Firm-Specific Capital and the New-Keynesian Phillips Curve[*]

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1 Introduction

A popular specification in recent analyses of alternative monetary policies is the "new-Keynesian" Phillips curve,

$$\pi_t = \xi \hat{s}_t + \beta E_t \pi_{t+1}, \tag{1.1}$$

where π_t is the rate of inflation, $\hat{s}_t$ is the departure of the (average) log of real marginal cost from its steady-state value, the coefficient $\xi > 0$ depends on the degree of stickiness of prices, and $0 < \beta < 1$ is a utility discount factor that, under an empirically realistic calibration, must nearly equal 1. As is well known, this relation follows (in a log-linear approximation) from the Calvo model of staggered price-setting under certain assumptions.[1] The implications of (1.1) for the co-movement of the general level of prices and marginal cost have been subject to extensive econometric testing, beginning with the work of Gali and Gertler (1999) and Sbordone (2002).

In standard derivations, (1.1) follows from the optimal pricing problem of a firm that adjusts the price of its product at random intervals, under the assumption that the marginal cost $S_t(i)$ of supplying a given good i in period t is given by a function of the form

$$S_t(i) = S(y_t(i); X_t), \tag{1.2}$$

where $y_t(i)$ is the quantity sold of good i in that period, and X_t is a vector of variables that firm i takes to be unaffected by its pricing decision. Under the further assumption of a demand curve of the form $y_t(i) = Y(p_t(i); X_t)$, this implies that marginal cost can be expressed as a function of the price $p_t(i)$ that i chooses to charge in that period, together with variables that are unaffected by its actions.

The specification (1.2) is in turn correct as long as all factors of production are either purchased on a spot market (at a price that is independent of the quantity used by i), or completely fixed. In particular, one can treat the case

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[1] See, *e.g.*, Woodford (2003, chap. 3, sec. 2.2).

in which capital is not a variable factor of production (and output is simply a concave function of the variable labor input, as in Woodford, 2003, chap. 3), or the case in which capital is variable, but capital services are obtained on a rental market (as in Gali and Gertler, 1999, and the baseline case considered in Sbordone, 2002).[2] Matters are more complex, however, under the more realistic assumption that capital is endogenous and *firm-specific.* That is, we shall assume that each firm accumulates capital for its own use only, and that (as in standard neoclassical investment theory) there are convex costs of more rapid adjustment of an individual firm's capital stock. In this case, $S_t(i)$ will depend not only on the quantity that firm i produces in period t, but also on the firm's capital stock in that period, and this latter variable depends on the firm's decisions in previous periods, including its previous pricing decisions. The dynamic linkages in a firm's optimal price-setting decision are therefore more complex in this case than is assumed in standard derivations of the new-Keynesian Phillips curve.

Here I treat the optimal price-setting problem in a model with firm-specific capital, and show that once again a relation of the form (1.1) can be derived.[3] Hence the econometric estimates reported by authors such as Gali and Gertler (1999) and Sbordone (2002) can be interpreted without making assumptions as restrictive as those papers had appeared to rely upon. However, the coefficient ξ is a more complex function of underlying model parameters, such as the frequency with which prices are re-optimized, in the case that capital is firm-specific.

This is potentially of considerable importance for the interpretation of econometric estimates of the coefficient ξ. Estimates of ξ are often interpreted in terms of the frequency of price of adjustment that they imply, given estimated or calibrated values for other model parameters. (Indeed, in many papers in the literature, beginning with Gali and Gertler, 1999, equation (1.1) is estimated in a form that results directly in an estimate of the

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[2]Both assumptions lead to a relation of the form (1.1). However, the interpretation of the coefficient ξ in terms of underlying model parameters is different in the two cases, as discussed in Sbordone (2002).

[3]The derivation here corrects the analysis given in Woodford (2003, chap. 5, sec. 3), to take account of an error in the original calculations noted by Sveen and Weinke (2004a).

frequency of price adjustment rather than of the elasticity ξ.) Furthermore, it is often argued that estimated values of ξ are so small as to imply that prices are sticky for an implausibly long length of time; and this is taken to cast doubt on the realism of the Calvo pricing model and hence of the aggregate-supply specification (1.1). But the mapping between the frequency of price adjustment and the value of ξ is different in the case of firm-specific capital than under the more common assumption of a rental market for capital services.[4] The assumption of a rental market for capital substantially weakens the degree of strategic complementarity among the pricing decisions of different firms — or alternatively, it reduces the importance of real rigidities in the sense of Ball and Romer (1990) — with the consequence that ξ is larger for any given frequency of price adjustment. It then follows that a small estimated value of ξ will be taken to imply very infrequent price adjustment. But allowing for firm-specific capital can make the implied frequency of price adjustment much greater, as shown in section 4.4 below.

The fact that an assumption that capital is firm-specific will lead to a lower estimate of the degree of price stickiness was first demonstrated by Sbordone (1998), and also illustrated by Gali, Gertler and Lopez-Salido (2001). However, in these papers, the treatment of capital as firm-specific is accompanied (at least implicitly) by an assumption that the capital stock of each firm is exogenously given, as in the analysis in Woodford (2003, chap. 3), rather than responding endogenously to the firm's incentives to invest. This is because it is only in this case that a specification of the form (1.2) remains consistent with the assumption of firm-specific capital. The analysis here instead presents an analysis of aggregate supply in the case that capital is both firm-specific and endogenous.[5] This case is a good deal more complicated to

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[4]It is also different under the assumption of a fixed quantity of capital for each firm, as noted above; but that simple model is disconfirmed by the observation that capital varies over time, and indeed that investment spending is substantially affected by monetary disturbances.

[5]Subsequent to the first circulation of these notes, Eichenbaum and Fisher (2004), Altig *et al.* (2005), and Matheron (2005) have built on the analysis here to examine the consequences of endogenous firm-specific capital for the estimated frequency of price adjustment in empirical versions of the new-Keynesian Phillips curve. These authors extend the present analysis to more complicated versions of (1.1) that allow a closer fit to

analyze, but it turns out still to be possible to derive an aggregate-supply relation that (in a log-linear approximation) takes the simple form (1.1).

The paper proceeds as follows. In section 2, I introduce a model of firm-specific investment demand with convex costs of adjustment of an individual firm's capital stock, with particular attention to the way in which standard neoclassical investment theory must be modified when the firm is not a price-taker in its product market, but instead fixes its price for a period of time and fills whatever orders it may receive. In section 3, I then consider the price-setting problem of such a firm, under the assumption that the price remains fixed for a random interval of time, and characterize the joint dynamics of the firm's price and its capital stock. Finally, in section 4, I derive the model's implications for the form of the aggregate-supply relation that connects the overall inflation rate with the overall level of real activity, and discuss the consequences for the inference about the frequency of price adjustment that can be drawn from an estimate of the elasticity ξ in (1.1).

2   Investment Demand when Prices are Sticky

I wish to analyze the relation between inflation and aggregate output in a model with staggered pricing (modeled after the fashion of Calvo (1983) and Yun (1996)) and endogenous capital accumulation. The main source of complication in this analysis is the assumption that the producers of individual differentiated goods (that adjust their prices at different dates) invest in firm-specific capital which is relatively durable, so that the distribution of capital stocks across different firms (as a result of differing histories of price adjustment) matters, and not simply the economy's aggregate capital stock. Nonetheless, I shall show that (in the same kind of log-linear approximation that is used in standard derivations of the New Keynesian Phillips curve) it is possible to derive structural relations that constitute the "aggregate supply block" of a macro model, which involve only the economy's aggregate capital stock, aggregate output, and overall index of prices.

A first task is to develop a model of optimizing investment demand by

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aggregate U.S. time series.

suppliers with sticky prices, and that are demand-constrained as a result. As in the sticky-price models with exogenous capital presented in Woodford (2003, chap. 3), there is a continuum of differentiated goods, each supplied by a single (monopolistically competitive) firm. The production function for good i is assumed to be of the form

$$y_t(i) = k_t(i)f(A_t h_t(i)/k_t(i)), \tag{2.1}$$

where f is an increasing, concave function, with f(0) = 0. I assume that each monopoly supplier makes an independent investment decision each period; there is a separate capital stock $k_t(i)$ for each good, that can be used only in the production of good i.

I also assume convex adjustment costs for investment by each firm, of the usual kind assumed in neoclassical investment theory. Increasing the capital stock to the level $k_{t+1}(i)$ in period $t+1$ requires investment spending in the amount $I_t(i) = I(k_{t+1}(i)/k_t(i))k_t(i)$ in period t. Here $I_t(i)$ represents purchases by firm i of the composite good, defined as the usual Dixit-Stiglitz aggregate over purchases of each of the continuum of goods (with the same constant elasticity of substitution $\theta > 1$ as for consumption purchases).[6] In this way, the allocation of investment expenditure across the various goods is in exactly the same proportion as consumption expenditure, resulting in a demand curve for each producer that is again of the form

$$y_t(i) = Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta}, \tag{2.2}$$

but where now aggregate demand is given by $Y_t = C_t + I_t + G_t$, in which expression C_t is the representative household's demand for the composite good for consumption purposes, G_t is the government's demand for the composite good (treated as an exogenous random variable), and I_t denotes the integral of $I_t(i)$ over the various firms i.

I assume as usual that the function $I(\cdot)$ is increasing and convex; the convexity implies the existence of costs of adjustment. I further assume that near a zero growth rate of the capital stock, this function satisfies $I(1) = \delta$,

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[6]See Woodford (2003, chap. 3) for discussion of this aggregator and its consequences for the optimal allocation of demand across alternative differentiated goods.

$I'(1) = 1$, and $I''(1) = \epsilon_\psi$, where $0 < \delta < 1$ and $\epsilon_\psi > 0$ are parameters. This implies that in the steady state to which the economy converges in the absence of shocks (which here involves a constant capital stock, as I abstract from trend growth), the steady rate of investment spending required to maintain the capital stock is equal to δ times the steady-state capital stock (so that δ can be interpreted as the rate of depreciation). It also implies that near the steady state, a marginal unit of investment spending increases the capital stock by an equal amount (as there are locally no adjustment costs). Finally, in my log-linear approximation to the equilibrium dynamics, ϵ_ψ is the parameter that indexes the degree of adjustment costs. A central goal of the analysis is consideration of the consequences of alternative values for ϵ_ψ; the model with exogenous firm-specific capital presented in Woodford (2003, chaps. 3, 4) is recovered as the limiting case of the present model in which ϵ_ψ is made unboundedly large.

Profit-maximization by firm i then implies that the capital stock for period $t + 1$ will be chosen in period t to satisfy the first-order condition

$$I'(g_t(i)) = E_t Q_{t,t+1} \Pi_{t+1} \{ \rho_{t+1}(i) +$$
$$g_{t+1}(i) I'(g_{t+1}(i)) - I(g_{t+1}(i)) \}, \qquad (2.3)$$

where $g_t(i) \equiv k_{t+1}(i)/k_t(i)$, $\rho_{t+1}(i)$ is the (real) shadow value of a marginal unit of additional capital for use by firm i in period $t + 1$ production, and $Q_{t,t+1} \Pi_{t+1}$ is the stochastic discount factor for evaluating real income streams received in period $t + 1$. Expressing the real stochastic discount factor as $\beta \lambda_{t+1}/\lambda_t$, where λ_t is the representative household's marginal utility of real income in period t and $0 < \beta < 1$ is the utility discount factor, and then log-linearizing (2.3) around the steady-state values of all state variables, we obtain

$$\hat{\lambda}_t + \epsilon_\psi(\hat{k}_{t+1}(i) - \hat{k}_t(i)) = E_t \hat{\lambda}_{t+1} +$$
$$[1 - \beta(1-\delta)] E_t \hat{\rho}_{t+1}(i) + \beta \epsilon_\psi E_t(\hat{k}_{t+2}(i) - \hat{k}_{t+1}(i)), \qquad (2.4)$$

where $\hat{\lambda}_t \equiv \log(\lambda_t/\bar{\lambda})$, $\hat{k}_t(i) \equiv \log(k_t(i)/\bar{K})$, $\hat{\rho}_t(i) \equiv \log(\rho_t(i)/\bar{\rho})$, and variables with bars denote steady-state values.

Note that $\rho_{t+1}(i)$ would correspond to the real "rental price" for capital services if a market existed for such services, though I do not assume one

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here.[7] It is *not* possible in the present model to equate this quantity with the marginal product, or even the marginal revenue product of capital (using the demand curve (2.2) to compute marginal revenue). For suppliers are demand-constrained in their sales, given the prices that they have posted; it is not possible to increase sales by moving down the demand curve. Thus the shadow value of additional capital must instead be computed as the reduction in labor costs through substitution of capital inputs for labor, while still supplying the quantity of output that happens to be demanded. In this way I obtain

$$\rho_t(i) = w_t(i) \left(\frac{f(\tilde{h}_t(i)) - \tilde{h}_t(i) f'(\tilde{h}_t(i))}{A_t f'(\tilde{h}_t(i))} \right),$$

where $w_t(i)$ is the real wage for labor of the kind hired by firm i and $\tilde{h}_t(i) \equiv A_t h_t(i)/k_t(i)$ is firm i's effective labor-capital input ratio.[8] I can alternatively express this in terms of the output-capital ratio for firm i (in order to derive an "accelerator" model of investment demand), by substituting (2.1) to obtain

$$\rho_t(i) = \frac{w_t(i)}{A_t} f^{-1}(y_t(i)/k_t(i))[\phi(y_t(i)/k_t(i)) - 1], \qquad (2.5)$$

where $\phi(y/k)$ is the reciprocal of the elasticity of the function f, evaluated at the argument $f^{-1}(y/k)$.

As in the baseline model treated in Woodford (2003, chap. 3), I shall assume a sector-specific labor market. In this case, the first-order condition for optimizing labor supply can be written in the form

$$w_t(i) = \frac{v_h(f^{-1}(y_t(i)/k_t(i))k_t(i)/A_t; \xi_t)}{\lambda_t}, \qquad (2.6)$$

where labor demand has been expressed as a function of the demand for good i. This can be log-linearized as

$$\hat{w}_t(i) = \nu(\hat{\tilde{h}}_t(i) - \bar{h}_t) - \hat{\lambda}_t,$$

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[7]The case in which there is a rental market for capital services is instead considered in section 4.2 below.

[8]Note that in the case of a flexible-price model, the ratio of $w_t(i)$ to the denominator would always equal marginal revenue, and so this expression would equal the marginal revenue product of capital, though it would be a relatively cumbersome way of writing it.

where $\nu > 0$ is the elasticity of the marginal disutility of labor with respect to labor supply, and $\bar{h}_t$ is an exogenous disturbance to preferences, indicating the percentage increase in labor supply needed to maintain a constant marginal disutility of working. Substituting (2.6) into (2.5) and log-linearizing, I obtain

$$\hat{\rho}_t(i) = \left(\nu\phi_h + \frac{\phi_h}{\phi_h - 1}\omega_p\right)(\hat{y}_t(i) - \hat{k}_t(i)) + \nu\hat{k}_t(i) - \hat{\lambda}_t - \omega q_t, \qquad (2.7)$$

where $\phi_h > 1$ is the steady-state value of $\phi(y/k)$, *i.e.*, the reciprocal of the elasticity of the production function with respect to the labor input, and $\omega_p > 0$ is the negative of the elasticity of the marginal product $f'(f^{-1}(y/k))$ with respect to y/k. The composite exogenous disturbance q_t is defined as

$$q_t \equiv \omega^{-1}[\nu\bar{h}_t + (1+\nu)a_t]$$

where $a_t \equiv \log A_t$; it indicates the percentage change in output required to maintain a constant marginal disutility of output supply, in the case that the firm's capital remains at its steady-state level.[9] Substituting (2.7) into (2.4), I then have an equation to solve for the dynamics of firm i's capital stock, given the evolution of demand $\hat{y}_t(i)$ for its product, the marginal utility of income $\hat{\lambda}_t$, and the exogenous disturbance q_t.

As the coefficients of these equations are the same for each firm, an equation of the same form holds for the dynamics of the aggregate capital stock (in our log-linear approximation). The equilibrium condition for the dynamics of the capital stock is thus of the form

$$\hat{\lambda}_t + \epsilon_\psi(\hat{K}_{t+1} - \hat{K}_t) = \beta(1-\delta)E_t\hat{\lambda}_{t+1} +$$
$$[1 - \beta(1-\delta)][\rho_y E_t\hat{Y}_{t+1} - \rho_k\hat{K}_{t+1} - \omega E_t q_{t+1}] + \beta\epsilon_\psi E_t(\hat{K}_{t+2} - \hat{K}_{t+1}),$$
$$(2.8)$$

where the elasticities of the marginal valuation of capital are given by

$$\rho_y \equiv \nu\phi_h + \frac{\phi_h}{\phi_h - 1}\omega_p > \rho_k \equiv \rho_y - \nu > 0.$$

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[9]That is, q_t measures the output change that would be required to maintain a fixed marginal disutility of supply given possible fluctuations in preferences and technology, but not taking account of the effect of possible fluctuations in the firm's capital stock. With this modification of the definition given in Woodford (2003, chap. 3) for the model with exogenous capital, q_t is again an exogenous disturbance term.

The implied dynamics of investment spending are then given by

$$\hat{I}_t = k[\hat{K}_{t+1} - (1 - \delta)\hat{K}_t], \tag{2.9}$$

where $\hat{I}_t$ is defined as the percentage deviation of investment from its steady-state level, as a share of steady-state output, and $k \equiv \bar{K}/\bar{Y}$ is the steady-state capital-output ratio.

Thus far I have derived investment dynamics as a function of the evolution of the marginal utility of real income of the representative household. This is in turn related to aggregate spending through the relation $\lambda_t = u_c(Y_t - I_t - G_t; \xi_t)$, which we may log-linearize as

$$\hat{\lambda}_t = -\sigma^{-1}(\hat{Y}_t - \hat{I}_t - g_t), \tag{2.10}$$

where the composite disturbance g_t reflects the effects both of government purchases and of shifts in private impatience to consume.[10] Finally, because of the relation between the marginal utility of income process and the stochastic discount factor that prices bonds,[11] the nominal interest rate must satisfy

$$1 + i_t = \{\beta E_t[\lambda_{t+1}/(\lambda_t \Pi_{t+1})]\}^{-1},$$

which one may log-linearize as

$$\hat{i}_t = E_t \pi_{t+1} + \hat{\lambda}_t - E_t \hat{\lambda}_{t+1}. \tag{2.11}$$

The system of equations (2.8) – (2.11) then comprise the "IS block" of the model. These jointly suffice to determine the paths of the variables $\{\hat{Y}_t, \hat{I}_t, \hat{K}_t, \lambda_t\}$, given an initial capital stock and the evolution of short-term real interest rates $\{\hat{i}_t - E_t \pi_{t+1}\}$. The nature of the effects of real interest-rate expectations on these variables is discussed further in Woodford (2004).

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[10]Note that the parameter σ in this equation is not precisely the intertemporal elasticity of substitution in consumption, but rather $\bar{C}/\bar{Y}$ times that elasticity. In a model with investment, these quantities are not exactly the same, even in the absence of government purchases.

[11]See Woodford (2003, chaps. 2, 4) for further discussion of the stochastic discount factor and the Fisher relation between the nominal interest rate and expected inflation.

3 Optimal Price-Setting with Endogenous Capital

I turn next to the implications of an endogenous capital stock for the price-setting decisions of firms. The capital stock affects a firm's marginal cost, of course; but more subtly, a firm considering how its future profits will be affected by the price it sets must also consider how its capital stock will evolve over the time that its price remains fixed.

I begin with the consequences for the relation between marginal cost and output. Real marginal cost can be expressed as the ratio of the real wage to the marginal product of labor,

$$s_t(i) = \frac{w_t(i)}{A_t f'(f^{-1}(y_t(i)/k_t(i)))}. \tag{3.1}$$

Again writing the factor input ratio as a function of the capital/output ratio, and using (2.6) for the real wage, we obtain

$$s_t(i) = \frac{v_h(f^{-1}(y_t(i)/k_t(i))k_t(i)/A_t; \xi_t)}{\lambda_t A_t f'(f^{-1}(y_t(i)/k_t(i)))} \tag{3.2}$$

for the real marginal cost of supplying good i. This can be log-linearized to yield

$$\hat{s}_t(i) = \omega(\hat{y}_t(i) - \hat{k}_t(i) - q_t) + \nu\hat{k}_t(i) - \hat{\lambda}_t, \tag{3.3}$$

where $\hat{s}_t(i) \equiv \log(s_t(i)/\bar{s})$, and $\omega \equiv \omega_w + \omega_p \equiv \nu\phi_h + \omega_p > 0$ is the elasticity of marginal cost with respect to a firm's own output.

Letting $\hat{s}_t$ without the index i denote the average level of real marginal cost in the economy as a whole, I note that (3.3) implies that

$$\hat{s}_t(i) = \hat{s}_t + \omega(\hat{y}_t(i) - \hat{Y}_t) - (\omega - \nu)(\hat{k}_t(i) - \hat{K}_t). \tag{3.4}$$

Then using (2.2) to substitute for the relative output of firm i in (3.4), one obtains

$$\hat{s}_t(i) = \hat{s}_t - (\omega - \nu)\tilde{k}_t(i) - \omega\theta\tilde{p}_t(i), \tag{3.5}$$

where $\tilde{p}_t(i) \equiv \log(p_t(i)/P_t)$ is the firm's log relative price, and $\tilde{k}_t(i) \equiv \hat{k}_t(i) - \hat{K}_t$ is its log relative capital stock. Note also that the average level of real marginal cost satisfies

$$\hat{s}_t = \omega(\hat{Y}_t - \hat{K}_t - q_t) + \nu\hat{K}_t - \hat{\lambda}_t. \tag{3.6}$$

Following the same logic as in Woodford (2003, chap. 3), the Calvo price-setting framework implies that if a firm i resets its price in period t, it chooses a price that satisfies the (log-linear approximate) first-order condition

$$\sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i [\tilde{p}_{t+k}(i) - \hat{s}_{t+k}(i)] = 0, \tag{3.7}$$

where $0 < \alpha < 1$ is the fraction of prices that are not reset in any period. Here I introduce the notation $\hat{E}_t^i$ for an expectation conditional on the state of the world at date t, but integrating *only over those future states in which i has not reset its price since period t*. Note that in the case of any *aggregate* state variable x_t (*i.e.*, a variable the value of which depends only on the history of aggregate disturbances, and not on the individual circumstances of firm i), $\hat{E}_t^i x_T = E_t x_T$, for any date $T \geq t$. However, the two conditional expectations differ in the case of variables that depend on the relative price or relative capital stock of firm i. For example,

$$\hat{E}_t^i \tilde{p}_{t+k}(i) = \tilde{p}_t(i) - \sum_{j=1}^{k} E_t \pi_{t+j}, \tag{3.8}$$

for any $k \geq 1$, since firm i's price remains unchanged along all of the histories that are integrated over in this case. Instead, the expectation when one integrates over all possible future states conditional upon the state of the world at date t is given by

$$E_t \tilde{p}_{t+1}(i) = \alpha[\tilde{p}_t(i) - E_t \pi_{t+1}] + (1-\alpha) E_t \hat{p}_{t+1}^*(i), \tag{3.9}$$

where $\hat{p}_t^*(i)$ is the (log) relative price chosen when i reconsiders its price at date t. (Similar expressions can be given for horizons $k > 1$.)

Substituting (3.5) for $s_{t+k}(i)$ and (3.8) for $\hat{E}_t^i \tilde{p}_{t+k}(i)$ in (3.7), one obtains

$$(1 + \omega\theta)\hat{p}_t^*(i) =$$

$$(1 - \alpha\beta) \sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i \left[\hat{s}_{t+k} + (1 + \omega\theta) \sum_{j=1}^{k} \pi_{t+j} - (\omega - \nu)\tilde{k}_{t+k}(i) \right] \tag{3.10}$$

for the optimal relative price that should be chosen by a firm that resets its price at date t. This relation differs from the result obtained in Woodford

(2003, chap. 3) for a model with exogenous capital only in the presence of the $\hat{E}_t^i \tilde{k}_{t+k}(i)$ terms.

The additional terms complicate the analysis in several respects. Note that the first two terms inside the square brackets are aggregate state variables, so that the distinction between $\hat{E}_t^i$ and E_t would not matter in this expression, were it not for the dependence of marginal cost on i's relative capital stock; it is for this reason that the alternative form of conditional expectation did not have to be introduced in Woodford (2003, chap. 3). However, in the model with endogenous capital, it is important to make this distinction when evaluating the $\hat{E}_t^i \tilde{k}_{t+k}(i)$ terms.[12] Furthermore, these new terms will not have the same value for all firms i that reset their prices at date t, for they will depend on i's relative capital stock $\tilde{k}_t(i)$ at the time that prices are reconsidered; hence $p_t^*(i)$ is no longer independent of i, as in the model with exogenous capital (or a model with an economy-wide rental market for capital). And finally, (3.10) is not yet a complete solution for the optimal price-setting rule, since the value of the right-hand side still depends on the expected evolution of i's relative capital stock; and this in turn depends on the expected evolution of i's relative price, which depends on the choice of $\hat{p}_t^*(i)$. A complete solution for this decision rule requires that one consider the effect of a firm's relative price on the evolution of its relative capital stock.

3.1   Dynamics of the Relative Capital Stock

Equation (2.8) implies that i's relative capital stock must evolve in accordance with the relation

$$\epsilon_\psi(\tilde{k}_{t+1}(i) - \tilde{k}_t(i)) = [1 - \beta(1-\delta)][\rho_y E_t(\hat{y}_{t+1}(i) - \hat{Y}_t) - \rho_k \tilde{k}_{t+1}(i)]$$

$$+ \beta\epsilon_\psi E_t(\tilde{k}_{t+2}(i) - \tilde{k}_{t+1}(i)).$$

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[12]It is the failure to distinguish between $\hat{E}_t^i$ and E_t in evaluating these terms that results in the incorrect calculations in the treatment of the present model in Woodford (2003, chap. 5) noted by Sveen and Weinke (2004a).

Again using i's demand curve to express relative output as a function of the firm's relative price, this can be written as

$$E_t[Q(L)\tilde{k}_{t+2}(i)] = \Xi E_t \tilde{p}_{t+1}(i), \qquad (3.11)$$

where the lag polynomial is

$$Q(L) \equiv \beta - [1 + \beta + (1 - \beta(1 - \delta))\rho_k \epsilon_\psi^{-1}]L + L^2,$$

and

$$\Xi \equiv (1 - \beta(1 - \delta))\rho_y \theta \epsilon_\psi^{-1} > 0.$$

I note for later reference that the lag polynomial can be factored as

$$Q(L) = \beta(1 - \mu_1 L)(1 - \mu_2 L).$$

Given that $Q(0) = \beta > 0$, $Q(\beta) < 0$, $Q(1) < 0$, and that $Q(z) > 0$ for all large enough $z > 0$, one sees that μ_1, μ_2 must be two real roots that satisfy $0 < \mu_1 < 1 < \beta^{-1} < \mu_2$.

Equation (3.11) can not yet be solved for the expected evolution of the relative capital stock, because of the dependence of the expected evolution of i's relative price (the "forcing term" on the right-hand side) on the expected evolution of the relative capital stock itself, for reasons just discussed. However, one may note that insofar as i's decision problem is locally convex, so that the first-order conditions characterize a locally unique optimal plan, the optimal decision for i's relative price in the event that the price is reset at date t must depend *only* on i's relative capital stock at date t and on the economy's aggregate state. Thus a log-linear approximation to i's pricing rule must take the form

$$\hat{p}_t^*(i) = \hat{p}_t^* - \psi \tilde{k}_t(i), \qquad (3.12)$$

where $\hat{p}_t^*$ depends only on the aggregate state (and so is the same for all i), and ψ is a coefficient to be determined below.

Note that the assumption that the firms that reset prices at date t are drawn with uniform probability from the entire population implies that the average value of $\tilde{k}_t(i)$ over the set of firms that reset prices is zero (just as it is over the entire population of firms). Hence $\hat{p}_t^*$ is also the average relative

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price chosen by firms that reset prices at date t, and the overall rate of price inflation will be given (in our log-linear approximation) by

$$\pi_t = \frac{1-\alpha}{\alpha}\hat{p}_t^*. \tag{3.13}$$

Substitution of this, along with (3.12), into (3.9) then yields

$$E_t\tilde{p}_{t+1}(i) = \alpha\tilde{p}_t(i) - (1-\alpha)\psi\tilde{k}_{t+1}(i). \tag{3.14}$$

Similarly, the optimal quantity of investment in any period t must depend only on i's relative capital stock in that period, its relative price (which matters as a separate argument of the decision rule in the event that the price is *not* reset in period t), and the economy's aggregate state. Thus a log-linear approximation to i's investment rule must imply an expression of the form

$$\tilde{k}_{t+1}(i) = \lambda\tilde{k}_t(i) - \tau\tilde{p}_t(i), \tag{3.15}$$

where the coefficients λ and τ remain to be determined. This in turn implies that

$$\begin{aligned}
E_t\tilde{k}_{t+2}(i) &= \lambda\tilde{k}_{t+1}(i) - \tau E_t\tilde{p}_{t+1}(i) \\
&= [\lambda + (1-\alpha)\tau\psi]\tilde{k}_{t+1}(i) - \alpha\tau\tilde{p}_t(i),
\end{aligned}$$

using (3.14) to substitute for $E_t\tilde{p}_{t+1}(i)$ in the second line. Using this to substitute for $E_t\tilde{k}_{t+2}(i)$ in (3.11), and again using (3.14) to substitute for $E_t\tilde{p}_{t+1}(i)$, we obtain a linear relation that can be solved for $\tilde{k}_{t+1}(i)$ as a linear function of $\tilde{k}_t(i)$ and $\tilde{p}_t(i)$. The conjectured solution (3.15) satisfies this equation, so that the first-order condition (3.11) is satisfied, if and only if the coefficients λ and τ satisfy

$$R(\lambda; \psi) = 0, \tag{3.16}$$

$$(1 - \alpha\beta\lambda)\tau = \Xi\alpha\lambda, \tag{3.17}$$

where

$$R(\lambda; \psi) \equiv (\beta^{-1} - \alpha\lambda)Q(\beta\lambda) + (1-\alpha)\Xi\psi\lambda$$

is a cubic polynomial in λ, with a coefficient on the linear term that depends on the value of the (as yet unknown) coefficient ψ. Condition (3.16) involves

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only λ (given the value of ψ); given a solution for λ, (3.17) then yields a unique solution for τ, as long as $\lambda \neq (\alpha\beta)^{-1}$.[13]

The dynamics of the the relative capital stock given by (3.15), together with (3.14), imply an expected joint evolution of i's relative price and relative capital stock satisfying

$$\left[\begin{array}{c} E_t \tilde{p}_{t+1}(i) \\ \tilde{k}_{t+1}(i) \end{array} \right] = \left[\begin{array}{cc} \alpha + (1-\alpha)\tau\psi & -(1-\alpha)\psi\lambda \\ -\tau & \lambda \end{array} \right] \left[\begin{array}{c} \tilde{p}_t(i) \\ \tilde{k}_t(i) \end{array} \right]. \tag{3.18}$$

This implies convergent dynamics — so that both the means and variances of the distribution of possible future values for i's relative price and relative capital stock remain bounded no matter how in the future one looks, as long as the fluctuations in the average desired relative price $\hat{p}_t^*$ are bounded — if and only if both eigenvalues of the matrix in this equation are inside the unit circle. This stability condition is satisfied if and only if

$$\lambda < \alpha^{-1}, \tag{3.19}$$

$$\lambda < 1 - \tau\psi, \tag{3.20}$$

and

$$\lambda > -1 - \frac{1-\alpha}{1+\alpha}\tau\psi. \tag{3.21}$$

These conditions must be satisfied if the implied dynamics of firm i's capital stock and relative price are to remain forever near enough to the steady-state values around which I have log-linearized the first-order conditions for the solution to the linearized equations to accurately approximate a solution to the exact first-order conditions. Hence the firm's decision problem has a solution that can be characterized using the local methods employed above only if equations (3.16) – (3.17) have a solution (λ, τ) satisfying (3.19) – (3.21). I show below that a unique solution consistent with these bounds exists, in the case of large enough adjustment costs.

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[13]It is obvious from (3.17) that no solution with $\lambda = (\alpha\beta)^{-1}$ is possible, as long as $\Xi > 0$, as we assume here (*i.e.*, there exists some cost of adjusting capital). Even in the case that $\Xi = 0$, such a solution would violate condition (3.19) below, so one can exclude this possibility.

3.2 The Optimal Pricing Rule

I return now to an analysis of the first-order condition for optimal price-setting (3.10). The term that depends on firm i's own intended future behavior is proportional to

$$\sum_{k=0}^{\infty}(\alpha\beta)^k \hat{E}_t^i \tilde{k}_{t+k}(i).$$

It is now possible to write this term as a function of i's relative capital stock at the time of the pricing decision and of the expected evolution of aggregate variables, allowing me to obtain an expression of the form (3.12) for the optimal pricing rule.

Equation (3.15) for the dynamics of the relative capital stock implies that

$$\hat{E}_t^i \tilde{k}_{t+k+1}(i) = \lambda \hat{E}_t^i \tilde{k}_{t+k}(i) - \tau[\tilde{p}_t(i) - E_t \sum_{j=1}^{k} \pi_{t+j}]$$

for each $k \geq 0$, using (3.8) to substitute for $\hat{E}_t^i \tilde{p}_{t+k}(i)$. This can be integrated forward (given that[14] $|\lambda| < (\alpha\beta)^{-1}$), to obtain

$$\sum_{k=0}^{\infty}(\alpha\beta)^k \hat{E}_t^i \tilde{k}_{t+k}(i) = (1 - \alpha\beta\lambda)^{-1}\tilde{k}_t(i)$$

$$-\tau\frac{\alpha\beta}{(1-\alpha\beta)(1-\alpha\beta\lambda)}\left[\tilde{p}_t(i) - \sum_{k=1}^{\infty}(\alpha\beta)^k E_t \pi_{t+k}\right]. \quad (3.22)$$

Substitution of this into (3.10) then yields

$$\phi\hat{p}_t^*(i) = (1-\alpha\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k E_t \hat{s}_{t+k} + \phi\sum_{k=1}^{\infty}(\alpha\beta)^k E_t \pi_{t+k} - (\omega-\nu)\frac{1-\alpha\beta}{1-\alpha\beta\lambda}\tilde{k}_t(i),$$

where

$$\phi \equiv 1 + \omega\theta - (\omega-\nu)\tau\frac{\alpha\beta}{1-\alpha\beta\lambda}. \quad (3.23)$$

The solution to this equation is a pricing rule of the conjectured form (3.12) if and only if the process $\hat{p}_t^*$ satisfies

$$\phi\hat{p}_t^* = (1-\alpha\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k E_t \hat{s}_{t+k} + \phi\sum_{k=1}^{\infty}(\alpha\beta)^k E_t \pi_{t+k}, \quad (3.24)$$

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[14]Note that (3.20) – (3.21) jointly imply that $\lambda > -\alpha^{-1}$. Hence any solution consistent with the stability conditions derived in the previous section must imply convergence of the infinite sum in (3.22).

where $\hat{s}_t$ is defined by (3.6), and the coefficient ψ satisfies

$$\phi\psi = (\omega - \nu)\frac{1 - \alpha\beta}{1 - \alpha\beta\lambda}. \tag{3.25}$$

Note that this last equation can be solved for ψ, given the values of λ and τ; however, the equations given earlier to determine λ and τ depend on the value of ψ. Hence equations (3.16), (3.17), and (3.25) comprise a system of three equations that jointly determine the coefficients λ, τ, and ψ of the firm's optimal decision rules.

This system of equations can be reduced to a single equation for λ in the following manner. First, note that for any conjectured value of $\lambda \neq 0$, (3.16) can be solved for ψ. This defines a function[15]

$$\psi(\lambda) \equiv -\frac{(1 - \alpha\beta\lambda)Q(\beta\lambda)}{(1 - \alpha)\beta\Xi\lambda}.$$

Similarly, (3.17) defines a function[16]

$$\tau(\lambda) \equiv \frac{\alpha\Xi\lambda}{1 - \alpha\beta\lambda}. \tag{3.26}$$

Substituting these functions for ψ and τ in (3.25), one obtains an equation in which λ is the only unknown variable. Multiplying both sides of this equation by $(1 - \alpha)\beta(1 - \alpha\beta\lambda)\Xi\lambda$,[17] one obtains the equation

$$V(\lambda) = 0, \tag{3.27}$$

where $V(\lambda)$ is the quartic polynomial

$$V(\lambda) \equiv [(1 + \omega\theta)(1 - \alpha\beta\lambda)^2 - \alpha^2\beta(\omega - \nu)\Xi\lambda]Q(\beta\lambda) + \beta(1 - \alpha)(1 - \alpha\beta)(\omega - \nu)\Xi\lambda. \tag{3.28}$$

Finally, one can write the inequalities (3.19) – (3.21) as restrictions upon the value of λ alone. One observes from the above discussion that the product

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[15]The function is not defined if $\lambda = 0$. However, since $Q(0) \neq 0$, it is clear from (3.16) that $\lambda \neq 0$, for any economy with some adjustment costs (so that Ξ is finite).

[16]The function is not defined if $\lambda = (\alpha\beta)^{-1}$, but that value of λ would be inconsistent with (3.20) and (3.21) holding jointly, as noted above.

[17]This expression is necessarily non-zero in the case of the kind of solution that we seek, for the reasons noted in the previous two footnotes.

$\tau(\lambda)\psi(\lambda)$ is well-defined for all λ, and equal to $-(\alpha/1-\alpha)\beta^{-1}Q(\beta\lambda)$. Using this function of λ to replace the terms $\tau\psi$ in the previous inequalities, one obtains an equivalent set of three inequalities,

$$\lambda < \alpha^{-1}, \tag{3.29}$$

$$\frac{\alpha}{1+\alpha}\beta^{-1}Q(\beta\lambda) - 1 < \lambda < \frac{\alpha}{1-\alpha}\beta^{-1}Q(\beta\lambda) + 1, \tag{3.30}$$

that λ must satisfy.

I can then summarize my characterization of a firm's optimal pricing and investment behavior as follows.

PROPOSITION 1. Suppose that the firm's decision problem has a solution in which, for any small enough initial log relative capital stock and log relative price of the individual firm, and in the case that the exogenous disturbance q_t and the aggregate variables $\hat{Y}_t$, $\hat{K}_t$, $\hat{\lambda}_t$, and π_t forever satisfy tight enough bounds, both the conditional expectation $\mathrm{E}_t\hat{k}_{t+j}(i)$ and the conditional variance $\mathrm{var}_t\hat{k}_{t+j}(i)$ remain bounded for all j, with bounds that can be made as tight as one likes by choosing sufficiently tight bounds on the initial conditions and the evolution of the aggregate variables.[18] Then the firm's optimal decision rules can be approximated by log-linear rules of the form (3.12) for $\hat{p}_t^*(i)$ in periods when the firm re-optimizes its price and (3.15) for the investment decision $\tilde{k}_{t+1}(i)$ each period. The coefficient λ in (3.15) is a root of the quartic equation (3.27), that satisfies the inequalities (3.29) – (3.30). The coefficient τ in (3.15) is furthermore equal to $\tau(\lambda)$, where the function $\tau(\cdot)$ is defined by (3.17), and the coefficient ψ in (3.12) is equal to $\psi(\lambda)$, where the function $\psi(\cdot)$ is defined by (3.25). Finally, the intercept $\hat{p}_t^*$ in (3.12) is given by (3.24), in which expression the process $\{\hat{s}_t\}$ is defined by (3.6).

This result gives a straightforward algorithm that can be used to solve for the firm's decision rules, in the case that local methods suffice to give an approximate characterization of optimal behavior in the event of small enough

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[18]Note that this is the only condition under which local log-linearizations of the kind used above can suffice to approximately characterize the solution to the firm's problem.

disturbances and a small enough initial departure of the individual firm's situation from that of an average firm. The two decision rules (3.12) and (3.15), together with the law of motion

$$\tilde{p}_t(i) = \tilde{p}_{t-1}(i) - \pi_t$$

for any period t in which i does not re-optimize its price, then allow a complete solution for the evolution of the firm's relative capital stock and relative price, given its initial relative capital stock and relative price and given the evolution of the aggregate variables $\{\hat{Y}_t, \hat{K}_t, \lambda_t, \pi_t, q_t\}$.

3.3   Existence of a Solution

Proposition 1 does not guarantee the existence of a non-explosive solution to the firm's decision problem. The following result, however, shows that at least in the case of large enough adjustment costs there is a solution of the kind characterized in Proposition 1.

PROPOSITION 2. Let household preferences, the production function, the rate of depreciation of capital, and the frequency of price changes all be fixed, but consider alternative specifications of the investment adjustment-cost function $I(\cdot)$, all of which are twice differentiable, increasing, convex, and satisfy $I(1) = \delta$, $I'(1) = 1$. Then for any adjustment-cost function for which the value of $\epsilon_\psi \equiv I''(1) > 0$ is large enough, the polynomial (3.27) has a unique real root λ satisfying (3.29) – (3.30). It follows that the firm decision problem has a solution of the kind described in Proposition 1. Furthermore, in this solution $0 < \lambda < 1$, and τ, ϕ, and ψ are all positive. In the limit as $\Xi \to 0$, $\lambda \to 1$, $\tau \to 0$, $\phi \to 1 + \omega\theta$, and

$$\phi \to \frac{\omega - \nu}{1 + \omega\theta} > 0.$$

This result can be established by considering the way in which the polynomial (3.27) depends on the value of Ξ, which in turn varies inversely with ϵ_ψ. Note that the steady state allocation associated with zero inflation (or

flexible prices) is determined independently of the assumed degree of adjustment costs, and so the values of the parameters $\alpha, \beta, \delta, \nu, \omega, \theta, \rho_y$, and ρ_k are all given, regardless of the variation considered in the value of ϵ_ψ. The coefficient Ξ is then equal to a positive constant divided by ϵ_ψ, so that one may equivalently consider the consequences of varying the value of Ξ while holding fixed the values of the parameters listed above. I am then interested in the roots of $V(\lambda)$ as the value of Ξ approaches zero.

Since the definition (3.28) involves the polynomial $Q(z)$, it is first necessary to consider how this polynomial depends on the value of Ξ. One observes that

$$Q(z) = z^2 - (1 + \beta + c\Xi)z + \beta,$$

where

$$c \equiv \frac{\rho_k}{\rho_y \theta} > 0.$$

One can then write

$$V(\lambda; \Xi) = \bar{V}(\lambda) + V_\Xi(\lambda)\Xi + \frac{1}{2} V_{\Xi\Xi}(\lambda)\Xi^2,$$

where the polynomials

$$\bar{V}(\lambda) \equiv (1 + \omega\theta)(1 - \alpha\beta\lambda)^2 \beta(1 - \lambda)(1 - \beta\lambda),$$

$$V_\Xi(\lambda) \equiv \beta(1 - \alpha\beta\lambda)[1 - \alpha(1 + \beta) + \alpha\beta\lambda](\omega - \nu)\lambda - (1 + \omega\theta)(1 - \alpha\beta\lambda)^2 c\lambda,$$

and $V_{\Xi\Xi}(\lambda)$ are each independent of the value of Ξ.

When $\Xi = 0$, the roots of $V(\lambda)$ are simply the roots of $\bar{V}(\lambda)$, which are easily seen to be $\lambda_1 = 1$, $\lambda_2 = \beta^{-1}$, and $\lambda_3 = \lambda_4 = (\alpha\beta)^{-1}$. By continuity, any real roots in the case of a small enough positive value of Ξ will also have to be close to one of the roots of $\bar{V}(\lambda)$.

It is easily seen that no such root can satisfy the inequalities (3.29) – (3.30), unless it is a root near 1. Because $Q(\beta\lambda_2; 0)Q(1; 0) = 0$, the rightmost term in (3.30) is equal to 1, so that the second inequality is violated when $\lambda = \lambda_2, \Xi = 0$. By continuity, the second inequality of (3.30) will also necessarily be violated by any root near λ_2 in the case of any small enough value of Ξ. Similarly, because $Q(\beta\lambda_3; 0) = Q(\alpha^{-1}; 0) = \alpha^{-1}(\alpha^{-1} - \beta)(1 - \alpha)$, the right-most term is negative, and the second inequality is again violated,

when $\lambda = \lambda_3 = \lambda_4$, $\Xi = 0$. Hence any roots near these will also violate the inequality in the case of any small enough value of Ξ. Thus there can be at most one root of (3.27) that satisfies the inequalities for small positive values of Ξ, and it must be near 1.

Because $\bar{V}'(1) < 0$, $V(\lambda)$ will continue to have a real root $\lambda_1(\Xi)$ near 1 for all small enough values of Ξ, and the implicit function theorem implies that

$$\frac{d\lambda_1}{d\Xi}(0) = -\frac{V_\Xi(1)}{\bar{V}'(1)}.$$

Since

$$
\begin{aligned}
V_\Xi(1) &= \beta(1 - \alpha\beta)(1 - \alpha)(\omega - \nu) - (1 + \omega\theta)(1 - \alpha\beta)^2 c \\
&< (1 - \alpha\beta)^2[(\omega - \nu) - (1 + \omega\theta)c] \\
&= (1 - \alpha\beta)^2[(\omega\rho_y^{-1} - 1)\nu - c] < 0,
\end{aligned}
$$

using the fact that $\rho_y > \omega$ in the final line, and

$$\bar{V}'(1) = -(1 + \omega\theta)\beta(1 - \beta)(1 - \alpha\beta)^2 < 0,$$

it follows that

$$\frac{d\lambda_1}{d\Xi}(0) < 0.$$

Thus there is a real root $0 < \lambda_1 < 1$ for all small enough positive values of Ξ. This root necessarily also satisfies (3.29).

Since $Q(\beta; 0) = 0$, the left-most term of (3.30) is near -1 for all small enough values of Ξ; hence the first inequality of (3.30) is satisfied by the root λ_1 as well. However, both sides of the second inequality are equal to 1 when $\Xi = 0$; thus in order to determine whether the inequality holds when $\Xi > 0$, one must determine the sign of the derivative

$$D \equiv \frac{d}{d\Xi}\left[\lambda_1(\Xi) - \frac{\alpha}{1 - \alpha}\frac{Q(\beta\lambda_1(\Xi);\Xi)}{\beta}\right]$$

at $\Xi = 0$. Since

$$\frac{d}{d\Xi}Q(\beta\lambda_1(\Xi);\Xi) = -\beta(1 - \beta)\frac{d\lambda_1}{d\Xi} - \beta c$$

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Table 1: Numerical parameter values.

β	0.99
ν	0.11
ϕ_h^{-1}	0.75
ω_p	0.33
$(\theta - 1)^{-1}$	0.15
δ	0.12

at $\Xi = 0$, it follows that

$$
\begin{aligned}
D &= \frac{1 - \alpha\beta}{1 - \alpha}\frac{d\lambda_1}{d\Xi} + \frac{\alpha}{1 - \alpha}c \\
&= \frac{(\omega - \nu) - (1 + \omega\theta)c}{(1 - \beta)(1 + \omega\theta)} \\
&= \frac{[(\omega\rho_y^{-1} - 1)\nu - c]}{(1 - \beta)(1 + \omega\theta)} < 0.
\end{aligned}
$$

Thus for all small enough $\Xi > 0$, the second inequality of (3.30) holds as well, and $\lambda = \lambda_1(\Xi)$ is the solution asserted to exist in the proposition.

It then follows from (3.26) that associated with this solution is a positive value of τ, and that $\tau \to 0$ as $\Xi \to 0$. It similarly follows from (3.23) that the associated value of ϕ is positive for all small enough values of Ξ, and that $\phi \to 1 + \omega\theta$ as $\Xi \to 0$. Finally, it follows from these results and (3.25) that the associated value of ψ is positive,[19] and that it approaches the positive limit stated in the proposition as $\Xi \to 0$. Proposition 2 is thus established.

Proposition 2 guarantees that a solution to the firm's optimization problem that can be characterized using the local methods employed above will exist for at least some economies, namely, those in which adjustment costs are large enough. The proposition also implies that in the limit of large adjustment costs, the optimal price-setting rule approaches the one derived in Woodford (2003, chap. 3) under the assumption of an exogenously given capital stock for each firm. Thus the exogenous-capital model represents a

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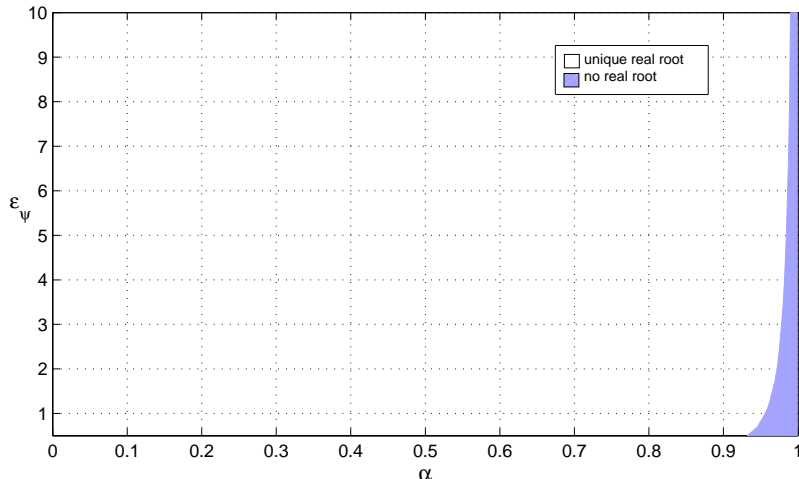[19]Recall that our assumptions require that $\omega > \nu$.

Figure 1: Values of α and ϵ_ψ for which a solution of the kind characterized in Proposition 1 exists.

useful approximation to the equilibrium dynamics in a model with endogenous capital accumulation, if adjustment costs are large enough.

Numerical exploration of the properties of the polynomial (3.27) suggests that adjustment costs do not have to be large in order for the analysis given above to apply. In Figure 1, model parameters are assigned the values given in Table 1,[20] while the values of α and ϵ_ψ are allowed to vary. The figure indicates for which part of the $\alpha - \epsilon_\psi$ plane the polynomial (3.27) has a unique real root satisfying the bounds (3.29) – (3.30). Except in the case of very high values of α ($\alpha > 0.93$, corresponding to an average interval between price changes longer than 3.5 years), a unique real root of this kind exists in the case of any $\epsilon_\psi > 0$. If we suppose that $\epsilon_\psi = 3$ (the calibration used in Woodford, 2004), then a solution exists in the case of any α less than 0.978 (*i.e.*, as long as prices are changed at least once every 11 years, on average).

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[20]These are the same parameter values used in the numerical illustrations in Woodford (2004), which are in turn chosen for comparability with the numerical analyses of related models in Woodford (2003). (The justification for interest in these values is discussed in both of those sources.) Thus, for example, in Figure 1, one sees that if $\alpha = 0.66$, a unique solution exists for all possible values of ϵ_ψ; this explains why it is possible to present solutions for alternative values of ϵ_ψ in Figure 1 of Woodford (2004). In this calibration of the model, periods are understood to correspond to quarters.

In the case of very high values of α, a solution does *not* exist, except in the case of very high values of ϵ_ψ;[21] and when it does not, the solution to the firm's problem cannot be characterized using the local methods employed above.[22] But such high values of α are clearly not empirically realistic, so we need not be concerned with this case.

4 Inflation Dynamics

I now consider the implications of the analysis above for the evolution of the overall inflation rate. I show that the model of price-setting presented above implies the existence of a new-Keynesian Phillips curve of the form (1.1), and then consider the interpretation of empirical estimates of the slope coefficient ξ in this relation.

4.1 A New-Keynesian Phillips Curve

Recall that the average log relative price set by firms that reoptimize at date t is given by (3.24). This equation can be quasi-differenced (after dividing by ϕ[23]) to yield

$$\hat{p}_t^* = (1 - \alpha\beta)\phi^{-1}\hat{s}_t + \alpha\beta E_t \pi_{t+1} + \alpha\beta E_t \hat{p}_{t+1}^*.$$

Then, using (3.13) to substitute for $\hat{p}_t^*$, one obtains a relation of the form (1.1), where

$$\xi \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha\phi}. \tag{4.1}$$

Equation (1.1) is the corrected form of equation (3.17) in Woodford (2003, chap. 5). Together with (3.6), it provides a complete characterization of

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[21]It may appear from the figure that no solution is possible when α exceeds 0.99, but this is because the vertical axis is truncated at $\epsilon_\psi = 10$. If $\alpha = 0.995$, a solution exists in the case of all $\epsilon_\psi > 22.2$; if $\alpha = 0.999$, a solution exists in the case of all $\epsilon_\psi > 88.2$. Thus a solution does always exist in the case of large enough adjustment costs, in accordance with Proposition 2.

[22]This may, for example, be due to a failure of the firm's problem to be locally convex. I do not further investigate the problem here, as it does not appear to arise in cases of practical interest.

[23]It follows from (3.25) that $\phi \neq 0$, given that (as already discussed) $\lambda \neq (\alpha\beta)^{-1}$.

the equilibrium dynamics of inflation, given the evolution of $\hat{Y}_t$, $\hat{K}_t$, and $\hat{\lambda}_t$. This pair of equations can be thought of as constituting the "aggregate supply block" of the model with endogenous capital. They generalize the aggregate-supply equation of the constant-capital model (expounded in Woodford, 2003, chap. 3) to take account of the effects of changes in the capital stock on real marginal cost, and hence on the short-run tradeoff between inflation and output.

In the constant-capital model, (3.6) (after using (2.10) to substitute for $\hat{\lambda}_t$) reduces to

$$\hat{s}_t = \omega(\hat{Y}_t - q_t) + \sigma^{-1}(\hat{Y}_t - g_t),$$

which can be equivalently written as

$$\hat{s}_t = (\omega + \sigma^{-1})\tilde{Y}_t, \tag{4.2}$$

where $\tilde{Y}_t$ is the "output gap," defined as the (log) difference between actual and flexible-price equilibrium output. Substituting this relation into (1.1), one obtains the familiar output-gap formulation of the new-Keynesian Phillips curve,

$$\pi_t = \kappa\tilde{Y}_t + \beta E_t \pi_{t+1}, \tag{4.3}$$

where $\kappa \equiv (\omega + \sigma^{-1})\xi > 0$.

In the model with endogenous (and firm-specific) capital, instead, (4.2) takes the more general form

$$\hat{s}_t = (\omega + \sigma^{-1})\tilde{Y}_t - \sigma^{-1}\tilde{I}_t, \tag{4.4}$$

where $\tilde{I}_t$ indicates the gap between actual investment (specifically, the value of $\hat{I}_t$) and its flexible-price equilibrium level.[24] If one substitutes this relation instead into (1.1), one obtains a generalization of (4.3),

$$\pi_t = \kappa\tilde{Y}_t - \kappa_I\tilde{I}_t + \beta E_t \pi_{t+1},$$

where κ is defined as before, but now $\kappa_I \equiv \sigma^{-1}\xi > 0$. Thus while (1.1) continues to apply, the relation between inflation and real activity is no longer

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[24]See Woodford (2004) for further discussion of the definition of this and related "gap" variables in this model.

as simple as (4.3). This is a further reason (in addition to the lack of simple empirical measures of the flexible-price equilibrium level of output) why it has been appropriate for the empirical literature to focus more on estimation of the inflation equation (1.1) than of the corresponding aggregate-supply relation.

As with equation (3.17) in Woodford (2003, chap. 5), equation (1.1) implies that one can solve for the inflation rate as a function of current and expected future real marginal cost, resulting in a relation of the form

$$\pi_t = \sum_{j=0}^{\infty} \Psi_j E_t \hat{s}_{t+j}. \tag{4.5}$$

The correct formula for these coefficients is given by

$$\Psi_j = \xi \beta^j,$$

just as in the model with constant capital discussed in Woodford (2003, chap. 3). Hence the coefficients do not decay as rapidly with increasing j as is shown in Figure 5.6 of Woodford (2003), in the case of finite adjustment costs. Nor do the coefficients ever change sign with increasing j, as occurs in the figure. In the case that $\xi > 0$ (as implied by the calibrated parameter values proposed below), an increase in the expected future level of real marginal costs unambiguously requires that inflation increase; and the degree to which inflation determination is forward-looking is even greater than is indicated by the figure in Woodford (2003).

4.2 The Case of a Rental Market for Capital

I now briefly compare the results obtained above to those that would be obtained under the assumption of a competitive rental market for capital services.[25] In the literature, when models of staggered pricing have allowed for endogenous capital accumulation (as, for example, in Yun, 1996, or Chari *et al.*, 2000), they have typically assumed that firms purchase capital services on a competitive rental market, rather than accumulating firm-specific

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[25]Sveen and Weinke (2004b) similarly compare the consequences of these two assumptions, but instead focussing on the differences that result for the implied impulse responses to disturbances in a complete DSGE model.

capital as in the model above. This alternative assumption is of considerable convenience, since it allows price-setting decisions to be analyzed separately from the decision to accumulate capital.[26] However, while the assumption of an economy-wide rental market for capital is purely a convenience in the case of standard real business-cycle models (*i.e.*, one-sector models with a competitive goods market), it is no longer innocuous in a model where firms are price-setters, and so must consider the consequences for their profits of setting a price different from that of their competitors. As we shall see, alternative assumptions about the way in which capital services can be obtained (with a production technology that is otherwise the same) lead to different conclusions regarding aggregate dynamics. In particular, the predicted slope of the Phillips-curve tradeoff can be affected to an extent that is quantitatively significant.

I shall consider two versions of a model with a competitive rental market for capital services. In each case, the production technology and the technology of capital accumulation are as described in section 1, except that now capital goods are either accumulated by households and rented to the firms that produce the goods that are used for consumption and investment, or they are accumulated by a special set of firms that accumulate capital and then rent capital services to the goods-producing firms. (Our equilibrium relations will be the same, whether capital is accumulated by households or by a special set of firms.) There is assumed to be a competitive market for capital services each period, with rental rate ρ_t in period t. (Note that this rental rate is no longer indexed by the firm that uses the capital.)

It follows that for each household or firm i that accumulates capital, its holdings of capital $\{k_t(i)\}$ must evolve in accordance with the first-order condition (2.3), except that now the firm-specific shadow value $\rho_{t+1}(i)$ is replaced by the market rental rate ρ_{t+1}, with the same value for all i. Log-linearization of this condition again leads to a relation of the form (2.4) for each i, but with $\hat{\rho}_{t+1}(i)$ replaced simply by $\hat{\rho}_{t+1}$. Assuming that one starts from a symmetric distribution of capital $k_0(i) = K_0$ for all i, one will similarly have a common capital stock $k_t(i) = K_t$ in all subsequent periods, since each

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[26]The same assumption was used, for example, in the DSGE model with oligopolistic pricing of Rotemberg and Woodford (1992).

household or firm solves an identical optimization problem. The aggregate capital stock will then also evolve in accordance with (2.3) or, up to a log-linear approximation, in accordance with (2.4).

An optimal demand for capital services by a goods-producing firm i [now not to be confused with a firm i that accumulates capital!] again requires that the firm's output/capital ratio satisfy (2.5), though (2.5) is now a first-order condition for a firm that takes as given the cost of capital services ρ_t, rather than a definition of the shadow value of additional capital services; and $\rho_t(i)$ must now be replaced by the common rental rate ρ_t for all i. There are two possible assumptions that may be made regarding labor inputs. In the literature, when a rental market for capital services is assumed, it is often also assumed that all sectors hire the same kind of labor, and that there is a single economy-wide labor market as well; this is the case of "homogeneous factor markets" treated in Woodford (2003, chap. 3).[27] In this case, every firm i faces a common wage, so that $w_t(i) = w_t$. It then follows from (2.5) that each firm i will choose a common output/capital ratio; firms with higher demand for their products (because of lower prices) will choose to use a proportionately higher quantity of capital services, and a proportionately higher quantity of labor as well. It then follows from (3.1) that the marginal cost of output supply will be the same for all firms i, and independent of the quantity produced by any firm, so that $s_t(i) = s_t$ for all i, where the common real marginal cost s_t is an increasing function of both ρ_t and w_t. Equation (3.5) then reduces simply to

$$\hat{s}_t(i) = \hat{s}_t.$$

In this case, frequently assumed in previous derivations of the new-Keynesian

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[27]It is the case assumed in the derivation of a new-Keynesian Phillips curve in Gali and Gertler (1999), and in the baseline case considered in Sbordone (1998, 2002). Note that a single economy-wide labor market is also assumed in the analysis of the consequences of an exogenous firm-specific capital stock in Sbordone (1998, 2002) and in Gali, Gertler and Lopez-Salido (2001). For this reason, the formula for $\xi(\alpha)$ presented by those authors for the case of firm-specific capital differs from the one derived in Woodford (2003, chap. 3) under the assumption of industry-specific labor markets. Eichenbaum and Fisher (2004) also assume an economy-wide labor market even in their model with firm-specific capital, though in their case each firm's capital stock is endogenous as in the model developed here.

Phillips curve, (3.7) implies that the optimal relative price that should be chosen by a firm that resets its price at date t is given by

$$\hat{p}_t^*(i) = (1 - \alpha\beta) \sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i \left[\hat{s}_{t+k} + \sum_{j=1}^{k} \pi_{t+j} \right] \qquad (4.6)$$

instead of (3.10). In this case, the quantities inside the brackets are not firm-specific, and there is no need to distinguish between the conditional expectations $\hat{E}_t^i[\cdot]$ and $E_t[\cdot]$. Nor is there any need to solve for the dynamics of a firm's relative capital stock in order to evaluate the right-hand side of (4.6). The right-hand side of (4.6) is the same for all i, and thus gives the value of $\hat{p}_t^*$. Equation (4.6) then leads directly to an inflation equation of the form (1.1), with

$$\xi_h \equiv \frac{(1 - \alpha)(1 - \alpha\beta)}{\alpha} > 0. \qquad (4.7)$$

Alternatively, we may assume the existence of a sector-specific labor market for each sector, as in the model developed in this paper for the case of firm-specific capital, or the model of "specific factor markets" treated in Woodford (2003, chap. 3). In this case, the real wage for the type of labor hired by firm i is given by a sector-specific labor supply equation (2.6). Substituting this into (2.5) and log-linearizing, we again obtain equilibrium relation (2.7) for each firm i, except that $\hat{\rho}_t(i)$ must now be replaced by the common rental rate $\hat{\rho}_t$. Because ρ_t is now the same for all firms, this conditional for cost-minimizing production by firm i implies that the firm's relative capital stock will be a monotonic function of its relative sales, so that

$$\rho_k(\hat{k}_t(i) - \hat{K}_t) = \rho_y(\hat{y}_t(i) - \hat{Y}_t) \qquad (4.8)$$

for all i at any date.

The marginal cost of production of each firm i is again given by (3.4), but we can now use (4.8) to substitute for the firm's relative demand for capital as a function of its relative sales. Then, again using (2.2) to substitute for the relative sales of firm i, one obtains

$$\hat{s}_t(i) = \hat{s}_t - \chi\theta\tilde{p}_t(i) \qquad (4.9)$$

instead of (3.5), where

$$\chi \equiv \frac{\omega\rho_k - (\omega - \nu)\rho_y}{\rho_k} = \frac{\nu\omega_p}{\rho_k(\phi_h - 1)} > 0.$$

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Note that there is no longer any dependence on the firm's relative capital stock (which is no longer a state variable for the firm's optimization problem).

Once again substituting (4.9) for $s_{t+k}(i)$ and (3.8) for $\hat{E}_t^i \tilde{p}_{t+k}(i)$ in (3.7), one now obtains

$$(1 + \chi\theta)\hat{p}_t^*(i) = (1 - \alpha\beta) \sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i \left[\hat{s}_{t+k} + (1 + \chi\theta) \sum_{j=1}^{k} \pi_{t+j} \right] \qquad (4.10)$$

for the optimal relative price that should be chosen by a firm that resets its price at date t. One can again replace the conditional expectation $\hat{E}_t^i[\cdot]$ by $E_t[\cdot]$, and one again observes that $\hat{p}_t^*(i)$ is the same for all i, so that one can replace $\hat{p}_t^*(i)$ by $\hat{p}_t^*$. Relation (4.10) is then of the same form as relation (3.24) for the model above with endogenous but firm-specific capital, but with the coefficient ϕ in the earlier equation here replaced by $1 + \chi\theta$. One again obtains a pricing relation of the form (1.1), but with elasticity

$$\xi_r \equiv \frac{(1 - \alpha)(1 - \alpha\beta)}{\alpha(1 + \chi\theta)} > 0. \qquad (4.11)$$

Thus each model leads to a Phillips-curve relation of the same form (1.1), except that in each case the elasticity $\xi > 0$ is a different function of underlying model parameters. The quantitative difference made by the alternative assumptions can be illustrated through a numerical example. Let us again assume the parameter values given in Table 1, and furthermore now specify that $\epsilon_\psi = 3$, as assumed in Woodford (2004). Figure 2 then plots the value of ξ corresponding to any given frequency of price change (indicated by the value of α on the horizontal axis), under each of four possible assumptions. The function $\xi_h(\alpha)$ defined in (4.7) indicates how the elasticity ξ in (1.1) varies with α in the case of homogeneous factor markets. The function $\xi_r(\alpha)$ defined in (4.11) applies instead in the case of industry-specific labor but an economy-wide rental market for capital. The function $\xi_f(\alpha)$ defined in (4.1) applies instead in the case of industry-specific labor and firm-specific capital.[28] And finally, the function $\xi_c(\alpha)$ is the corresponding relation derived in Woodford (2003, chap. 3) for the case of the model with industry-specific

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[28]As shown in Figure 1, the function ξ_f is only defined for values of α lower than a critical value on the order of 0.978. The other functions are defined for all values of α between 0 and 1.
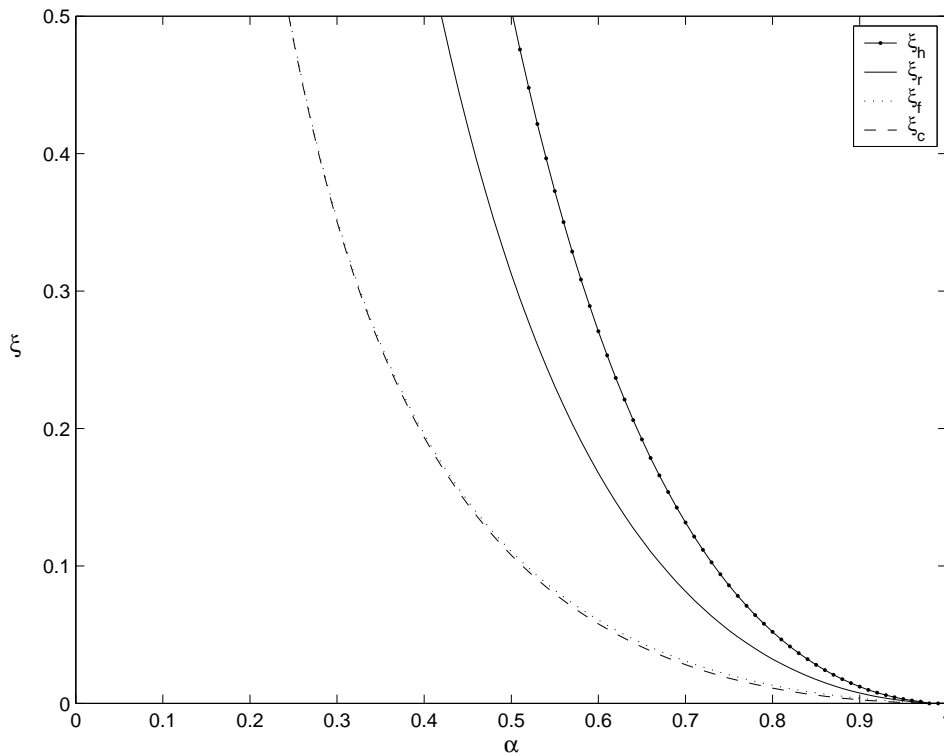
Figure 2: The relation between ξ and α under four alternative assumptions about factor markets.

labor and a constant quantity of firm-specific capital.[29] The function $\xi_c(\alpha)$ corresponds to the limit of $\xi_f(\alpha)$ as ϵ_ψ is made unboundedly large; it follows from Proposition 1 that this is given by

$$\xi_c \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha(1+\omega\theta)} > 0.$$

We see from the figure that any given value of α (in the range for which all four functions are defined), the model with homogeneous factor markets implies the highest value of ξ, as in this case the model possesses the fewest sources of "real rigidities" in the sense of Ball and Romer (1990). The fact that an increase in demand in one part of the economy bids up the price of factor inputs throughout the economy creates a source of "strategic substitutability" between the pricing decisions in different sectors of the economy

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[29]This is called the model with "specific factor markets" in Woodford (2003, chap. 3).

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(the fact that others keep their prices low increases your marginal cost of production, and so gives you a reason for *higher* prices, rather than lower ones); this speeds up the rate of adjustment of the aggregate price index to changes in demand conditions.[30] There are greater real rigidities, and hence a flatter Phillips curve, in the case of industry-specific labor markets, even if we continue to assume an economy-wide rental market for capital services; for in this case, an increase in demand in one part of the economy still bids up the price of capital services throughout the economy, but does not similarly affect wages in other sectors. There are still greater real rigidities, and a still flatter Phillips curve, if we assume firm-specific investment, because in this case an increase in demand in one part of the economy which increases the shadow value of capital there has no immediate effect on the shadow cost of capital services in other parts of the economy.

Real rigidities are the greatest if we assume, as in the model with "specific factor markets" in Woodford (2003, chap. 3), that the capital stock of each firm is exogenously given, and hence *never* affected by differential shadow values of capital in different sectors. In the model with endogenous firm-specific capital developed here, a sustained higher shadow value of capital in part of the economy will eventually raise the shadow value of capital services everywhere, as a result of differential rates of investment in the sectors with differing shadow values of capital. Thus capital is still reallocated among sectors in response to rate-of-return differentials, albeit with a delay, as long as investment adjustment costs are not too large. However, the figure shows that in our calibrated example, an empirically realistic level of adjustment costs result in a value of ξ that is quite close to what would be implied by the exogenous-capital model with firm-specific capital (though slightly larger), while it is considerably lower than would be implied by the assumption of instantaneous reallocation of capital across sectors so as to equalize the shadow value of capital services. Thus the implicit assumption of an exogenously evolving capital stock in derivations of the Phillips curve for

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[30]See Woodford, 2003, chap. 3, for further discussion of why the Phillips curve is relatively steep in this case, building upon the seminal treatment by Kimball (1995). The discussion there, conducted under the assumption of an exogenously given capital stock, still gives the essential insight into why the specificity of factor markets matters.

models with firm-specific capital by authors such as Sbordone (1998) appears not to have been a source of any great inaccuracy.[31] The endogeneity of the capital stock is instead of greater significance for predictions about the equilibrium responses of inflation or output to aggregate disturbances, as shown in Woodford (2004), because of the effects of the endogenous rate of investment on the evolution of real marginal cost, as indicated by equation (4.4).

4.3 Additional Sources of Real Rigidities

Even the model with firm-specific capital developed above still abstracts from a number of possible sources of real rigidities. Here I briefly consider the effects of two generalizations that are discussed in more detail (though in the context of a model with exogenous capital) in Woodford (2003, chap. 3, sec. 1.4).

First, I shall now suppose that each differentiated good is produced using not only labor and capital, but also intermediate inputs produced by other industries. As in Rotemberg and Woodford (1995), I assume a production function of the form

$$y_t(i) = \min \left[\frac{k_t(i)f(A_t h_t(i)/k_t(i))}{1 - s_m}, \ \frac{m_t(i)}{s_m} \right],$$

generalizing (2.1), where $f(\cdot)$ has the same properties as before, $m_t(i)$ denotes the quantity of materials inputs used by firm i in period t, and $0 \leq s_m < 1$ is a parameter of the production technology that can be identified, for purposes of calibration, with the share of materials costs in the value of gross output. The materials inputs are measured in units of the composite good.

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[31]Coenen and Levin (2004) also discuss the role of firm-specific capital in increasing real rigidities, in the context of a model with Taylor-style fixed-period price commitments, which allows separate econometric identification of the length of time between price changes, on the one hand, and the elasticity of a firm's desired relative price with respect to aggregate output, on the other. They are concerned with whether the estimated value of the latter elasticity can be reconciled with the microfoundations of the firm's pricing decision, and argue that allowing for firm-specific capital is important in doing so. Like Sbordone (1998) and Gali, Gertler and Lopez-Salido (2001), they assume that each firm's capital stock is fixed in analyzing this issue.

The shadow value to firm i of an additional unit of capital is then given by

$$\rho_t(i) = \frac{w_t(i)}{A_t} f^{-1}((1 - s_m)y_t(i)/k_t(i))[\phi((1 - s_m)y_t(i)/k_t(i)) - 1],$$

generalizing (2.5), where $\phi(\cdot)$ is the same function as before. But because the log deviation of $(1 - s_m)y_t(i)/k_t(i)$ from its steady-state value is equal to the log deviation of $y_t(i)/k_t(i)$ from its steady-state value, the log-linear relation (2.7) continues to apply, regardless of the size of the materials share.

With intermediate inputs, the real marginal cost of production can be written as

$$s_t(i) = (1 - s_m)s_t^{VA}(i) + s_m, \tag{4.12}$$

where $s_t^{VA}(i)$ is the real marginal cost of producing a unit of 'real value added', by which I mean the homogeneous-degree-one aggregate of primary factors of production given by $k_t(i)f(A_t h_t(i)/k_t(i))$. Equation (3.2) furthermore takes the more general form

$$s_t^{VA}(i) = \frac{v_h(f^{-1}((1 - s_m)y_t(i)/k_t(i))k_t(i)/A_t; \xi_t)}{\lambda_t A_t f'(f^{-1}((1 - s_m)y_t(i)/k_t(i)))}.$$

Substituting this into (4.12) and log-linearizing, I obtain

$$\hat{s}_t(i) = (1 - \mu s_m)[\omega(\hat{y}_t(i) - \hat{k}_t(i) - q_t) + \nu \hat{k}_t(i) - \hat{\lambda}_t], \tag{4.13}$$

generalizing (3.3), where $\mu \equiv \theta/(\theta - 1) > 1$ is the steady-state markup (ratio of price to marginal cost).[32] The reduced elasticity of real marginal cost with respect to the firm's level of production when s_m is positive (but less than μ^{-1}) indicates greater real rigidities.

Second, I shall suppose that substitution possibilities among the differentiated goods are no longer necessarily described by the familiar Dixit-Stiglitz aggregator that leads to the constant-elasticity demand function (2.2) for individual goods. If I instead assume only that the aggregator belongs to the more general family of homogeneous-degree-one functions considered by

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[32] In the case of the model with generalized preferences introduced in the next paragraph, this relation still applies, but $\theta > 1$ indicates the steady-state elasticity of substitution among differentiated goods, rather than a coefficient of the Dixit-Stiglitz aggregator.

Kimball (1995), the elasticity of demand varies with the relative price of (and hence the relative demand for) individual good i. The relative demand for an individual good is again a decreasing function of the relative price,[33] but the function need not be a constant-elasticity function, as in (2.2).

As a result, the desired markup of the supplier's price over the marginal cost of supply will no longer be a constant $\mu > 1$, but rather a function $\mu(y_t(i)/Y_t)$ of the relative output of the good, where Y_t is aggregate output, defined using the Kimball aggregator. To a log-linear approximation, the deviation of the log desired markup from its steady-state level (that I shall again call μ) is equal to $\epsilon_\mu \tilde{y}_t(i)$, where ϵ_μ is the elasticity of the function $\mu(\cdot)$, evaluated at the steady state (*i.e.*, at a relative output of 1), and $\tilde{y}_t(i)$ is the log relative output. The demand function can again be log-linearized to yield

$$\tilde{y}_t(i) = -\theta \tilde{p}_t(i),$$

where $\theta > 1$ is now the steady-state elasticity of demand (and not necessarily also the elasticity near a relative price other than 1). One can then show that the first-order condition for optimal price-setting under Calvo staggering of price changes takes the form

$$\sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i [\tilde{p}_{t+k}(i) - (1 + \theta\epsilon_\mu)^{-1} \hat{s}_{t+k}(i)] = 0, \qquad (4.14)$$

generalizing (3.7), just as in Woodford (2003, chap. 3). The only difference here is that real marginal cost will depend on the firm's endogenous, firm-specific capital stock in the way treated above. One observes directly from (4.14) that a value $\epsilon_\mu > 0$ will increase the degree of real rigidities, by reducing the sensitivity of the desired relative price to variations in real marginal cost, and hence to variations in the firm's output.

Substituting (4.13) into (4.14), I now obtain

$$\Gamma_1 \hat{p}_t^*(i) = (1 - \alpha\beta) \sum_{k=0}^{\infty} (\alpha\beta)^k \hat{E}_t^i \left[\hat{s}_{t+k} + \Gamma_1 \sum_{j=1}^{k} \pi_{t+j} - \Gamma_2 \tilde{k}_{t+k}(i) \right], \qquad (4.15)$$

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[33]As in the model developed above, it is assumed that both household preferences and the production technology of firms depends only on the quantity purchased of the composite good defined by this aggregator. Hence the purchases of each buyer, for whatever purpose, will be distributed across differentiated goods in the same proportions.

generalizing (3.10), where

$$\Gamma_1 \equiv 1 + \theta\epsilon_\mu + (1 - \mu s_m)\omega\theta, \qquad \Gamma_2 \equiv (1 - \mu s_m)(\omega - \nu).$$

Here only the expressions for Γ_1 and Γ_2 have become more complex. One can then show, using the same reasoning as above, that the solution to the firm's optimization problem is characterized by Proposition 1, *except* that (3.23) must be replaced by

$$\phi \equiv \Gamma_1 - \Gamma_2\tau\frac{\alpha\beta}{1 - \alpha\beta\lambda}, \tag{4.16}$$

and (3.28) must be replaced by

$$V(\lambda) \equiv [\Gamma_1(1 - \alpha\beta\lambda)^2 - \alpha^2\beta\Gamma_2\Xi\lambda]Q(\beta\lambda) + \beta(1 - \alpha)(1 - \alpha\beta)\Gamma_2\Xi\lambda. \tag{4.17}$$

One can similarly obtain once again an aggregate-supply relation of the form (1.1), where the elasticity ξ is defined by (4.1), but now using the generalized definition (4.16) of ϕ. Alternatively, one can write the aggregate-supply relation as

$$\pi_t = \xi\hat{s}_t^{VA} + \beta E_t\pi_{t+1}, \tag{4.18}$$

in which case

$$\xi \equiv \frac{(1 - \mu s_m)(1 - \alpha)(1 - \alpha\beta)}{\alpha\phi}, \tag{4.19}$$

where ϕ is defined by (4.16), using the fact that

$$\hat{s}_t = (1 - \mu s_m)\hat{s}_t^{VA},$$

from a log-linearization of (4.12). The alternative form (4.18) is actually the one that is estimated in the literature, since (under the assumption of a Cobb-Douglas production function for 'value added') it is s_t^{VA} rather than s_t that is proportional to real unit labor cost (the proxy for "marginal cost" that is used in empirical work).

The additional sources of real rigidities affect the value of ξ associated with a given average frequency of price adjustment, as shown in Figure 3. As in Woodford (2003, chap. 3, sec. 1.4), I shall consider the consequences of an intermediate input share such that $\mu s_m = 0.6$, and a non-constant elasticity of substitution among differentiated goods such that $\theta\epsilon_\mu = 1$. The figure
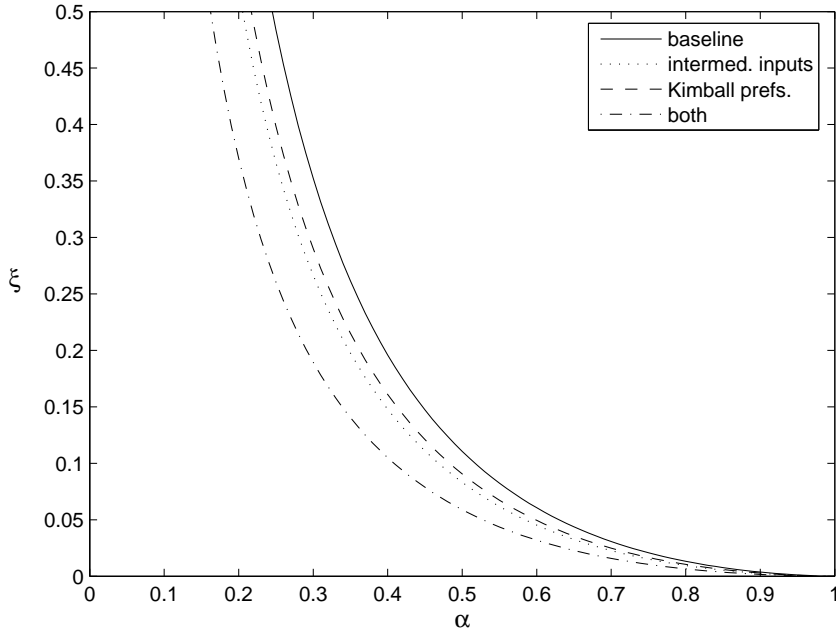
Figure 3: The relation between ξ and α with additional sources of real rigidities.

plots the functional relation $\xi(\alpha)$ defined by (4.1) for each of four possible combinations of parameter values: the baseline case, in which $s_m = 0, \epsilon_\mu = 0$; a case with intermediate inputs, in which $\mu s_m = 0.6$, though again $\epsilon_\mu = 0$; a case with Kimball preferences, in which $s_m = 0$ but $\theta \epsilon_\mu = 1$; and finally, a case with both additional sources of real rigidities, in which $\mu s_m = 0.6, \theta \epsilon_\mu = 1$. In all four cases, it is assumed that labor markets are industry-specific, capital is endogenous and firm-specific, and the numerical parameters other than those just listed are as in the case with firm-specific capital plotted in Figure 2. (The function $\xi(\alpha)$ in the baseline case here corresponds to the function $\xi_f(\alpha)$ in Figure 2.)

One observes that for each of the values of α considered, either intermediate inputs or Kimball preferences with $\epsilon_\mu > 0$ lowers the implied value of ξ, and if both departures from the baseline model are considered simultaneously, the implied value of ξ is still lower. Hence allowance for either of these empirically plausible additional sources of real rigidities further reduces

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the implied slope of the Phillips curve, without any change in the assumed frequency of price changes. The results obtained here are quite similar to those obtained in Woodford (2003, chap. 3) for the case of a model in which each firm's capital stock is given exogenously (see Table 3.1 there).

4.4   Consequences for Estimates of the Frequency of Price Adjustment

Because alternative assumptions about the specificity of factor markets affect the location of the curve $\xi(\alpha)$, as shown in Figure 2, it follows that the consequences of an estimate of ξ for the frequency of price adjustment are correspondingly different in the different cases. (One should note estimation of the aggregate-supply relation (1.1) only allows an estimate of the elasticity ξ, and provides no direct evidence regarding the frequency of price adjustment, nor any way of testing which of the alternative possible assumptions about the specificity of factor markets is the correct one.) An assumption of specific factor markets — either that labor markets are industry-specific, or that capital is firm-specific — increases the degree of real rigidities, relative to an assumption of an economy-wide market for the services of that factor, and so lowers the value of ξ corresponding to any given value of α. Conversely, it follows that the value of α required to explain any given value of ξ — and hence the value of α implied by any given estimate of ξ — is *lower* the greater the degree of specificity of factors. Hence a given degree of sluggishness in the adjustment of the overall price index to changes in aggregate conditions can be reconciled with a greater degree of firm-level flexibility of prices in the case that one assumes more specific factors of production.

This is illustrated by the calculations reported in Table 2. The numerical parameter values given in Table 1 are again assumed, and in addition (in the case of the model with firm-specific capital) it is assumed that $\epsilon_\psi = 3$, as in the baseline case considered in Woodford (2004). The first panel of the table then indicates the value of α that would be implied by a given estimate of the elasticity ξ, under each of three different possible assumptions about factor markets: homogeneous factor markets; industry-specific labor markets but a rental market for capital services; and industry-specific labor

Table 2: Interpretation of the estimated value of ξ under alternative assumptions about factor markets.

	Implied values of α		
ξ	homo. fact.	rental mkt	firm-spec.
0.05	.804	.757	.630
0.04	.823	.779	.663
0.03	.845	.806	.703
0.02	.872	.840	.754

	Implied values of T		
ξ	homo. fact.	rental mkt	firm-spec.
0.05	4.57	3.59	2.16
0.04	5.13	4.01	2.43
0.03	5.94	4.65	2.84
0.02	7.32	5.71	3.55

markets together with endogenous, firm-specific capital. The range of values considered for ξ in the table corresponds to the range of values found in empirical estimates of the new-Keynesian Phillips curve (1.1) for the US.[34]

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[34]For example, Gali and Gertler (1999) report an estimate of 0.023 when they estimate the "reduced form" equation (1.1) using US data. When they use an alternative GMM estimation approach that yields estimates of α (under the assumption of homogeneous factor markets) rather than of ξ, the values of ξ implied by the reported estimates (that vary depending on the sample and moment conditions used) are mostly in the range of 0.02 to 0.04. (It should noted that when Gali and Gertler report estimates of α, they are really only estimating a nonlinear transformation of the elasticity ξ that would correspond to α under the assumption of homogeneous factor markets, and do not attempt any test of the homogeneous-factor assumption.) Gali, Gertler and Lopez-Salido (2001) similarly report estimates of α using US data that imply values of ξ equal to 0.03 or 0.04. Sbordone (2002) obtains an estimate of 0.055 for US data using a different estimation technique, while Sbordone (2004) obtains an estimate of 0.025 for US data using yet another approach.

The second panel of the table shows the implied values of

$$T \equiv \frac{-1}{\log \alpha},$$

the average time (in quarters) that a price remains fixed,[35] for each of the same possible estimates of ξ under each of the same three possible assumptions about factor markets.

One observes that the assumption made regarding factor markets has a substantial effect on the implied frequency of price adjustment, given any estimate of the slope of the Phillips curve ξ. If, for example, one estimates a slope $\xi = 0.02$ — and some estimates using US data are this low, though most reported estimates have been at least somewhat larger — then under the assumption of homogeneous factor markets (and the other parametric assumptions in Table 1), one would conclude that the estimate implied an average time between price changes of over 7 quarters. This is implausibly long, given microeconomic evidence on the frequency of price changes, so that one might well conclude that the model cannot account for the observed facts about price adjustment, no matter how well it might fit the joint evolution of overall inflation and average marginal cost. Assuming instead that labor markets are industry-specific, however, would reduce the implied average time between price changes to less than 6 quarters, even if one continues to assume a rental market for capital services. And allowing for firm-specific capital would further reduce the implied average time between price changes, to only three and a half quarters. This is no longer so implausible, given the evidence in surveys such as that of Blinder *et al.* (1998) that many prices in the US are changed only once a year or less.

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[35]In the literature, estimates of α are often converted into estimates of the average time between price changes using the alternative formula $T = 1/(1 - \alpha)$. This latter formula is correct if one takes the discrete-time model (in which all prices change, if they change at all, at a single time each quarter) literally; but it has the unappealing feature that no matter how flexible prices may be (and how steep the estimated Phillips curve may be as a result), T must always equal at least 3 months. The formula here assumes instead that there is a constant hazard rate ρ in continuous time for price changes, and that an estimate of α is an estimate of $e^{-\rho}$. This means that if one estimates a steep enough Phillips curve, and hence infers a value of α close enough to 1, the inferred value of T may be arbitrarily small.

My finding that an assumption that capital is firm-specific reduces the average time between price changes implied by estimates of the aggregate-supply relation (1.1) confirms the previous results of Sbordone (1998) and Gali, Gertler and Lopez-Salido (2001), obtained under an implicit assumption that each firm's capital stock is constant, or evolves exogenously. Figure 2 shows that the assumption of a constant (or exogenous) capital stock would imply even slightly greater real rigidities than exist in the case of an endogenous but firm-specific capital stock; but the numerical error resulting from that simplifying assumption is not great, at least if investment adjustment costs are of the size assumed here.[36] For example, in the case that $\xi = 0.02$, under the assumptions of exogenous capital and industry-specific labor markets, the value of α would be 0.740 and the value of T would be 3.28, rather than the values shown in the third column of Table 2. Allowance for a realistic degree of endogenous adjustment of each firm's capital stock does not dramatically change those conclusions.

The implied average times between price adjustments shown in Table 2 may still seem a bit too long to square with microeconomic evidence, even in the case of firm-specific capital, especially if ξ is estimated to take a value between 0.2 and 0.3. (Blinder *et al.,* 1998, report a median time between price changes of 3 quarters; but Bils and Klenow, 2004, instead report a median of less than 2 quarters.) However, a value of ξ of this magnitude can be reconciled with even greater frequencies of adjustment of individual prices, if additional empirically plausible sources of real rigidities are taken into account.

Table 3 shows the values of α and T implied by alternative estimates of ξ, under alternative assumptions about the importance of intermediate inputs and the degree to which the aggregator that defines the composite good differs from the Dixit-Stiglitz form. All numerical parameters except s_m and ϵ_μ take the same values as in Table 2, and in each case it is now assumed that labor markets are industry-specific and capital is firm-specific. (Thus

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[36]As shown in Woodford (2004), adjustment costs of roughly this size are needed to explain the observed size of output response to an identified monetary policy shock, in the context of a simple new-Keynesian model that incorporates the model of investment and price-setting decisions developed here.

Table 3: Interpretation of the estimated value of ξ under alternative assumptions about input/output structure and substitutability of differentiated goods.

	Implied values of α			
ξ	baseline	intermed. inputs	Kimball	both
0.05	.630	.584	.598	.528
0.04	.663	.619	.633	.564
0.03	.703	.662	.674	.609
0.02	.754	.716	.728	.669

	Implied values of T			
ξ	baseline	intermed. inputs	Kimball	both
0.05	2.16	1.86	1.95	1.56
0.04	2.43	2.09	2.18	1.75
0.03	2.84	2.42	2.54	2.02
0.02	3.55	3.00	3.15	2.48

the "baseline" case in Table 3 corresponds to the "firm-specific" column of Table 2.) The values assumed for s_m and ϵ_μ in the alternative cases are the same as in Figure 3.

One observes the average time between price changes implied by any given estimate of ξ falls in the case that one assumes either of the additional sources of real rigidities, and falls by even more if one assumes both. Making corrections of both type that remain within the range of empirically plausible parameter values, one finds that a Phillips-curve slope of only 0.02 can be consistent with a average period between price changes that is less than 2.5 quarters. A Phillips-curve slope of 0.04 can instead be consistent with an average period between price changes that is well below 2 quarters. Since point estimates of this magnitude for ξ are obtained in a number of studies (and it is within the 95 percent confidence interval in an even larger number of

cases), one cannot say that estimates of ξ are too small to be consistent with microeconomic evidence regarding the frequency with which prices change.

I conclude that there is no necessary conflict between the parameter values that are required to explain the comovement between overall inflation and aggregate output, as indicated by Phillips curves estimated using aggregate time series, on the one hand, and the parameter values required for consistency with microeconomic observations, on the other. The appearance of a "micro/macro conflict" results from simplifying assumptions in familiar derivations of the new-Keynesian Phillips curve that are not actually necessary in order to obtain a relation between aggregate time series of that form, and that are not realistic, either. When one adopts more realistic (or at the very least, no less realistic) assumptions — industry-specific labor markets, firm-specific capital, intermediate inputs required for production, and a non-constant elasticity of substitution among differentiated goods for both consumption and investment purposes — the discrepancy between the frequency of price adjustment that is required to explain the aggregate comovements and the one that is indicated by microeconomic data disappears.

A similar conclusion is reached by Eichenbaum and Fisher (2004), Altig *et al.* (2005), and Matheron (2005) in the context of econometric models that allow for endogenous, firm-specific capital, following the analysis presented above. While the first two papers place particular stress on the role of firm-specific capital in reconciling the microeconomic and macroeconomic evidence, the assumption of an aggregator of the Kimball form that departs substantially from the Dixit-Stiglitz case is also important for the quantitative results of Eichenbaum and Fisher. Matheron stresses the importance of allowing for industry-specific labor as well as firm-specific capital. In the case of his analysis with euro-area data, a specification with firm-specific capital but homogeneous labor, as assumed by the other authors, reduces the estimated time between price revisions relative to the specification with economy-wide markets for both factors, but not by nearly enough to reconcile the model with microeconomic evidence on the frequency of price changes; allowing for both firm-specific capital and industry-specific labor, as proposed here, results in a substantial further reduction in the estimated time between time between price revisions.

I have given particular attention to the importance of allowing for firm-specific capital because, in the case that one allows for endogenous capital accumulation, the assumption that capital is firm-specific results in a non-trivial complication in the analysis. It turns out, however, that the same form of equilibrium relation between inflation dynamics and the evolution of average real marginal cost can be derived under this assumption. Moreover, the relation between the slope of the Phillips curve and the frequency of price adjustment that can be derived under the simpler assumption of an exogenously given capital stock for each firm turns out to be fairly accurate as an approximation to the correct relation in the case of an empirically realistic size of adjustment costs for investment. Hence the conclusions of the earlier literature (beginning with Sbordone, 1998) that drew inferences about the frequency of price adjustment from estimated Phillips curves under the implicit assumption of an exogenous capital stock are found to have been essentially correct, even if a more precise inference can be made using the analysis given here.

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