lambda_0 - \mu_j - i0)^{-1} - (H_0 - \lambda_0 - \mu_k - i0)^{-1}\right] \mathbf{P_c}\beta_j\psi_0\right).$$

By (8.7), $\Re\ \eta(t)$ given by (5.46) converges uniformly on $t \in \mathbb{R}$. Therefore, for each $t \in \mathbb{R}$ we may integrate the series term-by-term to obtain

$$\Re \int_0^t \eta(s)ds = \frac{1}{8}\varepsilon^2 \sum_{j,k\ j\neq k} \Re(e^{i(\mu_k-\mu_j)t} - 1)\rho_{j,k}. \tag{5.47}$$

Moreover, the modulus of the right-hand side in (5.47) is less or equal than $\frac{1}{4}\varepsilon^2 \sum_{j,k\ j\neq k} |\rho_{j,k}|$, which by (5.32) is bounded by $C\varepsilon^2|||W|||^2$ for some constant C depending only on $\mathcal{C}$ and r_1. Note that we derived (5.32) by using only hypothesis **(H3b)** and not relying on **(H6)**.

Thus we have

$$\Re \int_0^t \eta(s)ds \leq C\varepsilon^2|||W|||^2. \tag{5.48}$$

To summarize, we have split $\rho(t)$ into

$$\rho(t) = \eta(t) \ + \ \varepsilon^2\sigma(t)$$

such that (5.48) is valid. If we now define $\tilde{A}$ as in (5.40), then by (4.11) $\tilde{A}$ satisfies (5.41). Solving (5.41) we get

$$\begin{aligned} \tilde{A}(t) \ &= \ e^{-\varepsilon^2\Gamma t + \varepsilon^2 \int_0^t \sigma(s)ds}\tilde{A}(0) \ + \ \int_0^t e^{-\varepsilon^2\Gamma(t-\tau) + \varepsilon^2 \int_\tau^t \sigma(s)ds}\tilde{E}(s)\ d\tau \\ &\equiv \ e^{-\varepsilon^2\Gamma t + \varepsilon^2 \int_0^t \sigma(s)ds}\tilde{A}(0) \ + \ R_A(t). \end{aligned} \tag{5.49}$$

From (5.42) and (5.48) it is sufficient to estimate $R_A(t)$ in terms of $\tilde{E}(t)$.

Remark 5.2. The estimates of $R_a(t)$ which appear in the statement of Theorem 2.2 are related to those for $R_A(t)$ via

$$R_a(t) \ = \ e^{-i\lambda_0 t + \int_0^t \eta(s)ds} R_A(t) + \left(1 - e^{\varepsilon^2(\int_0^t \sigma(s)ds - \gamma t) + \Re \int_0^t \eta(s)ds}\right) e^{-\varepsilon^2(\Gamma-\gamma)t} e^{iw(t)} a(0). \tag{5.50}$$

Before we estimate $R_A(t)$, we review some properties of the function $\sigma(t)$.

The function $\sigma(t)$ is almost periodic since the sum of the moduli of its Fourier coefficients is finite. Namely, by (2.20), the terms in the series (5.36) defining $\sigma(t)$ are majorized by those of a convergent series (whose sum is $C\pi^{-1}|||W|||^2$). Therefore, the series in (5.36) is uniformly convergent. As the uniform limit of almost periodic functions, $\sigma(t)$ is then itself almost periodic, bounded by

$$\sup_{t\in\mathbb{R}} |\sigma(t)| \ \leq \ C|||W|||^2 \tag{5.51}$$

for some constant C; see also section 9. Moreover, $\sigma(t)$ has mean value zero since all the Fourier exponents are nonzero; see (5.36) and section 9. Therefore

$$\int_\tau^t \sigma(s)ds \leq \frac{\Gamma}{2}(t-\tau), \text{ for } \ t - \tau \geq \mathcal{M} \tag{5.52}$$

provided $\mathcal{M}$ is taken sufficiently large. It can be shown (see section 9 or [2, p. 42]) that (5.52) holds provided

$$\mathcal{M} \geq \frac{4\ \sup_{t\in\mathbb{R}}\{|\sigma(t)|\}\ L(\Gamma/4)}{\Gamma/2}, \tag{5.53}$$

where $L(\Gamma/4)$ (see Definition 9.1) is such that in each interval of length $L(\Gamma/4)$ there is at least one $\Gamma/4$ almost period for σ.

Using (5.51) and then **(H5)**, we can choose

$$\mathcal{M} = 8CL(\Gamma/4)/\theta_0 \tag{5.54}$$

independently of ε and still satisfy (5.53).

We now return to the estimation of R_A. We split the integral in (5.35) into two integrals, one from 0 to $t-\mathcal{M}$ and the other from $t-\mathcal{M}$ to t. For the former we use (5.52) while for the latter we use (5.51). The result is

$$\begin{aligned} |R_A(t)| \ \leq\ & \int_0^{t-\mathcal{M}} e^{-\frac{1}{2}\varepsilon^2\Gamma(t-\tau)}|\tilde{E}(\tau)|d\tau \\ & + \int_{t-\mathcal{M}}^{t} e^{\varepsilon^2(C|||W|||^2-\Gamma)(t-\tau)}|\tilde{E}(\tau)|d\tau. \end{aligned} \tag{5.55}$$

The first integral in (5.55) can be bounded exactly as the term $\int_0^t\ e^{-\varepsilon^2\Gamma(t-\tau)}|\tilde{E}(\tau)|d\tau$ in the proof of Proposition 5.1. The second integral in (5.55) is bounded in the following manner:

$$\begin{aligned} &\langle t\rangle^{r_1}\int_{t-\mathcal{M}}^{t} e^{\varepsilon^2(C|||W|||^2-\Gamma)(t-\tau)}|\tilde{E}(\tau)|d\tau \\ &\leq \frac{\langle t\rangle^{r_1}}{\langle t-\mathcal{M}\rangle^{r_1}}\int_{t-\mathcal{M}}^{t} e^{\varepsilon^2(C|||W|||^2-\Gamma)(t-\tau)}d\tau \sup_{t-\mathcal{M}\leq\tau\leq t}\left(\langle\tau\rangle^{r_1}|E(\tau)|\right) \\ &\leq D \sup_{(\varepsilon^2\Gamma/2)^{-\alpha}\leq\tau\leq t}\left(\langle\tau\rangle^{r_1}|E(\tau)|\right). \end{aligned} \tag{5.56}$$

Note that ε and consequently $\varepsilon^2\Gamma \sim \varepsilon^2|||W|||^2$ are small, so we can consider $\mathcal{M} \ll (\varepsilon^2\Gamma/2)^{-\alpha}$ and $D \sim \mathcal{M} \ll (\varepsilon^2\Gamma)^{-1}$. The result is (5.37). A simple bound, using the definition of $R_A(t)$, yields (5.38).

This completes the proof of Proposition 5.3.

6. Dispersive estimates and local decay. In this section we prove the local decay of ϕ_d and the decay in time of the remainder terms, $E(t)$, in bound state amplitude equation (4.11) of section 4. The arguments rely on hypotheses **(H1)**–**(H5)** and results of the previous section, so we will handle Theorem 2.1 first. However, due to the differences between Theorems 2.2 and 2.3 we separately finish their proofs in the final two subsections of this section. We will repeatedly use the following lemma.

LEMMA 6.1. *For any $\eta \in [0, r_1]$ and $j \in \mathbb{Z}$ we have*

$$\left\|\int_0^t w_- e^{-iH_0(t-s)}\mathbf{P_c}f(s)ds\right\| \ \leq\ C\langle t\rangle^{-\eta}\sup_{0\leq\tau\leq t}\left(\langle\tau\rangle^{\eta}\|w_+f(\tau)\|\right) \tag{6.1}$$

and

$$\left\|\int_0^t w_- e^{-iH_0(t-s)}\mathbf{P_c}(H_0-\lambda_0-\mu_j-i0)^{-1}f(s)\ ds\right\| \ \leq\ C\langle t\rangle^{-\eta}\sup_{0\leq\tau\leq t}\left(\langle\tau\rangle^{\eta}\,\|w_+f(\tau)\|\right). \tag{6.2}$$

Proof. The proof follows from the assumed local decay estimates on e^{-iH_0t}; see **(H3a)**. Namely, using that $r_1 > 1$,

$$\begin{aligned}\left\| \int_0^t w_- e^{-iH_0(t-s)} \mathbf{P_c} f(s)\, ds \right\| \;&\le\; \int_0^t \|w_- e^{-iH_0(t-s)}\, \mathbf{P_c} w_-\|_{\mathcal{L}(\mathcal{H})} \langle s\rangle^{-\eta}\, ds \\ &\quad \cdot \sup_{0\le\tau\le t} (\langle\tau\rangle^{\eta} \|w_+ f(\tau)\|) \\ &\le\; C \int_0^t \langle t-s\rangle^{-r_1} \langle s\rangle^{-\eta}\, ds \sup_{0\le\tau\le t} (\langle\tau\rangle^{\eta} \|w_+ f(\tau)\|) \\ &\le C\langle t\rangle^{-\eta} \sup_{0\le\tau\le t} (\langle\tau\rangle^{\eta}\, \|w_+ f(\tau)\|)\,,\end{aligned}$$

which proves (6.1). The proof of (6.2) is identical and uses the singular local decay estimate of **(H3b)**. □

We now define the norms

$$[A]_\alpha(T) \;=\; \sup_{0\le\tau\le T} \langle\tau\rangle^{\alpha} |A(\tau)| \tag{6.3}$$

and

$$[\phi_d]_{LD,\alpha}(T) \;=\; \sup_{0\le\tau\le T} \langle\tau\rangle^{\alpha} \|w_-\phi_d(\tau)\|. \tag{6.4}$$

Then we have the following.

PROPOSITION 6.1. *For any $T > 0$ and $\eta \in [0,\ r_1]$,*

$$[\phi_d]_{LD,\eta}(T) \le C\,(\ \|w_+\phi_d(0)\| \ +\ |\varepsilon|\ |||W|||\ [A]_\eta(T)\ )\,. \tag{6.5}$$

Proof. From (4.9) we get, using the assumed local decay estimate for e^{-iH_0t} and (6.1),

$$\begin{aligned}\|w_-\phi_d(t)\| \;&\le\; \sum_{j=0}^{2} \|w_-\phi_j(t)\| \\ &\le\; C\langle t\rangle^{-\eta}\, \|w_+\phi_d(0)\| + \; C|\varepsilon|\langle t\rangle^{-\eta}\, [A]_\eta(t) \sup_{0\le s\le t} \|w_+ W(s)\psi_0\| \\ &\qquad + C\ |\varepsilon|\ |||W|||\ \langle t\rangle^{-\eta}\, [\phi_d]_{LD,\eta}\,(t).\end{aligned} \tag{6.6}$$

Since $\|w_+W(s)\psi_0\| \le |||W|||\ \|\psi_0\| \;=\; |||W|||$ and $|\varepsilon|\ |||W|||$ is assumed to be small, multiplying both sides of this last equation by $\langle t\rangle^{\eta}$ and taking supremum over $t \le T$ yields (6.5). □

We now estimate $E(t)$.

PROPOSITION 6.2. *Let $T > 0$. For any $\eta \in [0, r_1]$*

$$[E]_\eta(T) \;\le\; C\left(\varepsilon^2 |||W|||^2\ |A(0)| \ +\ |\varepsilon|\ |||W|||\ \|w_+\phi_d(0)\| \ +\ |\varepsilon|^3 |||W|||^3\ [A]_\eta(T)\right). \tag{6.7}$$

Proof. $E(t)$ is defined in (4.13). From these equations it is seen that we need to bound the following terms:

$$R_1 \equiv \frac{1}{4}\varepsilon^2 |A(0)| \sum_{j,k\in\mathbb{Z}} \left|\left(\beta_k\psi_0,\ e^{-iH_0t}\ (H_0 - \lambda_0 - \mu_j - i0)^{-1} \mathbf{P_c}\beta_j\psi_0\right)\right|,$$

$$R_2 \equiv \frac{1}{4}\varepsilon^2 \sum_{j,k\in\mathbb{Z}} \left|\left(\beta_k\psi_0, \int_0^t\ e^{-iH_0(t-s)} (H_0 - \lambda_0 - \mu_j - i0)^{-1} \mathbf{P_c} e^{-i(\lambda_0+\mu_j)s} \partial_s A(s)\beta_j\psi_0 ds\right)\right|$$

and

$$|\varepsilon\,(\psi_0,\ W(t)\phi_0(t))| = \left|\varepsilon\left(W(t)\psi_0, e^{-iH_0t}\phi_d(0)\right)\right|,$$
$$|\varepsilon\,(\psi_0,\ W(t)\phi_2)| = \left|\varepsilon^2\left(W(t)\psi_0, \int_0^t e^{-iH_0(t-s)}\mathbf{P_c}W(s)\phi_d(s)ds\right)\right|.$$

The estimates of the above terms repeatedly use Lemma 6.1. Let $\eta \in [0, r_1]$.

Estimation of R_1.

$$R_1 = \frac{1}{4}\varepsilon^2|A(0)| \sum_{j,k\in\mathbb{Z}} \left|\left(w_+\beta_k\psi_0,\ w_-e^{-iH_0t}(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}w_-\ w_+\beta_j\psi_0\right)\right|$$
$$\leq C|A(0)|\ \varepsilon^2|||W|||^2\ \langle t\rangle^{-\eta} \tag{6.8}$$

by the local decay estimates (2.4).

Estimation of R_2. From (4.11) we have that

$$|\partial_s A(s)| \leq C|\varepsilon|\ |||W|||\ |A(s)| + |E(s)| \tag{6.9}$$

since $\Im\rho$ is linear in $|\varepsilon|\ |||W|||$ and $\Re\rho$, Γ are quadratic.

Applying Lemma 6.1 to R_2 we then get

$$R_2 = \tfrac{1}{4}\varepsilon^2 \textstyle\sum_{j,k\in\mathbb{Z}} \left|\left(w_+\beta_k\psi_0, \int_0^t w_-e^{-iH_0(t-s)}(H_0-\lambda_0-\mu_j-i0)^{-1}\mathbf{P_c}w_-\partial_s A(s)w_+\beta_j\psi_0 ds\right)\right|$$
$$\leq\ C\varepsilon^2|||W|||^2\ \langle t\rangle^{-\eta}\big(|\varepsilon|\ |||W|||\ [A]_\eta(t)\ +\ [E]_\eta(t)\big). \tag{6.10}$$

Estimation of $|\varepsilon\,(\psi_0,\ W(t)\phi_0(t))|$. Since, by definition, $\phi_d(0) = \mathbf{P_c}\phi_d(0)$ we can apply local decay estimates for e^{-iH_0t} to get

$$|\varepsilon\,(W(t)\psi_0,\ \phi_0(t))| \leq\ C|\varepsilon|\ |||W|||\ \langle t\rangle^{-\eta}\ \|w_+\phi_d(0)\|. \tag{6.11}$$

Estimation of $|\varepsilon\,(\psi_0,\ W(t)\phi_2)|$. Applying Lemma 6.1 as before we get, for $0 \leq t \leq T$,

$$|\varepsilon\,(\psi_0,\ W(t)\phi_2)| \leq C\varepsilon^2|||W|||^2\ \langle t\rangle^{-\eta}\ [\phi_d]_{LD,\eta}(T). \tag{6.12}$$

Using Proposition 6.1 to estimate $[\phi_d]_{LD,\eta}(t)$ in (6.12), we get

$$|\varepsilon\,(\psi_0,\ W(t)\phi_2)| \leq C\varepsilon^2|||W|||^2\ \langle t\rangle^{-\eta}\left\{\|w_+\phi_d(0)\|\ +\ |\varepsilon|\ |||W|||\ [A]_\eta(t)\right\}. \tag{6.13}$$

Finally, combining the above estimates, we can bound $[E]_\eta(T)$ for any $\eta \in [0, r_1]$ as follows:

$$[E]_\eta(T) \leq C\big\{\ \varepsilon^2\,|||W|||^2\ |A(0)| + |\varepsilon|\ |||W|||\ \|w_+\phi_d(0)\|$$
$$+\ \varepsilon^2|||W|||^2\ [E]_\eta(T) + |\varepsilon|^3|||W|||^3\ [A]_\eta(T)\big\}. \tag{6.14}$$

Since $|\varepsilon|\ |||W|||$ is assumed to be small, Proposition 6.2 follows. □

We can now complete the proof of Theorem 2.1. To prove the assertions concerning the infinite time behavior, the key is to establish local decay of ϕ_d, in particular, the uniform boundedness of $[\phi_d]_{LD,r_1}(T)$. This will follow directly from Proposition 6.1 if we prove the uniform boundedness $[A]_{r_1}(T)$, or equivalently, $[\tilde{A}]_{r_1}(T)$.

PROPOSITION 6.3. *Under the hypothesis of Theorem* 2.1*, there exists an* $\varepsilon_0 > 0$ *such that for each real number* ε*,* $|\varepsilon| < \varepsilon_0$ *there is a constant* C_* *with the property that for any* $T > 0$

$$[A]_{r_1}(T) \le C_*.$$

Proof. We begin with the expansion of $A(t)$ given in Proposition 5.3. Multiplying (5.34) by $\langle t\rangle^{r_1}$, and taking the supremum over $0 \le t \le T$ we have

$$\begin{aligned}[A]_{r_1}(T) \;\le\; C\Bigg(&|A(0)|\;(\varepsilon^2\Gamma/2)^{-r_1} \;+ \sup_{0\le\tau\le 2(\varepsilon^2\Gamma/2)^{-\alpha}} \langle\tau\rangle^{r_1}|R_A(\tau)| \\ &+ \sup_{2(\varepsilon^2\Gamma/2)^{-\alpha}\le\tau\le T} \langle\tau\rangle^{r_1}|R_A(\tau)|\;\Bigg).\end{aligned} \tag{6.15}$$

The right-hand side of (6.15) is estimated using Proposition 5.3.

$$\begin{aligned}[A]_{r_1}(T) \le C|A(0)|\;(\varepsilon^2\Gamma/2)^{-r_1} \;&+ D\;(\varepsilon^2\Gamma/2)^{-\alpha(r_1-1)}\;[E]_0(2(\varepsilon^2\Gamma/2)^{-\alpha}) \\ &+ C_1\;e^{-(\varepsilon^2\Gamma/2)^{-\delta}}\;[E]_0(2(\varepsilon^2\Gamma/2)^{-\alpha}) \;+ C_2\;(\varepsilon^2\Gamma/2)^{-1}[E]_{r_1}(T).\end{aligned}$$

Next, we apply Proposition 6.2 which yields

$$\begin{aligned}[A]_{r_1}(T) \le C|A(0)|\;(\varepsilon^2\Gamma/2)^{-r_1} \;&+\; D\;(\varepsilon^2\Gamma/2)^{-\alpha(r_1+1)}\;[E]_0(2(\varepsilon^2\Gamma/2)^{-\alpha}) \\ &+ C_1 e^{-(\varepsilon^2\Gamma/2)^{-\delta}}\;[E]_0(2(\varepsilon^2\Gamma/2)^{-\alpha}) \\ &+ C_2(\varepsilon^2\Gamma/2)^{-1}\big(\varepsilon^2|A(0)||||W|||^2 \;+\; |\varepsilon|\;|||W|||\;\|w_+\phi_d(0)\| \\ &\qquad + |\varepsilon|^3|||W|||^3\;[A]_{r_1}(T)\big).\end{aligned} \tag{6.16}$$

Note that by Proposition 6.2 and the simple bound

$$[A]_0(T) \;\le\; \|\phi_0\|,$$

$[E]_0(2(\varepsilon^2\Gamma/2)^{-\alpha})$ is bounded in terms of the initial data and $|\varepsilon|\;|||W|||$.

Choose ε_0 such that

$$1 - \frac{2C_2|||W|||^3}{\Gamma}\varepsilon_0 = 0,$$

where C_2 is the same as in (6).

Then, for $|\varepsilon| < \varepsilon_0$

$$[A]_{r_1}(T) \le C_*. \tag{6.17}$$

Here, C_* depends on $\|\phi_0\|,\ \|w_+\phi_0\|,\ r_1$, and ε.

This completes the proof of Proposition 6.3 and therewith the $t \to \infty$ asymptotics asserted in Theorems 2.1–2.3. □

It remains to finish the proofs of the Theorems 2.3 and 2.2. Due to some differences we consider them separately in the following two subsections.

6.1. Proof of Theorem 2.3. In order to obtain (2.23) we note that (4.7), (5.3), and (5.14) together with the definition of $\omega(t)$ in (2.16) already gives us

$$\begin{aligned} a(t) &= e^{-i\lambda_0 t + \int_0^t \rho(s)ds}\big(A(0)e^{-\varepsilon^2\Gamma t} + R_A(t)\big) \\ &= a(0)e^{-\varepsilon^2\Gamma t}e^{i\omega(t)} + R_a(t), \end{aligned}$$

which is in fact the second relation in (2.23). The third is a direct consequence of the second since $P(t) = |a(t)|^2$ while the fourth relation is exactly (4.9).

It remains to prove the intermediate time estimate (2.17). The ingredients are contained in (6.7) and its proof. First, by (5.15)

$$|R_a(t)| \ \leq C\ |R_A(t)| \ + \ \mathcal{O}(\varepsilon^2 |||W|||^2).$$

So, it suffices to prove an $\mathcal{O}(|\varepsilon|\ |||W|||)$ upper bound for R_A.

Using (5.4) and (5.13) we know that

$$|R_A(t)| \leq C \int_0^t \ e^{-\varepsilon^2\Gamma(t-\tau)}\ |E(\tau)|\ d\tau. \tag{6.18}$$

Let T_0 denote an arbitrary fixed positive number. We estimate the equation (6.18) for $t \in [0, T_0(\varepsilon^2\Gamma)^{-1}]$. We bound the exponential in the integrand by one (explicit integration would give something of order $(\varepsilon^2\Gamma)^{-1}$) and bound $|E(\tau)|$ by estimating the expressions in the proof of Proposition 6.2. First, the estimates of Proposition 6.2 for R_1 and $|\varepsilon\,(\psi_0,\ W(t)\phi_0(t))|$ are useful as is. Integration of the bounds (6.8) and (6.11) gives

$$\int_0^t \ e^{-\varepsilon^2\Gamma(t-\tau)} R_1\ d\tau \leq C\ \varepsilon^2 |||W|||^2\ \|w_+\phi(0)\|,$$

$$\int_0^t \ e^{-\varepsilon^2\Gamma(t-\tau)}\, |\varepsilon\,(\psi_0,\ W(t)\phi_0(t))|\ d\tau \leq C\ |\varepsilon|\ |||W|||\ \|w_+\phi(0)\|. \tag{6.19}$$

To estimate the contributions of R_2, first observe that by (6.9) and Proposition 6.2 with $\eta = 0$

$$|\partial_s A(s)| \ \leq\ C\ |\varepsilon|\ |||W|||\ \|w_+\phi(0)\|. \tag{6.20}$$

Therefore, using local decay estimates we have

$$\begin{aligned} \int_0^t \ e^{-\varepsilon^2\Gamma(t-\tau)} R_2\ d\tau \leq\ & C\ T_0(\varepsilon^2\Gamma)^{-1}\ |\varepsilon|^3 |||W|||^3\ \|w_+\phi(0)\| \\ \leq\ & D|\varepsilon|\ |||W|||\ \|w_+\phi(0)\|. \end{aligned}$$

Finally, we come to the contribution of $|\varepsilon\,(\psi_0,\ W(t)\phi_2)|$. We rewrite it as follows:

$$\begin{aligned} |\varepsilon\,(\psi_0,\ W(t)\phi_2)| =\ & \varepsilon^2 \left| \int_0^t \big(W(s)e^{iH_0(t-s)}\mathbf{P_c}W(t)\psi_0, \phi_d(s)\big)ds \right| \\ =\ & \left| \int_0^t \varepsilon^2 (w_+W(s)w_+ \cdot w_- e^{iH_0(t-s)}\mathbf{P_c}w_- \cdot w_+W(t)\psi_0,\ w_-\phi_d(s))\ ds \right|. \end{aligned} \tag{6.21}$$

Recall that by (4.9) $\phi_d = \phi_0 + \phi_1 + \phi_2$, where $\phi_0(t) = e^{-iH_0t}\phi_d(0)$. Using local decay estimates **(H3a)**, the contribution of the term $\phi_0(t)$ can be bounded

by $C\ \varepsilon^2|||W|||^2\ \|w_+\phi_d(0)\|\ \langle\tau\rangle^{-r_1}$. Multiplication of this bound by $e^{-\varepsilon^2\Gamma(t-\tau)}$ and integration with respect to t gives the bound $C\ \varepsilon^2|||W|||^2\|w_+\phi_d(0)\|$. To assess the contributions from $\phi_1+\phi_2$, note that local decay estimates **(H3a)** imply

$$\|w_-(\phi_1+\phi_2)\| \ \le\ C\ |\varepsilon|\ |||W|||\ \|w_+\phi(0)\|. \tag{6.22}$$

Putting together the contributions from ϕ_0 and from $\phi_1+\phi_2$, we have

$$\int_0^t e^{-\varepsilon^2\Gamma(t-\tau)}\,|\varepsilon\,(\psi_0,\ W(t)\phi_2)|\ d\tau\ \le\ C\left(\ \varepsilon^2|||W|||^2\ \|w_+\phi_d(0)\|\ +\ (\varepsilon^2\Gamma)^{-1}\ |\varepsilon|^3|||W|||^3\right). \tag{6.23}$$

The above estimates and (5.15) imply (2.17). Now, (2.18) is a direct consequence of (2.17) and the relation $P(t)=|a(t)|^2$.

This concludes the proof of Theorem 2.3.

6.2. Proof of Theorem 2.2. As in the proof of Theorem 2.3 relations (4.7), (5.34), (5.50), and the definition of $\omega(t)$ in (2.16) gives

$$\begin{aligned} a(t) &= e^{-i\lambda_0 t+\int_0^t\eta(s)ds}\left(A(0)e^{-\varepsilon^2\left(\Gamma t-\int_0^t\sigma(s)ds\right)}+R_A(t)\right)\\ &= a(0)e^{-\varepsilon^2(\Gamma-\gamma)t}e^{i\omega(t)}+R_a(t), \end{aligned}$$

which is the second relation in (2.15). In what follows, the only difference from the previous argument is in estimating $R_a(t)$.

We start with the relation (5.50):

$$R_a(t)\ =\ e^{-i\lambda_0 t+\int_0^t\eta(s)ds}R_A(t)+\left(1-e^{\varepsilon^2(\int_0^t\sigma(s)ds-\gamma t)+\Re\int_0^t\eta(s)ds}\right)e^{-\varepsilon^2(\Gamma-\gamma)t}e^{iw(t)}a(0). \tag{6.24}$$

Since $\sigma(t)$ is an almost periodic function with zero mean, for any $\gamma>0$ there is an $\mathcal{M}_\gamma>0$ such that whenever $|t|\ge\mathcal{M}_\gamma$

$$\int_0^t\sigma(s)ds\le\gamma t.$$

On the other hand for $|t|<\mathcal{M}_\gamma$, using (5.51) we have

$$\int_0^t\sigma(s)ds\le C\mathcal{M}_\gamma|||W|||^2.$$

So, in both cases,

$$\int_0^t\sigma(s)ds-\gamma t\le C\mathcal{M}_\gamma|||W|||^2.$$

Substituting now in (6.24) and tacking into account that by (5.48),

$$\Re\ \int_0^t\eta(s)ds\ \le\ C\varepsilon^2|||W|||^2$$

uniformly in t, we get

$$|R_a(t)|\ \le\ C\,|R_A(t)|+\mathcal{O}(\varepsilon^2|||W|||^2). \tag{6.25}$$

It remains to prove an $\mathcal{O}(|\varepsilon|\ |||W|||)$ for $R_A(t)$. Looking now at (5.55) we see that we can bound the exponential by $\max\{1,e^{\varepsilon^2(C|||W|||^2-\Gamma)\mathcal{M}}\}$. Now, the same argument as in the end of the previous subsection will give us the required result.

This completes the proof of Theorem 2.2.

7. Generalizations. In the previous sections we considered perturbations of the form $\varepsilon W(t)$, with $W(t)$ independent of ε. In this section, we shall extend our theory to a more general class of potentials, W_ε, which are small for small ε but which may deform nontrivially as ε varies.

Consider a family of perturbations $\mathcal{W}$ and the general system

$$\begin{aligned} i\partial_t\phi(t) &= (H_0 + W(t))\,\phi(t), \\ \phi|_{t=0} &= \phi(0), \end{aligned} \tag{7.1}$$

where $W \in \mathcal{W}$ (compare to (2.1). The results are as follows.

THEOREM 7.1. *Suppose that H_0 and any $W \in \mathcal{W}$ satisfy hypotheses* **(H1)**–**(H5)**. *In addition assume the following:*

(H7) *Equi-almost periodicity. There exists a positive constant L_{θ_0}, independent of $W \in \mathcal{W}$, such that in any interval of real numbers of length L_{θ_0}, the function $|||W|||^{-2}\ \sigma(t)$ ($|||W||| \neq 0$), where*

$$\sigma(t) \equiv -\frac{\pi}{4}\Re \sum_{j\in I_{res}, j\neq k\in\mathbb{Z}} e^{i(\mu_k-\mu_j)t}\,(\beta_k\psi_0, \delta(H_0-\lambda_0-\mu_j)\beta_j\psi_0) \tag{7.2}$$

has a $\theta_0/4$ almost period, θ_0 is given by **(H5)**. *More precisely, there exists $L_{\theta_0} > 0$ which does not depend on W such that in any interval of length L_{θ_0} there is a number $\tau = \tau(\theta_0/4)$ such that for all $t \in \mathbb{R}$*

$$\big|\ |||W|||^{-2}\sigma(t+\tau)\ -\ |||W|||^{-2}\sigma(t)\ \big|\ \leq\ \theta_0/4. \tag{7.3}$$

If $w_+\phi(0) \in \mathcal{H}$, then there exists an $\varepsilon_0 > 0$ (depending on $\mathcal{C}$, r_1, θ_0, and L_{θ_0}) such that whenever $|||W||| < \varepsilon_0$, the solution of (7.1) *satisfies the local decay estimate (identical with the one in Theorem* 2.1)

$$\|w_-\ \phi(t)\| \leq C\langle t\rangle^{-r_1}\|w_+\ \phi_0\|, \quad t \in \mathbb{R}. \tag{7.4}$$

Sketch of the proof. Once we drop ε from all expressions (since it is not present in the actual setting), the arguments in the previous sections hold in this case except the analysis of $\sigma(t)$ in Proposition 5.3. Formulas (5.36) and (7.2) are the same, but now $\mu_j,\ \beta_j,\ j \in \mathbb{Z}$, are not fixed as they define W by **(H4)** and W sweeps a general class $\mathcal{W}$. This may prevent us from finding a fixed time interval, $\mathcal{M}$, independent of $W \in \mathcal{W}$, after which $\sigma(t)$ is within $\Gamma/2$ distance from its mean; see relations (5.52)–(5.54).

Nevertheless, **(H7)** is exactly what we need to overcome the difficulty. A straightforward calculation shows that any $\theta_0/4$ almost period of $|||W|||^{-2}\sigma(t)$ is a $\Gamma/4$ almost period for $\sigma(t)$. Consequently, $L(\Gamma/4)$ in (5.54) is bounded above by $L(\theta_0)$ given in **(H7)**. But the latter is fixed, so we can choose

$$\mathcal{M} = 8CL(\theta_0)/\theta_0 \tag{7.5}$$

independent of $W \in \mathcal{W}$ and still satisfy (5.53) hence (5.52).

Finally, we can close the arguments exactly as we did for Theorem 2.1.

Remark 7.1. Theorems analogous to Theorem 2.2 (respectively, Theorem 2.3) can be proved under hypotheses **(H1)-(H5), (H7)** (respectively, **(H1)**–**(H6)**).

Examples. **(H7)** holds trivially for

(1) $\mathcal{W} = \{\varepsilon W(t,x) :\ \varepsilon \in \mathbb{R},\ W \text{ fixed}\}$ or

(2) $\mathcal{W} = \{\varepsilon W(\varepsilon^{-1}t, x) : \ \varepsilon \in \mathbb{R} - \{0\}, \ |\varepsilon| \leq 1, \ \ W \text{ fixed}\}$.

In Example (1), ε cancels in the formula $|||W|||^{-2}\sigma(t)$ while in Example (2) we have a time dilation which shrinks the gaps between the almost periods, so the $L(\theta_0)$ valid for $W(t,x)$ is good for the entire family.

(3) There are more general families of perturbations $\mathcal{W}$ for which **(H7)** holds. For example, if $\mathcal{W}$ is equi-almost periodic, see section 9.

8. Appendix: Singular operators. In this section we present the definition and the properties we needed previously for the singular operators

$$e^{-iH_0t}\,(H_0 - \Lambda - i0)^{-1}\,\mathbf{P_c},\ \delta\,(H_0 - \Lambda)\,\mathbf{P_c},\ \text{P.V.}\,(H_0 - \Lambda)^{-1}\,\mathbf{P_c}$$

and establish the identities

$$(H_0 - \Lambda \mp i0)^{-1}\,\mathbf{P_c} \ = \ \text{P.V.}\,(H_0 - \Lambda)^{-1}\,\mathbf{P_c} \pm i\pi\delta\,(H_0 - \Lambda)\,\mathbf{P_c}$$

suggested by the well-known distributional identities

$$(x \mp i0)^{-1} \ = \ \text{P.V.}\ \frac{1}{x} \ \pm i\pi\ \delta(x).$$

Recall that we are in the complex Hilbert space $\mathcal{H}$ with self-adjoint "weights" $w_\pm$ and projection operator $\mathbf{P_c}$ satisfying (i), (ii), and (iii). We can then construct the complex Hilbert space $\mathcal{H}_+$ as the closure of the domain of w_+ under the scalar product $(f,g)_+ = (w_+f, w_+g)$ and the complex Hilbert space $\mathcal{H}_-$ as the closure of $\mathbf{P_c}\mathcal{H}$ under the scalar product $(f,g)_- = (w_-f, w_-g)$.

By the hypotheses of section 2, H_0 is a self-adjoint operator on $\mathcal{H}$ and satisfies the local decay estimate (2.3). Based on this property, in [11, 22, 23] it is proved that for Λ in the continuous spectrum of H_0 and $t \in \mathbb{R}$

$$T_t \equiv i \lim_{\eta \searrow 0} \int_t^\infty e^{-i(H_0 - \Lambda - i\eta)s} ds \mathbf{P_c},$$
$$T_t^* \equiv -i \lim_{\eta \searrow 0} \int_{-\infty}^{-t} e^{-i(H_0 - \Lambda + i\eta)s} ds \mathbf{P_c}$$

are well defined linear bounded operators from $\mathcal{H}_+$ to $\mathcal{H}_-$. We then define

$$e^{-iH_0t}\,(H_0 - \Lambda - i0)^{-1}\,\mathbf{P_c} \equiv e^{-i\Lambda t}T_t, \tag{8.1}$$

$$e^{+iH_0t}\,(H_0 - \Lambda + i0)^{-1}\,\mathbf{P_c} \equiv e^{+i\Lambda t}T_t^*, \tag{8.2}$$

and

$$\text{P.V.}\,(H_0 - \Lambda)^{-1}\,\mathbf{P_c} \equiv \frac{1}{2}(T_0 + T_0^*), \tag{8.3}$$

$$\delta\,(H_0 - \Lambda)\,\mathbf{P_c} \equiv \frac{1}{2\pi i}(T_0 - T_0^*). \tag{8.4}$$

Note that the definitions imply the identities

$$(H_0 - \Lambda \mp i0)^{-1}\,\mathbf{P_c} = \text{P.V.}\,(H_0 - \Lambda)^{-1}\,\mathbf{P_c} \pm i\pi\delta\,(H_0 - \Lambda)\,\mathbf{P_c}. \tag{8.5}$$

Particularly important properties of these operators are their symmetries when viewed as quadratic forms on $\mathcal{H}_+ \times \mathcal{H}_+$. For example, on any $f,\ g \in \mathcal{H}_+$ the quadratic form induced by T_t is given by

$$(f,g) \mapsto (w_+f, w_-T_tg).$$

Note that

$$\lim_{\eta\searrow 0}(f,T_t^\eta g)\equiv\lim_{\eta\searrow 0}\left(f,i\int_t^\infty e^{-i(H_0-\Lambda-i\eta)s}ds\mathbf{P_c}g\right)\;=\;(w_+f,w_-T_tg) \tag{8.6}$$

by the following calculation:

$$\begin{aligned}\lim_{\eta\searrow 0}\left(f,i\int_t^\infty e^{-i(H_0-\Lambda-i\eta)s}ds\mathbf{P_c}g\right)&=\lim_{\eta\searrow 0}\left(f,\mathbf{P_c}i\int_t^\infty e^{-i(H_0-\Lambda-i\eta)s}ds\mathbf{P_c}g\right)\\&=\lim_{\eta\searrow 0}\left(f,w_+w_-\mathbf{P_c}i\int_t^\infty e^{-i(H_0-\Lambda-i\eta)s}ds\mathbf{P_c}g\right)\\&=\lim_{\eta\searrow 0}\left(w_+f,w_-i\int_t^\infty e^{-i(H_0-\Lambda-i\eta)s}ds\mathbf{P_c}g\right)\\&=(w_+f,w_-T_tg),\end{aligned}$$

where we used that $\mathbf{P_c}$ is a projection operator commuting with the integral operator, the identity $w_+w_-\mathbf{P_c}=\mathbf{P_c}$ on $\mathcal{H}$, the self-adjointness of $w_\pm$ and $\mathbf{P_c}$, and $\lim_{\eta\searrow 0}w_-T_t^\eta=w_-T_t$ in $\mathcal{L}(\mathcal{H}_+,\mathcal{H})$.

Identity (8.6) suggests the notation

$$(f,g)\mapsto(f,T_tg)$$

for the quadratic form induced by T_t, where $(\cdot,\cdot)$ can formally be treated as the scalar product in $\mathcal{H}$. Moreover, (8.6) implies

$$(f,T_tg)=(T_t^*f,g).$$

Therefore, the quadratic form induced by $\mathrm{P.V.}(H_0-\Lambda)^{-1}\mathbf{P_c}$ is the symmetric part of the one induced by T_0 while $\delta(H_0-\lambda)\mathbf{P_c}$ induces the skew-symmetric part of it divided by the factor $i\pi$. As a consequence both the forms corresponding to the last two operators are symmetric.

In conclusion, for any $f,g\in Domain(w_+),\ t\in\mathbb{R}$, and $\Lambda\in\sigma_{\mathrm{cont}}(H_0)$ we have

$$\begin{aligned}(f,e^{\mp iH_0t}(H_0-\Lambda\mp i0)^{-1}\mathbf{P_c}g)&\equiv\left(w_+f,w_-e^{\mp iH_0t}(H_0-\Lambda\mp i0)^{-1}\mathbf{P_c}g\right)\\&\le C_t\ \|w_+f\|\ \|w_+g\|,\end{aligned} \tag{8.7}$$

$$(f,\delta(H_0-\Lambda)\mathbf{P_c}g)\equiv(w_+f,w_-\delta(H_0-\Lambda)\mathbf{P_c}g)\le\ \frac{C_0}{\pi}\ \|w_+f\|\ \|w_+g\|, \tag{8.8}$$

$$(f,\mathrm{P.V.}(H_0-\Lambda)^{-1}\mathbf{P_c}g)\equiv\left(f,w_-\mathrm{P.V.}(H_0-\Lambda)^{-1}\mathbf{P_c}g\right)\le\ C_0\ \|w_+f\|\ \|w_+g\|. \tag{8.9}$$

The inequalities are due to the boundedness of T_t, where C_t denotes the norm of T_t in $\mathcal{L}(\mathcal{H}_+,\mathcal{H}_-)$. Moreover, the following symmetry properties hold:

$$\begin{aligned}(f,e^{\mp iH_0t}(H_0-\Lambda\mp i0)^{-1}\mathbf{P_c}g)&=(e^{\pm iH_0t}(H_0-\Lambda\pm i0)^{-1}\mathbf{P_c}f,g),\\(f,\delta(H_0-\Lambda)\mathbf{P_c}g)&=(\delta(H_0-\Lambda)\mathbf{P_c}f,g),\\(f,\mathrm{P.V.}(H_0-\Lambda)^{-1}\mathbf{P_c}g)&=(\mathrm{P.V.}(H_0-\Lambda)^{-1}\mathbf{P_c}f,g).\end{aligned}$$

9. Appendix: Almost periodic functions. In this section we present the definition and the properties of almost periodic functions we used throughout this paper. We will confine to functions of the form $f:\mathbb{R}\to X$, where X is a complex Banach space with norm denoted by $\|\cdot\|$.

DEFINITION 9.1. *We say that*

$$f : \mathbb{R} \to X$$

is almost periodic if and only if it is continuous and for each $\varepsilon > 0$ *there exists a length* $L(\varepsilon, f) > 0$ *such that in any closed interval of length greater or equal than* $L(\varepsilon, f)$ *there is at least one* τ *with the property that for all* $t \in \mathbb{R}$ *we have*

$$\|f(t+\tau) - f(t)\| \leq \varepsilon. \tag{9.1}$$

The number τ *with the property above is called an* ε *almost period for* f*.*

We say that the family, $\mathcal{F}$ *of almost periodic functions is equi-almost periodic if* $L(\varepsilon, f)$ *can be choosen independently of* $f \in \mathcal{F}$*.*

Example. Any continuous periodic function is almost periodic since for any $\varepsilon > 0$ we can choose the length $L(\varepsilon)$ to be the period of the function.

THEOREM 9.1. *Any almost periodic function has a relative compact image.*

The proof of the theorem can be found in [9, Property 1, p. 2]. In particular, any almost periodic function $f : \mathbb{R} \to X$ is in the Banach space of all bounded and continuous functions on $\mathbb{R}$ with values in X, $C(X)$, endowed with the uniform norm. The next result is Bochner's characterization of almost periodic functions; see, for example, [9, Bochner's theorem, p. 4].

THEOREM 9.2 (Bochner). *Let* $f : \mathbb{R} \to X$ *be a continuous function. For* f *to be almost periodic it is necessary and sufficient that the family of functions* $\{f(t+h)\}$, $-\infty < h < \infty$*, is relatively compact in* $C(X)$*.*

As a consequence of Bochner's criterion and Property 4 from [9, p. 3] we have the following.

THEOREM 9.3. *Suppose* $X_1,\ X_2, \ldots, X_{k+1}$ *are Banach spaces,* $f_i : \mathbb{R} \to X_i,\ 1 \leq i \leq k$ *are almost periodic functions, and* $g : \prod_{i=1}^{k} \to X_{k+1}$ *is continuous. Then* $g(f_1(t),\ f_2(t), \ldots, f_k(t))$ *is an almost periodic function.*

The last theorem has very important consequences in the theory of almost periodic functions. We will list only those which are useful in our presentation.

COROLLARY 9.1. *A finite sum of almost periodic functions with values in the same Banach space is an almost periodic function.*

COROLLARY 9.2. *A product between a complex valued almost periodic function and an arbitrary almost periodic function is an almost periodic function.*

COROLLARY 9.3. *If* $\mathcal{H}$ *is a complex Hilbert space,* $\mathcal{L}(\mathcal{H})$ *is the Banach space of the bounded linear operators on* $\mathcal{H}$*, and* $W : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ *is an almost periodic function, then for any* $\varphi,\ \psi \in \mathcal{H}$ *the following functions are almost periodic:*

$$\begin{aligned} &t \to W(t)\varphi, \\ &t \to (\psi,\ W(t)\varphi)\,, \\ &t \to (W(t)\psi,\ W(t)\varphi)\,, \end{aligned}$$

where $(\cdot, \cdot)$ *denotes the scalar product on* $\mathcal{H}$*.*

Another essential result in the theory of almost periodic functions is (see, for example, [9, Property 3, p. 3]) the following.

THEOREM 9.4. *Any uniform convergent sequence of almost periodic functions converges towards an almost periodic function.*

COROLLARY 9.4. *If* $\{\mu_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ *and* $\{\beta_j\}_{j\in\mathbb{Z}} \subseteq X$ *satisfies* $\sum_{j\in\mathbb{Z}} \|\beta_j\| < \infty$*, then*

$$\sum_{j\in\mathbb{Z}} e^{i\mu_j t}\beta_j$$

is an X-valued almost periodic function of t.

Proof. According to Weierstrass's criterion the series $\sum_{j\in\mathbb{Z}} e^{i\mu_j t}\beta_j$ is uniformly convergent on $\mathbb{R}$.

By Corollary 9.1 and the example above the partial sums of the above series are almost periodic. The result follows now from Theorem 9.4. □

We continue with the harmonic analysis results for almost periodic functions.

THEOREM 9.5 (mean value). *If $f:\mathbb{R}\to X$ is almost periodic, then the following limit exists and it is approached uniformly with respect to $a\in\mathbb{R}$:*

$$\lim_{t\to\infty}\frac{1}{t}\int_a^{a+t} f(s)ds = M(f)\in X.$$

Moreover, whenever

$$t\geq \frac{4\sup_{s\in\mathbb{R}}\|f(s)\|L(\varepsilon/2,f)}{\varepsilon}$$

we have

$$\left\|M(f)-\frac{1}{t}\int_a^{a+t} f(s)ds\right\|\leq\varepsilon$$

for all $a\in\mathbb{R}$.

The proof of the mean value theorem in this form can be found in [2, pp. 39–44]. Note that although Bohr's book considers only complex valued almost periodic functions the proof can be carried on to Banach space valued functions by simply replacing the modulus by the norm and the Lebesque's integral for complex valued functions by the Bochner's integral.

The results of the next theorem are presented in [9, Chapter 2].

THEOREM 9.6 (fundamental theorem). *If $f,\ g:\mathbb{R}\to X$ are almost periodic, then*
(a) *for any $\mu\in\mathbb{R}$,*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(s)e^{-i\mu s}ds = a(\mu,f)$$

exists and is nonzero for at most a denumerable set of μ's; if $a(\mu,f)\neq 0$, then $a(\mu,f)$ is called a Fourier coefficient for f while μ is called a Fourier exponent;

(b) *$a(\mu,f)=a(\mu,g)$ for all $\mu\in\mathbb{R}$ if and only if $f\equiv g$;*

(c) *let $\Lambda(f)=\{\mu:\ a(\mu,f)\neq 0\}$ denote the set of Fourier exponents for f; then there is an ordering on $\Lambda(f)$, $\Lambda(f)=\{\mu_1,\ \mu_2,\ldots\}$ independent of the Fourier coefficients, such that for any $\varepsilon>0$ there exist the numbers $N(\varepsilon)\in\mathbb{N}$, $0\leq k_{n,\varepsilon}\leq 1$, $n\in\mathbb{N}$, with the property that the trigonometric polynomial*

$$P_\varepsilon(t)=\sum_{n=1}^{N(\varepsilon)} k_{n,\varepsilon}a(\mu_n,f)e^{i\mu_n t}$$

satisfies

$$\|f(t)-P_\varepsilon(t)\|\leq\varepsilon\quad\text{for all } t\in\mathbb{R}.$$

Moreover, $k_{n,\varepsilon}$ can be choosen such that for any fixed n, $\lim_{\varepsilon\searrow 0}k_{n,\varepsilon}=1$.

In this paper we use a less general result than the above fundamental theorem, namely, the following.

COROLLARY 9.5. *If* $f(t) = \sum_{j\in\mathbb{Z}} e^{i\mu_j t}\beta_j$, *where* $\{\mu_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ *and* $\sum_{j\in\mathbb{Z}} \|\beta_j\| < \infty$, *then* $\Lambda(f) = \{\mu_j\}_{j\in\mathbb{Z}}$, $a(\mu_j, f) = \beta_j$, $j \in \mathbb{Z}$, *in particular if* $\mu_j \neq 0$, $j \in \mathbb{Z}$, *then* $M(f) = 0$. *Moreover, we can arbitrarily order* $\Lambda(f)$ *and still have that for any* $\varepsilon > 0$ *there exists a natural number* $N(\varepsilon)$ *such that*

$$\|f(t) - \sum_{j=-N(\varepsilon)}^{j=N(\varepsilon)} e^{i\mu_j t}\beta_j\| \leq \varepsilon$$

or, in other words, in this particular case the conclusion in part (c) of the fundamental theorem is valid even if we have an arbitrary order on $\Lambda(f)$ *and we choose* $k_{j,\varepsilon} \equiv 1$.

Proof. By the Weierstrass criterion the series

$$f(t)e^{-i\mu t} = \sum_{j\in\mathbb{Z}} e^{i(\mu_j-\mu)t}\beta_j$$

is uniformly convergent on $\mathbb{R}$. So, when we compute $a(\mu, f)$ we can integrate term by term and therefore use the identities

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-i\lambda s} ds = \begin{cases} 0 & \text{if} \quad \lambda \neq 0, \\ 1 & \text{if} \quad \lambda = 0 \end{cases}$$

to get the first part of the corollary. The last part is a direct consequence of the fact that f is an absolute and uniform convergent series. □

Acknowledgments. The authors wish to thank A. Soffer and P. D. Miller for discussions on this work.

REFERENCES

[1] V. BACH, J. FRÖHLICH, I.M. SIGAL, AND A. SOFFER, *Positive commutators and the spectrum of Pauli-Fierz Hamiltonians of atoms and molecules*, Comm. Math. Phys., 207 (1999), pp. 557–587.
[2] H. BOHR, *Almost Periodic Functions*, Chelsea, New York, 1951.
[3] C. COHEN-TANNOUDJI, J. DUPONT-ROC, AND G. GRYNBERG, *Atom-Photon Interactions*, Wiley, New York, 1992.
[4] A. GALINDO AND P. PASCUAL, *Quantum Mechanics*, II, Springer, Berlin, 1991.
[5] A. JENSEN AND T. KATO, *Spectral properties of Schrödinger operators and time-decay of wave functions*, Duke Math. J., 46 (1979), pp. 583–611.
[6] J.-L. JOURNÉ, A. SOFFER, AND C. SOGGE, $L^p \to L^{p'}$ *estimates for the time dependent Schrödinger equation*, Bull. Amer. Math. Soc., 23 (1990), pp. 519–524.
[7] E. KIRR AND M.I. WEINSTEIN, *Almost Periodic Perturbations of Hamiltonian PDEs with Multiple Bound States*, in preparation.
[8] W. KOHLER AND G.C. PAPANICOLAOU, *Wave propagation in a randomly inhomogeneous ocean*, in Wave Propagation in Underwater Acoustics, J.B Keller and J.S. Papadakis, eds., Lecture Notes in Phys. 70, Springer, Berlin, 1977.
[9] B.M. LEVITAN AND V.V. ZHIBOV, *Almost Periodic Functions and Differential Equations*, Cambridge, University Press, London, New York, 1982.
[10] L.D. LANDAU AND E.M. LIFSHITZ, *Quantum Mechanics: Non-relativistic Theory*, Course of Theoretical Physics, Vol. 3, Pergamon Press, Oxford, UK, 1965.
[11] M. MERKLI AND I.M. SIGAL, *A time-dependent theory of quantum resonances*, Comm. Math. Phys., 201 (1999), pp. 549–576.
[12] P.D. MILLER, A. SOFFER, AND M.I. WEINSTEIN, *Metastability of breather modes of time dependent potentials*, Nonlinearity, 13 (2000), pp. 507–568.
[13] D. MARCUSE, *Theory of Dielectric Optical Waveguides*, Academic Press, New York, 1974.
[14] M. MURATA, *Rate of decay of local energy and spectral properties of elliptic operators*, Japan J. Math. (N.S.), 6 (1980), pp. 77–127.

[15] R.G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed., Springer-Verlag, New York, 1982.
[16] J. Rauch, *Local decay of scattering solutions to Schrödinger's equation*, Comm. Math. Phys., 61 (1978), pp. 149–168.
[17] M. Reed and B. Simon, *Methods in Modern Mathematical Physics*, I, *Functional Analysis,* Academic Press, New York, 1972.
[18] T. Schonbek, *Decay of solutions of Schrödinger equations*, Duke Math. J., 46 (1979), pp. 203–213.
[19] I.M. Sigal, *Nonlinear wave and Schrödinger equations* I, *Instability of time-periodic and quasiperiodic solutions*, Comm. Math. Phys., 153 (1993), pp. 297–320.
[20] I.M. Sigal, *General characteristics of nonlinear dynamics*, in Spectral and Scattering Theory, M. Ikawa, ed., Proceedings of the Taniguchi International Workshop, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1994, pp. 197–217.
[21] A. Soffer and M.I. Weinstein, *Dynamic theory of quantum resonances and perturbation theory of embedded eigenvalues*, in Partial Differential Equations and Their Applications, P. Greiner, V. Ivrii, L. Seco, and C. Sulem, eds., CRM Proc. Lecture Notes 12, AMS, Providence, 1997, pp. 277–282.
[22] A. Soffer and M.I. Weinstein, *Nonautonomous Hamiltonians*, J. Statist. Phys., 93 (1998), pp. 359–391.
[23] A. Soffer and M.I. Weinstein, *Time dependent resonance theory*, Geom. Funct. Anal., 8 (1998), pp. 1086–1128.
[24] A. Soffer and M.I. Weinstein, *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations*, Invent. Math., 136 (1999), pp. 9–74.
[25] B. Vainberg, *Scattering of waves in a medium depending periodically on time*, Asterisque, 210 (1992), pp. 327–340.
[26] K. Yajima, *Scattering theory for Schrödinger operators with potentials periodic in time*, J. Math. Soc. Japan, 29 (1977), pp. 729–743.
[27] K. Yajima, *Resonances for the AC-Stark effect*, Comm. Math. Phys., 78 (1982), pp. 331–352.
[28] K. Yajima, *A multichannel scattering theory for some time dependent Hamiltonians, charge transfer problem*, Comm. Math. Phys., 75 (1980), pp. 153–178.