Inflation and Output Dynamics with Firm-Specific Investment*

Michael Woodford
Princeton University

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This note corrects the analysis given in Woodford (2003, chap. 5) of a model with staggered pricing and endogenous capital accumulation, to take account of an error in the original calculations noted by Sveen and Weinke (2004). The main source of complication in this analysis is the assumption that the producers of individual differentiated goods (that adjust their prices at different dates) invest in firm-specific capital which is relatively durable, so that the distribution of capital stocks across different firms (as a result of differing histories of price adjustment) matters, and not simply the economy's aggregate capital stock. This additional complications that result from this assumption are worth pursuing, not only because an economy-wide market for the rental of existing capital goods does not exist in practice, but because the non-existence of such a rental market has important implications for the degree of strategic complementarity among the pricing decisions of the producers of different goods, and hence for the speed of aggregate price adjustment, as discussed in Woodford (2003, chaps. 3, 5).

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1 A Model of Investment Demand with Sticky Prices

I shall begin by recapitulating the structure of the model with firm-specific investment proposed in Woodford (2003, chap. 5). A first task is to develop a model of optimizing investment demand by suppliers with sticky prices, and that are demand-constrained as a result. As in the sticky-price models with exogenous capital presented in Woodford (2003, chap. 3), there is a continuum of differentiated goods, each supplied by a single (monopolistically competitive) firm. The production function for good i is assumed to be of the form

$$y_t(i) = k_t(i) f(A_t h_t(i)/k_t(i)), \tag{1.1}$$

where f is an increasing, concave function, with f(0) = 0. I assume that each monopoly supplier makes an independent investment decision each period; there is a separate capital stock $k_t(i)$ for each good, that can be used only in the production of good i.

I also assume convex adjustment costs for investment by each firm, of the usual kind assumed in neoclassical investment theory. Increasing the capital stock to the level $k_{t+1}(i)$ in period $t+1$ requires investment spending in the amount $I_t(i) = I(k_{t+1}(i)/k_t(i))k_t(i)$ in period t. Here $I_t(i)$ represents purchases by firm i of the composite good, defined as the usual Dixit-Stiglitz aggregate over purchases of each of the continuum of goods (with the same constant elasticity of substitution $\theta > 1$ as for consumption purchases).[1] In this way, the allocation of investment expenditure across the various goods is in exactly the same proportion as consumption expenditure, resulting in a demand curve for each producer that is again of the form

$$y_t(i) = Y_t \left(\frac{p_t(i)}{P_t} \right)^{-\theta}, \tag{1.2}$$

but where now aggregate demand is given by $Y_t = C_t + I_t + G_t$, in which expression C_t is the representative household's demand for the composite

[1]See Woodford (2003, chap. 3) for discussion of this aggregator and its consequences for the optimal allocation of demand across alternative differentiated goods.

good for consumption purposes, G_t is the government's demand for the composite good (treated as an exogenous random variable), and I_t denotes the integral of $I_t(i)$ over the various firms i. I assume as usual that the function $I(\cdot)$ is increasing and convex; the convexity implies the existence of costs of adjustment. I further assume that near a zero growth rate of the capital stock, this function satisfies $I(1) = \delta$, $I'(1) = 1$, and $I''(1) = \epsilon_\psi$, where $0 < \delta < 1$ and $\epsilon_\psi > 0$ are parameters. This implies that in the steady state to which the economy converges in the absence of shocks (which here involves a constant capital stock, as I abstract from trend growth), the steady rate of investment spending required to maintain the capital stock is equal to δ times the steady-state capital stock (so that δ can be interpreted as the rate of depreciation). It also implies that near the steady state, a marginal unit of investment spending increases the capital stock by an equal amount (as there are locally no adjustment costs). Finally, in my log-linear approximation to the equilibrium dynamics, ϵ_ψ is the parameter that indexes the degree of adjustment costs. A central goal of the analysis is consideration of the consequences of alternative values for ϵ_ψ; the model with exogenous firm-specific capital presented in Woodford (2003, chaps. 3, 4) is recovered as the limiting case of the present model in which ϵ_ψ is made unboundedly large.

Profit-maximization by firm i then implies that the capital stock for period $t+1$ will be chosen in period t to satisfy the first-order condition

$$\begin{aligned} I'(k_{t+1}(i)/k_t(i)) \quad &= \quad E_t Q_{t,t+1}\Pi_{t+1}\left\{\rho_{t+1}(i) \; + \right. \\ &\left. (k_{t+2}(i)/k_{t+1}(i)) I'(k_{t+2}(i)/k_{t+1}(i)) - I(k_{t+2}(i)/k_{t+1}(i))\right\}, \end{aligned}$$

where $\rho_{t+1}(i)$ is the (real) shadow value of a marginal unit of additional capital for use by firm i in period $t+1$ production, and $Q_{t,t+1}\Pi_{t+1}$ is the stochastic discount factor for evaluating real income streams received in period $t+1$. Expressing the real stochastic discount factor as $\beta\lambda_{t+1}/\lambda_t$, where λ_t is the representative household's marginal utility of real income in period t and $0 < \beta < 1$ is the utility discount factor, and then log-linearizing this condition around the steady-state values of all state variables, we obtain

$$\hat{\lambda}_t + \epsilon_\psi(\hat{k}_{t+1}(i) - \hat{k}_t(i)) \quad = \quad E_t\hat{\lambda}_{t+1} \; +$$

$$[1-\beta(1-\delta)]E_t\hat{\rho}_{t+1}(i) + \beta\epsilon_\psi E_t(\hat{k}_{t+2}(i) - \hat{k}_{t+1}(i)), \qquad (1.3)$$

where $\hat{\lambda}_t \equiv \log(\lambda_t/\bar{\lambda})$, $\hat{k}_t(i) \equiv \log(k_t(i)/\bar{K})$, $\hat{\rho}_t(i) \equiv \log(\rho_t(i)/\bar{\rho})$, and variables with bars denote steady-state values.

Note that $\rho_{t+1}(i)$ would correspond to the real "rental price" for capital services if a market existed for such services, though I do not assume one here. It is *not* possible in the present model to equate this quantity with the marginal product, or even the marginal revenue product of capital (using the demand curve (1.2) to compute marginal revenue). For suppliers are demand-constrained in their sales, given the prices that they have posted; it is not possible to increase sales by moving down the demand curve. Thus the shadow value of additional capital must instead be computed as the reduction in labor costs through substitution of capital inputs for labor, while still supplying the quantity of output that happens to be demanded. In this way I obtain

$$\rho_t(i) = w_t(i)\left(\frac{f(\tilde{h}_t(i)) - \tilde{h}_t(i)f'(\tilde{h}_t(i))}{A_t f'(\tilde{h}_t(i))}\right),$$

where $w_t(i)$ is the real wage for labor of the kind hired by firm i and $\tilde{h}_t(i) \equiv A_t h_t(i)/k_t(i)$ is firm i's effective labor-capital input ratio.[2] I can alternatively express this in terms of the output-capital ratio for firm i (in order to derive an "accelerator" model of investment demand), by substituting (1.1) to obtain

$$\rho_t(i) = \frac{w_t(i)}{A_t} f^{-1}(y_t(i)/k_t(i))[\phi(y_t(i)/k_t(i)) - 1], \qquad (1.4)$$

where $\phi(y/k)$ is the reciprocal of the elasticity of the function f, evaluated at the argument $f^{-1}(y/k)$.

As in Woodford (2003, chap. 3), the first-order condition for optimizing labor supply can be written in the form

$$w_t(i) = \frac{v_h(f^{-1}(y_t(i)/k_t(i))k_t(i)/A_t;\xi_t)}{\lambda_t}, \qquad (1.5)$$

[2]Note that in the case of a flexible-price model, the ratio of $w_t(i)$ to the denominator would always equal marginal revenue, and so this expression would equal the marginal revenue product of capital, though it would be a relatively cumbersome way of writing it.

where labor demand has been expressed as a function of the demand for good i. This can be log-linearized as

$$\hat{w}_t(i) = \nu(\hat{h}_t(i) - \bar{h}_t) - \hat{\lambda}_t,$$

where $\nu > 0$ is the elasticity of the marginal disutility of labor with respect to labor supply, and $\bar{h}_t$ is an exogenous disturbance to preferences, indicating the percentage increase in labor supply needed to maintain a constant marginal disutility of working. Substituting (1.5) into (1.4) and log-linearizing, I obtain

$$\hat{\rho}_t(i) = \left(\nu\phi_h + \frac{\phi_h}{\phi_h - 1}\omega_p\right)(\hat{y}_t(i) - \hat{k}_t(i)) + \nu\hat{k}_t(i) - \hat{\lambda}_t - \omega q_t, \tag{1.6}$$

where $\phi_h > 1$ is the steady-state value of $\phi(y/k)$, *i.e.*, the reciprocal of the elasticity of the production function with respect to the labor input, and $\omega_p > 0$ is the negative of the elasticity of the marginal product $f'(f^{-1}(y/k))$ with respect to y/k. The composite exogenous disturbance q_t is defined as

$$q_t \equiv \omega^{-1}[\nu\bar{h}_t + (1+\nu)a_t]$$

where $a_t \equiv \log A_t$; it indicates the percentage change in output required to maintain a constant marginal disutility of output supply, in the case that the firm's capital remains at its steady-state level.[3] Substituting (1.6) into (1.3), I then have an equation to solve for the dynamics of firm i's capital stock, given the evolution of demand $\hat{y}_t(i)$ for its product, the marginal utility of income $\hat{\lambda}_t$, and the exogenous disturbance q_t.

As the coefficients of these equations are the same for each firm, an equation of the same form holds for the dynamics of the aggregate capital stock (in our log-linear approximation). The equilibrium condition for the dynamics of the capital stock is thus of the form

$$\hat{\lambda}_t + \epsilon_\psi(\hat{K}_{t+1} - \hat{K}_t) \quad = \quad \beta(1-\delta)E_t\hat{\lambda}_{t+1} \; +$$

[3]That is, q_t measures the output change that would be required to maintain a fixed marginal disutility of supply given possible fluctuations in preferences and technology, but not taking account of the effect of possible fluctuations in the firm's capital stock. With this modification of the definition given in Woodford (2003, chap. 3) for the model with exogenous capital, q_t is again an exogenous disturbance term.

$$[1-\beta(1-\delta)][\rho_y E_t \hat{Y}_{t+1} - \rho_k \hat{K}_{t+1} - \omega E_t q_{t+1}] + \beta\epsilon_\psi E_t(\hat{K}_{t+2} - \hat{K}_{t+1}), \tag{1.7}$$

where the elasticities of the marginal valuation of capital are given by

$$\rho_y \quad \equiv \quad \nu\phi_h + \frac{\phi_h}{\phi_h - 1}\omega_p \quad > \quad \rho_k \quad \equiv \quad \rho_y - \nu \quad > \quad 0.$$

The implied dynamics of investment spending are then given by

$$\hat{I}_t = k[\hat{K}_{t+1} - (1-\delta)\hat{K}_t], \tag{1.8}$$

where $\hat{I}_t$ is defined as the percentage deviation of investment from its steady-state level, as a share of steady-state output, and $k \equiv \bar{K}/\bar{Y}$ is the steady-state capital-output ratio.

Thus far I have derived investment dynamics as a function of the evolution of the marginal utility of real income of the representative household. This is in turn related to aggregate spending through the relation $\lambda_t = u_c(Y_t - I_t - G_t; \xi_t)$, which we may log-linearize as

$$\hat{\lambda}_t = -\sigma^{-1}(\hat{Y}_t - \hat{I}_t - g_t), \tag{1.9}$$

where the composite disturbance g_t reflects the effects both of government purchases and of shifts in private impatience to consume.[4] Finally, because of the relation between the marginal utility of income process and the stochastic discount factor that prices bonds,[5] the nominal interest rate must satisfy

$$1 + i_t = \{\beta E_t[\lambda_{t+1}/(\lambda_t \Pi_{t+1})]\}^{-1},$$

which one may log-linearize as

$$\hat{\imath}_t \quad = \quad E_t\pi_{t+1} + \hat{\lambda}_t - E_t\hat{\lambda}_{t+1}. \tag{1.10}$$

The system of equations (1.7) – (1.10) then comprise the "IS block" of the model. These jointly suffice to determine the paths of the variables

[4]Note that the parameter σ in this equation is not precisely the intertemporal elasticity of substitution in consumption, but rather $\bar{C}/\bar{Y}$ times that elasticity. In a model with investment, these quantities are not exactly the same, even in the absence of government purchases.

[5]See Woodford (2003, chaps. 2, 4) for further discussion of the stochastic discount factor and the Fisher relation between the nominal interest rate and expected inflation.

$\{\hat{Y}_t, \hat{I}_t, \hat{K}_t, \lambda_t\}$, given an initial capital stock and the evolution of short-term real interest rates $\{\hat{\imath}_t - E_t\pi_{t+1}\}$. The nature of the effects of real interest-rate expectations on these variables is discussed further in section xx below.

2 Optimal Price-Setting with Endogenous Capital

I turn next to the implications of an endogenous capital stock for the price-setting decisions of firms. The capital stock affects a firm's marginal cost, of course; but more subtly, a firm considering how its future profits will be affected by the price it sets must also consider how its capital stock will evolve over the time that its price remains fixed.

I begin with the consequences for the relation between marginal cost and output. Real marginal cost can be expressed as the ratio of the real wage to the marginal product of labor. Again writing the factor input ratio as a function of the capital/output ratio, and using (1.5) for the real wage, we obtain

$$s_t(i) = \frac{v_h(f^{-1}(y_t(i)/k_t(i))k_t(i)/A_t;\xi_t)}{\lambda_t A_t f'(f^{-1}(y_t(i)/k_t(i)))}$$

for the real marginal cost of supplying good i. This can be log-linearized to yield

$$\hat{s}_t(i) = \omega(\hat{y}_t(i) - \hat{k}_t(i)) + \nu\hat{k}_t(i) - \hat{\lambda}_t - [\nu\bar{h}_t + (1+\nu)a_t], \qquad (2.1)$$

where $\hat{s}_t(i) \equiv \log(s_t(i)/\bar{s})$, and $\omega \equiv \omega_w + \omega_p \equiv \nu\phi_h + \omega_p > 0$ is the elasticity of marginal cost with respect to a firm's own output.

Letting $\hat{s}_t$ without the index i denote the average level of real marginal cost in the economy as a whole, I note that (2.1) implies that

$$\hat{s}_t(i) = \hat{s}_t + \omega(\hat{y}_t(i) - \hat{Y}_t) - (\omega - \nu)(\hat{k}_t(i) - \hat{K}_t).$$

Then using (1.2) to substitute for the relative output of firm i, one obtains

$$\hat{s}_t(i) = \hat{s}_t - (\omega - \nu)\tilde{k}_t(i) - \omega\theta\tilde{p}_t(i), \qquad (2.2)$$

where $\tilde{p}_t(i) \equiv \log(p_t(i)/P_t)$ is the firm's relative price, and $\tilde{k}_t(i) \equiv \hat{k}_t(i) - \hat{K}_t$ is its relative capital stock. Note also that the average level of real marginal cost satisfies

$$\hat{s}_t = \omega(\hat{Y}_t - \hat{K}_t) + \nu\hat{K}_t - \hat{\lambda}_t - [\nu\bar{h}_t + (1+\nu)a_t]. \tag{2.3}$$

Following the same logic as in Woodford (2003, chap. 3), the Calvo price-setting framework implies that if a firm i resets its price in period t, it chooses a price that satisfies the (log-linear approximate) first-order condition

$$\sum_{k=0}^{\infty}(\alpha\beta)^k \hat{E}_t^i[\tilde{p}_{t+k}(i) - \hat{s}_{t+k}(i)] = 0, \tag{2.4}$$

where $0 < \alpha < 1$ is the fraction of prices that are not reset in any period. Here I introduce the notation $\hat{E}_t^i$ for an expectation conditional on the state of the world at date t, but integrating *only over those future states in which i has not reset its price since period t*. Note that in the case of any *aggregate* state variable x_t (*i.e.*, a variable the value of which depends only on the history of aggregate disturbances, and not on the individual circumstances of firm i), $\hat{E}_t^i x_T = E_t x_T$, for any date $T \geq t$. However, the two conditional expectations differ in the case of variables that depend on the relative price or relative capital stock of firm i. For example,

$$\hat{E}_t^i \tilde{p}_{t+k}(i) = \tilde{p}_t(i) - \sum_{j=1}^{k} E_t \pi_{t+j}, \tag{2.5}$$

for any $k \geq 1$, since firm i's price remains unchanged along all of the histories that are integrated over in this case. Instead, the expectation when one integrates over all possible future states conditional upon the state of the world at date t is given by

$$E_t \tilde{p}_{t+1}(i) = \alpha[\tilde{p}_t(i) - E_t \pi_{t+1}] + (1-\alpha)E_t \hat{p}^*_{t+1}(i), \tag{2.6}$$

where $\hat{p}^*_t(i)$ is the (log) relative price chosen when i reconsiders its price at date t. (Similar expressions can be given for horizons $k > 1$.)

Substituting (2.2) for $s_{t+k}(i)$ and (2.5) for $\hat{E}_t^i \tilde{p}_{t+k}(i)$ in (2.4), one obtains

$$(1+\omega\theta)\hat{p}^*_t(i) =$$

$$(1-\alpha\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k \hat{E}_t^i \left[\hat{s}_{t+k} + (1+\omega\theta)\sum_{j=1}^{k}\pi_{t+j} - (\omega-\nu)\tilde{k}_{t+k}(i)\right] \tag{2.7}$$

for the optimal relative price that should be chosen by a firm that resets its price at date t. This relation differs from the result obtained in Woodford (2003, chap. 3) for a model with exogenous capital only in the presence of the $\hat{E}_t^i \tilde{k}_{t+k}(i)$ terms.

The additional terms complicate the analysis in several respects. Note that the first two terms inside the square brackets are aggregate state variables, so that the distinction between $\hat{E}_t^i$ and E_t would not matter in this expression, were it not for the dependence of marginal cost on i's relative capital stock; it is for this reason that the alternative form of conditional expectation did not have to be introduced in Woodford (2003, chap. 3). However, in the model with endogenous capital, it is important to make this distinction when evaluating the $\hat{E}_t^i \tilde{k}_{t+k}(i)$ terms.[6] Furthermore, these new terms will not have the same value for all firms i that reset their prices at date t, for they will depend on i's relative capital stock $\tilde{k}_t(i)$ at the time that prices are reconsidered; hence $p_t^*(i)$ is no longer independent of i, as in the model with exogenous capital (or a model with an economy-wide rental market for capital). And finally, (2.7) is not yet a complete solution for the optimal price-setting rule, since the value of the right-hand side still depends on the expected evolution of i's relative capital stock; and this in turn depends on the expected evolution of i's relative price, which depends on the choice of $\hat{p}_t^*(i)$. A complete solution for this decision rule requires that one consider the effect of a firm's relative price on the evolution of its relative capital stock.

2.1 Dynamics of the Relative Capital Stock

Equation (1.7) implies that i's relative capital stock must evolve in accordance with the relation

$$\epsilon_\psi(\tilde{k}_{t+1}(i) - \tilde{k}_t(i)) = [1 - \beta(1-\delta)][\rho_y E_t(\hat{y}_{t+1}(i) - \hat{Y}_t) - \rho_k \tilde{k}_{t+1}(i)]$$
$$+\beta\epsilon_\psi E_t(\tilde{k}_{t+2}(i) - \tilde{k}_{t+1}(i)).$$

[6]It is the failure to distinguish between $\hat{E}_t^i$ and E_t in evaluating these terms that results in the incorrect calculations in the treatment of the present model in Woodford (2003, chap. 5) noted by Sveen and Weinke (2004).

Again using i's demand curve to express relative output as a function of the firm's relative price, this can be written as

$$E_t[Q(L)\tilde{k}_{t+2}(i)] = \Xi E_t \tilde{p}_{t+1}(i), \tag{2.8}$$

where the lag polynomial is

$$Q(L) \equiv \beta - [1 + \beta + (1 - \beta(1-\delta))\rho_k \epsilon_\psi^{-1}]L + L^2,$$

and

$$\Xi \equiv (1 - \beta(1-\delta))\rho_y \theta \epsilon_\psi^{-1} > 0.$$

I note for later reference that the lag polynomial can be factored as

$$Q(L) = \beta(1 - \mu_1 L)(1 - \mu_2 L).$$

Given that $Q(0) = \beta > 0$, $Q(\beta) < 0$, $Q(1) < 0$, and that $Q(z) > 0$ for all large enough $z > 0$, one sees that μ_1, μ_2 must be two real roots that satisfy $0 < \mu_1 < 1 < \beta^{-1} < \mu_2$.

Equation (2.8) can not yet be solved for the expected evolution of the relative capital stock, because of the dependence of the expected evolution of i's relative price (the "forcing term" on the right-hand side) on the expected evolution of the relative capital stock itself, for reasons just discussed. However, one may note that insofar as i's decision problem is locally convex, so that the first-order conditions characterize a locally unique optimal plan, the optimal decision for i's relative price in the event that the price is reset at date t must depend *only* on i's relative capital stock at date t and on the economy's aggregate state. Thus a log-linear approximation to i's pricing rule must take the form

$$\hat{p}_t^*(i) = \hat{p}_t^* - \psi \tilde{k}_t(i), \tag{2.9}$$

where $\hat{p}_t^*$ depends only on the aggregate state (and so is the same for all i), and ψ is a coefficient to be determined below.

Note that the assumption that the firms that reset prices at date t are drawn with uniform probability from the entire population implies that the average value of $\tilde{k}_t(i)$ over the set of firms that reset prices is zero (just as it is over the entire population of firms). Hence p_t^* is also the average relative price chosen by firms that reset prices at date t, and the overall rate of price inflation will be given (in our log-linear approximation) by

$$\pi_t = \frac{1-\alpha}{\alpha}\hat{p}_t^*. \tag{2.10}$$

Substitution of this, along with (2.9), into (2.6) then yields

$$E_t\tilde{p}_{t+1}(i) = \alpha\tilde{p}_t(i) - (1-\alpha)\psi\tilde{k}_{t+1}(i). \tag{2.11}$$

Similarly, the optimal quantity of investment in any period t must depend only on i's relative capital stock in that period, its relative price (which matters as a separate argument of the decision rule in the event that the price is *not* reset in period t), and the economy's aggregate state. Thus a log-linear approximation to i's investment rule must imply an expression of the form

$$\tilde{k}_{t+1}(i) = \lambda\tilde{k}_t(i) - \tau\tilde{p}_t(i), \tag{2.12}$$

where the coefficients λ and τ remain to be determined. This in turn implies that

$$\begin{aligned} E_t\tilde{k}_{t+2}(i) &= \lambda\tilde{k}_{t+1}(i) - \tau E_t\tilde{p}_{t+1}(i) \\ &= [\lambda + (1-\alpha)\tau\psi]\tilde{k}_{t+1}(i) - \alpha\tau\tilde{p}_t(i), \end{aligned}$$

using (2.11) to substitute for $E_t\tilde{p}_{t+1}(i)$ in the second line. Using this to substitute for $E_t\tilde{k}_{t+2}(i)$ in (2.8), and again using (2.11) to substitute for $E_t\tilde{p}_{t+1}(i)$, we obtain a linear relation that can be solved for $\tilde{k}_{t+1}(i)$ as a linear function of $\tilde{k}_t(i)$ and $\tilde{p}_t(i)$. The conjectured solution (2.12) satisfies this equation, so that the first-order condition (2.8) is satisfied, if and only if the coefficients λ and τ satisfy

$$R(\lambda;\psi) = 0, \tag{2.13}$$

$$(1-\alpha\beta\lambda)\tau = \Xi\alpha\lambda, \tag{2.14}$$

where

$$R(\lambda;\psi) \equiv (\beta^{-1} - \alpha\lambda)Q(\beta\lambda) + (1-\alpha)\Xi\psi\lambda$$

is a cubic polynomial in λ, with a coefficient on the linear term that depends on the value of the (as yet unknown) coefficient ψ. Condition (2.13) involves only λ (given the value of ψ); given a solution for λ, (2.14) then yields a unique solution for τ, as long as $\lambda \neq (\alpha\beta)^{-1}$.[7]

The dynamics of the the relative capital stock given by (2.12), together with (2.11), imply an expected joint evolution of i's relative price and relative capital stock satisfying

$$\begin{bmatrix} E_t\tilde{p}_{t+1}(i) \\ \tilde{k}_{t+1}(i) \end{bmatrix} = \begin{bmatrix} \alpha + (1-\alpha)\tau\psi & -(1-\alpha)\psi\lambda \\ -\tau & \lambda \end{bmatrix} \begin{bmatrix} \tilde{p}_t(i) \\ \tilde{k}_t(i) \end{bmatrix}. \qquad (2.15)$$

This implies convergent dynamics — so that both the means and variances of the distribution of possible future values for i's relative price and relative capital stock remain bounded no matter how in the future one looks, as long as the fluctuations in the average desired relative price $\hat{p}_t^*$ are bounded — if and only if both eigenvalues of the matrix in this equation are inside the unit circle. This stability condition is satisfied if and only if

$$\lambda < \alpha^{-1}, \qquad (2.16)$$

$$\lambda < 1 - \tau\psi, \qquad (2.17)$$

and

$$\lambda > -1 - \frac{1-\alpha}{1+\alpha}\tau\psi. \qquad (2.18)$$

These conditions must be satisfied if the implied dynamics of firm i's capital stock and relative price are to remain forever near enough to the steady-state values around which I have log-linearized the first-order conditions for the solution to the linearized equations to accurately approximate a solution to the exact first-order conditions. Hence the firm's decision problem has a solution that can be characterized using the local methods employed above only if equations (2.13) – (2.14) have a solution (λ, τ) satisfying (2.16) – (2.18). I show below that a unique solution consistent with these bounds exists, in the case of large enough adjustment costs.

[7] It is obvious from (2.14) that no solution with $\lambda = (\alpha\beta)^{-1}$ is possible, as long as $\Xi > 0$, as we assume here (*i.e.,* there exists some cost of adjusting capital). Even in the case that $\Xi = 0$, such a solution would violate condition (2.16) below, so one can exclude this possibility.

2.2 The Optimal Pricing Rule

I return now to an analysis of the first-order condition for optimal price-setting (2.7). The term that depends on firm i's own intended future behavior is proportional to

$$\sum_{k=0}^{\infty}(\alpha\beta)^k \hat{E}_t^i \tilde{k}_{t+k}(i).$$

It is now possible to write this term as a function of i's relative capital stock at the time of the pricing decision and of the expected evolution of aggregate variables, allowing me to obtain an expression of the form (2.9) for the optimal pricing rule.

Equation (2.12) for the dynamics of the relative capital stock implies that

$$\hat{E}_t^i \tilde{k}_{t+k+1}(i) = \lambda \hat{E}_t^i \tilde{k}_{t+k}(i) - \tau[\tilde{p}_t(i) - E_t \sum_{j=1}^{k} \pi_{t+j}]$$

for each $k \geq 0$, using (2.5) to substitute for $\hat{E}_t^i \tilde{p}_{t+k}(i)$. This can be integrated forward (given that[8] $|\lambda| < (\alpha\beta)^{-1}$), to obtain

$$\sum_{k=0}^{\infty}(\alpha\beta)^k \hat{E}_t^i \tilde{k}_{t+k}(i) = (1-\alpha\beta\lambda)^{-1}\tilde{k}_t(i)$$

$$-\tau \frac{\alpha\beta}{(1-\alpha\beta)(1-\alpha\beta\lambda)}\left[\tilde{p}_t(i) - \sum_{k=1}^{\infty}(\alpha\beta)^k E_t \pi_{t+k}\right]. \quad (2.19)$$

Substitution of this into (2.7) then yields

$$\phi \hat{p}_t^*(i) = (1-\alpha\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k E_t \hat{s}_{t+k} + \phi \sum_{k=1}^{\infty}(\alpha\beta)^k E_t \pi_{t+k} - (\omega-\nu)\frac{1-\alpha\beta}{1-\alpha\beta\lambda}\tilde{k}_t(i),$$

where

$$\phi \equiv 1 + \omega\theta - (\omega-\nu)\tau \frac{\alpha\beta}{1-\alpha\beta\lambda}.$$

The solution to this equation is a pricing rule of the conjectured form (2.9) if and only if the process $\hat{p}_t^*$ satisfies

$$\phi \hat{p}_t^* = (1-\alpha\beta)\sum_{k=0}^{\infty}(\alpha\beta)^k E_t \hat{s}_{t+k} + \phi \sum_{k=1}^{\infty}(\alpha\beta)^k E_t \pi_{t+k}, \quad (2.20)$$

[8] Note that (2.17) – (2.18) jointly imply that $\lambda > -\alpha^{-1}$. Hence any solution consistent with the stability conditions derived in the previous section must imply convergence of the infinite sum in (2.19).

where $\hat{s}_t$ is defined by (2.3), and the coefficient ψ satisfies

$$\phi\psi = (\omega - \nu)\frac{1 - \alpha\beta}{1 - \alpha\beta\lambda}. \tag{2.21}$$

Note that this last equation can be solved for ψ, given the values of λ and τ; however, the equations given earlier to determine λ and τ depend on the value of ψ. Hence equations (2.13), (2.14), and (2.21) comprise a system of three equations that jointly determine the coefficients λ, τ, and ψ of the firm's optimal decision rules.

This system of equations can be reduced to a single equation for λ in the following manner. First, note that for any conjectured value of $\lambda \neq 0$, (2.13) can be solved for ψ. This defines a function[9]

$$\psi(\lambda) \equiv -\frac{(1 - \alpha\beta\lambda)Q(\beta\lambda)}{(1 - \alpha)\beta\Xi\lambda}.$$

Similarly, (2.14) defines a function[10]

$$\tau(\lambda) \equiv \frac{\alpha\Xi\lambda}{1 - \alpha\beta\lambda}.$$

Substituting these functions for ψ and τ in (2.21), one obtains an equation in which λ is the only unknown variable. Multiplying both sides of this equation by $(1 - \alpha)\beta(1 - \alpha\beta\lambda)\Xi\lambda$,[11] one obtains the equation

$$V(\lambda) = 0, \tag{2.22}$$

where $V(\lambda)$ is the quartic polynomial

$$V(\lambda) \equiv [(1+\omega\theta)(1-\alpha\beta\lambda)^2 - \alpha^2\beta(\omega-\nu)\Xi\lambda]Q(\beta\lambda) + \beta(1-\alpha)(1-\alpha\beta)(\omega-\nu)\Xi\lambda.$$

Finally, one can write the inequalities (2.16) – (2.18) as restrictions upon the value of λ alone. One observes from the above discussion that the product

[9] The function is not defined if $\lambda = 0$. However, since $Q(0) \neq 0$, it is clear from (2.13) that $\lambda \neq 0$, for any economy with some adjustment costs (so that Ξ is finite).

[10] The function is not defined if $\lambda = (\alpha\beta)^{-1}$, but that value of λ would be inconsistent with (2.17) and (2.18) holding jointly, as noted above.

[11] This expression is necessarily non-zero in the case of the kind of solution that we seek, for the reasons noted in the previous two footnotes.

$\tau(\lambda)\psi(\lambda)$ is well-defined for all λ, and equal to $-(\alpha/1-\alpha)\beta^{-1}Q(\beta\lambda)$. Using this function of λ to replace the terms $\tau\psi$ in the previous inequalities, one obtains an equivalent set of three inequalities,

$$\lambda < \alpha^{-1}, \tag{2.23}$$

$$\frac{\alpha}{1+\alpha}\beta^{-1}Q(\beta\lambda) - 1 < \lambda < \frac{\alpha}{1-\alpha}\beta^{-1}Q(\beta\lambda) + 1, \tag{2.24}$$

that λ must satisfy.

I can then summarize my characterization of a firm's optimal pricing and investment behavior as follows.

Proposition 1. Suppose that the firm's decision problem has a solution in which, for any small enough initial log relative capital stock and log relative price of the individual firm, and in the case that the exogenous disturbance q_t and the aggregate variables $\hat{Y}_t$, $\hat{K}_t$, $\hat{\lambda}_t$, and π_t forever satisfy tight enough bounds, both the conditional expectation $\mathrm{E}_t\hat{k}_{t+j}(i)$ and the conditional variance $\mathrm{var}_t\hat{k}_{t+j}(i)$ remain bounded for all j, with bounds that can be made as tight as one likes by choosing sufficiently tight bounds on the initial conditions and the evolution of the aggregate variables.[12] Then the firm's optimal decision rules can be approximated by log-linear rules of the form (2.9) for $\hat{p}^*_t(i)$ in periods when the firm re-optimizes its price and (2.12) for the investment decision $\tilde{k}_{t+1}(i)$ each period. The coefficient λ in (2.12) is a root of the quartic equation (2.22), that satisfies the inequalities (2.23) – (2.24). The coefficient τ in (2.12) is furthermore equal to $\tau(\lambda)$, where the function $\tau(\cdot)$ is defined by (2.14), and the coefficient ψ in (2.9) is equal to $\psi(\lambda)$, where the function $\psi(\cdot)$ is defined by (2.21). Finally, the intercept p^*_t in (2.9) is given by (2.20), in which expression the process $\{\hat{s}_t\}$ is defined by (2.3).

This result gives a straightforward algorithm that can be used to solve for the firm's decision rules, in the case that local methods suffice to give an approximate characterization of optimal behavior in the event of small enough

[12]Note that this is the only condition under which local log-linearizations of the kind used above can suffice to approximately characterize the solution to the firm's problem.

disturbances and a small enough initial departure of the individual firm's situation from that of an average firm. The two decision rules (2.9) and (2.12), together with the law of motion

$$\tilde{p}_t(i) = \tilde{p}_{t-1}(i) - \pi_t$$

for any period t in which i does not re-optimize its price, then allow a complete solution for the evolution of the firm's relative capital stock and relative price, given its initial relative capital stock and relative price and given the evolution of the aggregate variables $\{\hat{Y}_t, \hat{K}_t, \lambda_t, \pi_t, q_t\}$.

Proposition 1 does not guarantee the existence of a non-explosive solution to the firm's decision problem. The following result, however, shows that at least in the case of large enough adjustment costs there is a solution of the kind characterized in Proposition 1.

[MORE TO BE ADDED]

2.3 Inflation Dynamics

I now consider the implications of the analysis above for the evolution of the overall inflation rate. Recall that the average log relative price set by firms that reoptimize at date t is given by (2.20). This equation can be quasi-differenced (after dividing by ϕ[13]) to yield

$$\hat{p}_t^* = (1 - \alpha\beta)\phi^{-1}\hat{s}_t + \alpha\beta E_t \pi_{t+1} + \alpha\beta E_t \hat{p}_{t+1}^*.$$

Then, using (2.10) to substitute for $\hat{p}_t^*$, one obtains

$$\pi_t = \xi \hat{s}_t + \beta E_t \pi_{t+1}, \tag{2.25}$$

where

$$\xi \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha\phi}.$$

Equation (2.25) is the corrected form of equation (3.17) in Woodford (2003, chap. 5). Together with (2.3), it provides a complete characterization of the equilibrium dynamics of inflation, given the evolution of $\hat{Y}_t, \hat{K}_t$, and

[13] It follows from (2.21) that $\phi \neq 0$, given that (as already discussed) $\lambda \neq (\alpha\beta)^{-1}$.

$\hat{\lambda}_t$. This pair of equations can be thought of as constituting the "aggregate supply block" of the model with endogenous capital. They generalize the aggregate-supply equation of the constant-capital model (the so-called "new Keynesian Phillips curve") to take account of the effects of changes in the capital stock on real marginal cost, and hence on the short-run tradeoff between inflation and output. One may still say that inflation depends only on inflation expectations and the "output gap," if the latter is defined as the (log) difference between actual and flexible-price output; but now it is important to recognize that flexible-price equilibrium output depends on the (endogenously varying) capital stock, so that the output gap (under this definition) would no longer depend only on current output and the exogenous disturbances.

Note that (2.25) is a relation between the dynamics of inflation and average real marginal cost of exactly the same form as in the model with constant capital[14] (see equation (2.14) of Woodford, 2003, chap. 3); only the numerical magnitude of ξ is affected by the size of the adjustment costs for investment. Furthermore, the result mentioned in Woodford (2003, chap. 3, footnote 37), according to which average real marginal cost should be proportional to average real unit labor cost (or the labor share) under the assumption of a Cobb-Douglas production technology, continues to apply in the case of endogenous firm-level capital. Hence the relation between the dynamics of inflation and those of unit labor costs, derived by Sbordone (1998, 2002) and Gali and Gertler (1999) under assumptions either of constant firm-level capital or of an economy-wide rental market for capital services, and tested empirically by those authors and a large subsequent literature (see Woodford, 2003, chap. 3, sec. 2.1) applies equally to the model developed here.

As with equation (3.17) in Woodford (2003, chap. 5), equation (2.25) implies that one can solve for the inflation rate as a function of current and expected future real marginal cost, resulting in a relation of the form

$$\pi_t = \sum_{j=0}^{\infty} \Psi_j E_t \hat{s}_{t+j}. \tag{2.26}$$

[14]This conclusion differs from the one presented in Woodford (2003, chap. 5). Correction of the mistake in the calculations presented there yields a simpler result.

The correct formula for these coefficients is given by

$$\Psi_j = \xi \beta^j,$$

just as in the model with constant capital. Hence the coefficients do not decay as rapidly with increasing j as is shown in Figure 5.6 of the book, in the case of finite adjustment costs. Nor do the coefficients ever change sign with increasing j, as occurs in the figure. In the case that $\xi > 0$ (as implied by the calibrated parameter values proposed below), an increase in the expected future level of real marginal costs unambiguously requires that inflation increase; and the degree to which inflation determination is forward-looking is even greater than is indicated by Figure 5.6.

3 Comparison with the Basic Neo-Wicksellian Model

The complete model with endogenous capital accumulation then consists of the system of equations (1.7) – (1.10), (2.3), and (2.25), together with an interest-rate feedback rule specifying monetary policy. This is a system of seven expectational difference equations per period to determine the equilibrium paths of seven endogenous variables, namely the variables $\{\pi_t, \hat{\imath}_t, \hat{Y}_t, \hat{K}_t, \hat{I}_t, \hat{s}_t, \hat{\lambda}_t\}$, given the paths of three composite exogenous disturbances g_t, q_t, and the exogenously varying intercept of the interest-rate rule. It is useful to comment upon the extent to which the structure of the extended (variable-capital) model remains similar, though not identical, to that of the basic (constant-capital) model presented in Woodford (2003, chap. 4).

I have already noted that the equations of the extended model consist of an "IS block"(which allows one to solve for the paths of real output and of the capital stock, given the expected path of real interest rates and the initial capital stock), an "AS block" (which allows one to solve for the path of inflation given the paths of real output and of the capital stock), and a monetary policy rule (which implies a path for nominal interest rates given the paths of inflation and output). In this overall structure it is similar to the basic neo-Wicksellian model, except that the model involves an additional endogenous variable, the capital stock, which is determined by the "IS block" and taken as an input to the "AS block", along with the level of real activity. It also continues to be the case that real disturbances affect the determination of inflation and output only through their effects upon the two composite disturbances g_t and q_t. However, in the case of inflation determination alone (and determination of the output *gap*, as opposed to the level of output) it is possible in the case of the basic neo-Wicksellian model to further reduce these to a single composite disturbance, the implied variation in the Wicksellian "natural rate of interest." This is no longer possible in the case of the extended model, although, as is discussed below, it is still possible to explain inflation determination in terms of the gap between an actual and a "natural" real rate of interest. The problem is that with endogenous variation in the capital stock, the natural rate of interest is no longer a purely exogenous state variable.

I have already noted in the previous section that the inflation equation (2.25) of the extended model is as forward-looking as the corresponding equation of the basic model. The "IS block" of the extended model also implies that aggregate expenditure is as sensitive to expectations regarding future (short-term) real rates of interest as in the basic model, where all private expenditure is modelled as non-durable consumer expenditure. Because this feature of the "IS relation" of the basic model has often attracted criticism, it is worth elaborating further on this point.

It is first useful to consider the implied long-run average values for capital, output and investment as a function of the long-run average rate of inflation π_∞ implied by a given monetary policy. Equations (1.7) – (1.9) imply that the long-run average values of the various state variables must satisfy

$$\begin{aligned}
\hat{\lambda}_\infty &= \rho_y \hat{Y}_\infty - \rho_k \hat{K}_\infty, \\
\hat{I}_\infty &= \delta k \hat{K}_\infty, \\
\hat{\lambda}_\infty &= -\sigma^{-1}(\hat{Y}_\infty - \hat{I}_\infty).
\end{aligned}$$

These relations can be solved for $\hat{Y}_\infty$, $\hat{I}_\infty$ and $\hat{K}_\infty$ as multiples of $\hat{\lambda}_\infty$; this generalizes the relation between $\hat{Y}_\infty$ and $\hat{\lambda}_\infty$ obtained for the basic model. Equation (2.3) similarly implies that the long-run average level of real marginal cost must satisfy

$$\hat{s}_\infty = \omega \hat{Y}_\infty - (\omega - \nu)\hat{K}_\infty - \hat{\lambda}_\infty;$$

substituting the above solutions, I obtain $\hat{s}_\infty$ as a multiple of $\hat{\lambda}_\infty$ as well. Finally, (2.25) implies that[15]

$$\pi_\infty = \frac{\xi}{1-\beta}\,\hat{s}_\infty.$$

Using this together with the previous solution allows me to solve for $\hat{\lambda}_\infty$, and hence for $\hat{Y}_\infty, \hat{I}_\infty$ and $\hat{K}_\infty$ as well, as multiples of π_∞.

I turn next to the characterization of transitory fluctuations around these long-run average values. Using (1.8) – (1.9) to eliminate $\hat{Y}_{t+1}$ from (1.7), I obtain a relation of the form

$$E_t[A(L)\hat{K}_{t+2}] = E_t[B(L)\hat{\lambda}_{t+1}] + z_t, \tag{3.1}$$

where $A(L)$ is a quadratic lag polynomial, $B(L)$ is linear, and z_t is a linear combination of the disturbances g_t and q_t. For empirically realistic parameter values, the polynomial $A(L)$ can be factored as $(1-\tilde{\mu}_1 L)(1-\tilde{\mu}_2 L)$, where the two real roots satisfy $0 < \tilde{\mu}_1 < 1 < \tilde{\mu}_2$. It follows that there is a unique bounded solution for $\hat{K}_{t+1}$ as a linear function of $\hat{K}_t$, the expectations $E_t\hat{\lambda}_{t+j}$ for $j \geq 0$, and the expectations $E_t z_{t+j}$ for $j \geq 0$. Then solving (1.10) forward to obtain

$$\hat{\lambda}_t = \hat{\lambda}_\infty + \sum_{j=0}^{\infty} E_t(\hat{\imath}_{t+j} - \pi_{t+j+1}), \tag{3.2}$$

and using this to eliminate the expectations $E_t\hat{\lambda}_{t+j}$, I finally obtain a solution of the form

$$\hat{K}_{t+1} = (1-\tilde{\mu}_1)\hat{K}_\infty + \tilde{\mu}_1\hat{K}_t - \sum_{j=0}^{\infty}\tilde{\chi}_j E_t(\hat{\imath}_{t+j} - \pi_{t+j+1}) + e_t^k, \tag{3.3}$$

where the $\{\tilde{\chi}_j\}$ are constant coefficients and e_t^k is an exogenous disturbance term (a linear combination of the $\{E_t z_{t+j}\}$). This can be solved iteratively for the dynamics of the capital stock, starting from an initial capital stock and given the evolution of the exogenous disturbances and of real interest-rate expectations.

Equation (3.2) can also be substituted into (1.9) to yield

$$\hat{Y}_t = (\hat{Y}_\infty - \hat{I}_\infty) + \hat{I}_t + g_t - \sigma\sum_{j=0}^{\infty} E_t(\hat{\imath}_{t+j} - \pi_{t+j+1}),$$

[15] This equation corrects the one given at the middle of p. 365 of Woodford (2003).

Table 1: Numerical parameter values.

α	0.66
β	0.99
σ	1
ν	0.11
ϕ_h^{-1}	0.75
ω_w	0.14
ω_p	0.33
ω	0.47
$(\theta-1)^{-1}$	0.15
δ	0.12
ϵ_ψ	3
ρ	0.7

a direct generalization of the "intertemporal IS relation" of the basic neo-Wicksellian model,[16] which now however takes account of investment spending. Using (1.8) and (3.3) to substitute for $\hat{I}_t$, this expression takes the form

$$\hat{Y}_t = (\hat{Y}_\infty - \Sigma\hat{K}_\infty) + \Sigma\hat{K}_t - \sum_{j=0}^{\infty} \chi_j E_t(\hat{\imath}_{t+j} - \pi_{t+j+1}) + e_t^y, \tag{3.4}$$

where $\Sigma \equiv k[\tilde{\mu}_1 - (1-\delta)]$, $\{\chi_j\}$ is another set of constant coefficients, and e_t^y is another exogenous disturbance term (a linear combination of g_t and of the $\{E_t z_{t+j}\}$). The joint evolution of output and of the capital stock are then determined by the pair of equations (3.3) – (3.4), starting from an initial capital stock and given the evolution of the exogenous disturbances and of real interest-rate expectations.

Except for the need to jointly model the evolution of output and of the capital stock, this system of equations has implications rather similar to those of the "IS equation" of the basic neo-Wicksellian model. In particular, for typical parameter values, the coefficients $\{\chi_j\}$ in (3.4) are all positive, and even of roughly similar magnitude for all j. For example, consider the

[16]Compare equation (1.8) of Woodford (2003, chap. 4).

numerical calibration proposed in Woodford (2003, chap. 5), summarized in Table 1.[17] Then Figure 1 plots the coefficients $\{\chi_j\}$ for each of a range of alternative values for the parameter ϵ_ψ, measuring the size of the investment adjustment costs, holding fixed the values of the other parameters. In the limit of very large ϵ_ψ, the coefficients all approach the constant value σ (here assigned the value 1), as in equation (1.1) of Woodford (2003, chap. 5) for the constant-capital model. For most lower values of ϵ_ψ, the coefficients are not exactly equal in magnitude, and each coefficient is larger the smaller are the adjustment costs associated with investment spending. However, the coefficients all remain positive, and quite similar in magnitude to one another, especially for values of ϵ_ψ near 3.

One can show analytically that χ_j takes the same value for all j (though a value greater than σ) if it happens that $B(L)$ in (3.1) is of the form $-h(1-\tilde{\mu}_2 L)$, where $h > 0$ and $\tilde{\mu}_2$ is the root greater than one in the factorization of $A(L)$. In this case (3.1) is equivalent to

$$(1 - \tilde{\mu}_1 L)\hat{K}_{t+1} = -h\hat{\lambda}_t + e_t^k,$$

and substitution of (3.2) yields a solution for aggregate demand of the form (3.4) with $\chi_j = \sigma + h$ for all $j \geq 0$. For the parameter values given in Table 1, the root of $B(L)$ coincides with a root of $A(L)$ in this way if and only if ϵ_ψ happens to take a specific value, equal approximately to 3.23. This is in fact not an unrealistic value to assume. Perhaps more interesting, however, is the fact that the coefficients $\{\chi_j\}$ are all reasonably similar in magnitude even when ϵ_ψ is larger or smaller than this critical value.

Thus it continues to be true, as in the basic model, that changes in interest-rate expectations (due, for example, to a shift in monetary policy) affect aggregate demand through their effect upon a *very long* real rate; the existence of endogenous variation in investment spending simply makes the degree of sensitivity of aggregate demand to the level of the very long real

[17] The justification for these values is discussed further in Woodford (2003, chap. 5). In the calculations reported in Figure 1, I do not impose the value $\epsilon_\psi = 3$, indicated in the table, but instead consider a range of possible values for this parameter. The coefficient ρ of the monetary policy rule is also irrelevant for this calculation, as I assume no particular monetary policy in Figure 1.

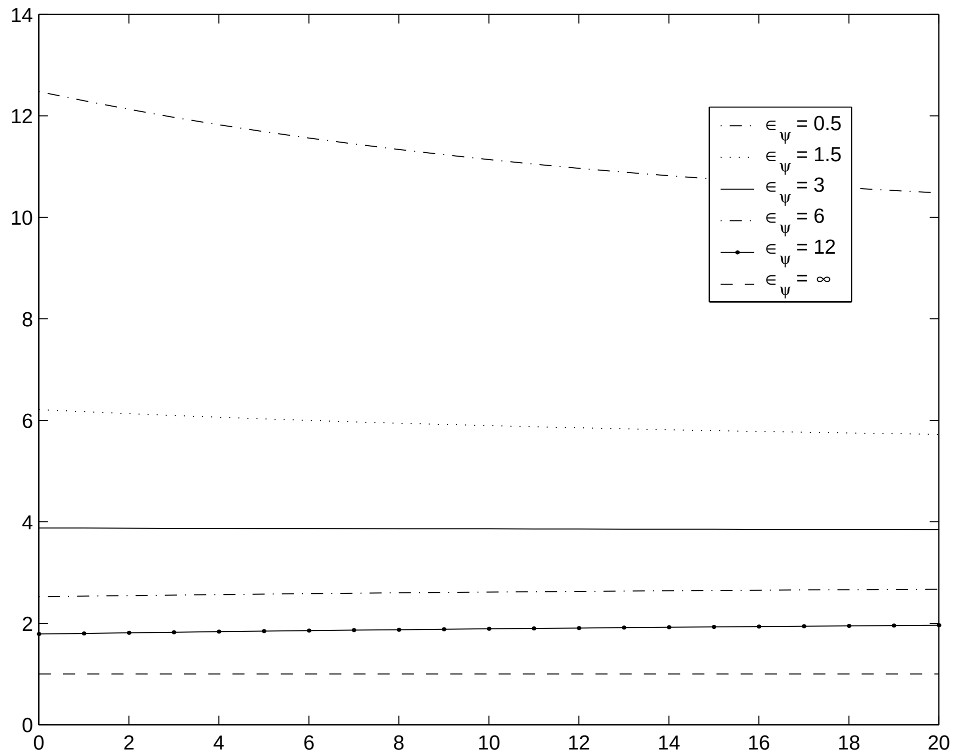

Figure 1: The coefficients χ_j in aggregate demand relation (3.4), for alternative sizes of investment adjustment costs.

rate greater. For example, the figure shows that when $\epsilon_\psi = 3$, the degree of interest-sensitivity of aggregate demand is about four times as large as if ϵ_ψ were extremely large; the response to interest-rate changes is thus roughly the same as in a constant-capital model with a value of σ near 4, rather than equal to 1 as assumed here. This justifies the use of a value of σ much larger than 1 in Woodford (2003, chap. 4) when calibrating the basic model.

However, even if one adjusts the value assumed for σ in this way, the predictions of the constant-capital model as to the effects of real interest rate changes are not exactly the same as those of the model with variable capital. This is because lower investment spending as a result of high long real rates of interest soon results in a lower capital stock, and once this occurs aggregate demand is affected through the change in the size of the $\Sigma \hat{K}_t$ term in (3.4). In the case of sufficiently moderate adjustment costs (the empirically realistic case), the value of Σ is negative; for given real interest-rate expectations, a higher existing capital stock depresses investment demand (because returns to existing capital are low).[18] Thus a sustained increase in long real rates of interest will initially depress aggregate demand, in the variable-capital model, by more than it does later on; once the capital stock has fallen this fact helps investment demand to recover, despite the continued high real rates.

[MORE TO BE ADDED]

3.1 Capital and the Natural Rate of Interest

I now consider the extent to which the concept of the "natural rate of interest", expounded in Woodford (2003, chap. 4) in connection with the basic neo-Wicksellian model, can be extended to a model which allows for endogenous variation in the capital stock. The most important difference in the case of the extended model is that the equilibrium real rate of return under flexible prices is no longer a function solely of current and expected future exogenous disturbances; it depends on the capital stock as well, which is now an endogenous state variable (and so a function of past monetary policy, among other things, when prices are sticky). Hence if one continues to define the natural rate of interest in this way, it ceases to refer to an exogenous process.

To be more precise, I shall define the "natural rates" of output and interest as those that would result from price flexibility now and in the future, *given* all exogenous and predetermined state variables at the present time, including the economy's capital stock.[19] Since the equilibrium with flexible prices at any date t depends only on the capital stock at that date[20] and current

[18]For the parameter values given in Table 1, the baseline value $\epsilon_\psi = 3$ implies that $\Sigma = -1.246$ in the case of a quarterly model. Note that the model does not require Σ to be negative; one can show that $\Sigma > 0$ (because $\mu_1 > 1 - \delta$) if and only if ϵ_ψ exceeds the critical value $\rho_k(1-\delta)/\delta > 0$. For the calibrated parameter values, this critical value is approximately equal to 114.5, and thus would imply a level of adjustment costs in investment that would be inconsistent with the observed degree of volatility of investment spending.

[19]For further discussion of this definition, see Woodford (2003, chap.5, sec. 3.4).

[20]This is true up to the log-linear approximation that we use here to characterize equi-

and expected future exogenous real disturbances, I can write a log-linear approximation to the solution in the form

$$\hat{Y}_t^n = \hat{Y}_t^{ncc} + \eta_y \hat{K}_t,$$

$$\hat{r}_t^n = \hat{r}_t^{ncc} + \eta_r \hat{K}_t,$$

and so on, where the terms $\hat{Y}_t^{ncc}$ and $\hat{r}_t^{ncc}$ refer to exogenous processes (functions solely of the exogenous real disturbances). These intercept terms in each expression indicate what the level of real output (or the real interest rate, and so on) *would* be, given current and expected future real disturbances, if prices were flexible *and* the capital stock did not differ from its steady-state level; I shall call this the *constant-capital natural rate* of output (or of interest, and so on). It is also useful to define a "natural rate" of investment $\hat{I}_t^n$ and of the marginal utility of income $\hat{\lambda}_t^n$ in a similar way. One can even define a "natural" capital stock $\hat{K}_{t+1}^n$, as what the capital stock in period $t+1$ would be if it had been chosen in a flexible-price equilibrium in period t, as a function of the actually existing capital stock $\hat{K}_t$ and the exogenous disturbances at that time; thus I similarly write

$$\hat{K}_{t+1}^n = \hat{K}_{t=1}^{ncc} + \eta_k \hat{K}_t.$$

Finally, I shall use tildes to indicate the "gaps" between the actual and "natural" values of these several variables: $\tilde{Y}_t \equiv \hat{Y}_t - \hat{Y}_t^n$, $\tilde{r}_t \equiv \hat{r}_t - \hat{r}_t^n$, and so on.

Just as in the constant-capital models discussed in Woodford (2003, chap. 3), in a flexible-price equilibrium, real marginal cost must at all times be equal to a constant, $(\theta - 1)/\theta$. It then follows from (2.3) that fluctuations in the natural rate of output satisfy

$$\hat{Y}_t^n = \frac{\omega - \nu}{\omega + \sigma^{-1}} \hat{K}_t + \frac{\sigma^{-1}}{\omega + \sigma^{-1}} \hat{I}_t + \frac{\sigma^{-1}}{\omega + \sigma^{-1}} g_t + \frac{\omega}{\omega + \sigma^{-1}} q_t.$$

This relation generalizes equation (2.2) of Woodford (2003, chap. 4) for the basic neo-Wicksellian model. Note that this does not allow one to solve for the natural rate of output as a function of the capital stock and the real

librium. More precisely, it would depend on the capital stock in place in each of the firms producing differentiated goods.

disturbances without also simultaneously solving for the natural rate of investment. However, comparison with (2.3) allows one to derive an expression for real marginal cost in terms of the "gaps",

$$\hat{s}_t = (\omega + \sigma^{-1})\tilde{Y}_t - \sigma^{-1}\tilde{I}_t, \tag{3.5}$$

generalizing equation (2.7) of Woodford (2003, chap. 3).

One can also write condition (1.7) in terms of the "gap" variables, obtaining the following result.

Proposition 3. Let $\tilde{Y}_t \equiv \hat{Y}_t - \hat{Y}_t^n$, where the natural rate of output $\hat{Y}_t^n$ represents the flexible-price equilibrium level of output given $\hat{K}_t$, and similarly let $\tilde{K}_{t+1} \equiv \hat{K}_{t+1} - \hat{K}_{t+1}^n$, where $\hat{K}_{t+1}^n$ represents the flexible-price equilibrium capital stock given $\hat{K}_t$ [the *actual* capital stock in period t, not what it *would* have been if prices had been flexible in earlier periods]. Then optimizing investment demand implies that the joint dynamics of the output and capital "gaps" $\{\tilde{Y}_t, \tilde{K}_{t+1}\}$ satisfy

$$[1-\beta(1-\delta)][\rho_y(E_t\tilde{Y}_{t+1}+\eta_y\tilde{K}_{t+1})-\rho_k\tilde{K}_{t+1}]+\beta\epsilon_\psi[(E_t\tilde{K}_{t+2}+\eta_k\tilde{K}_{t+1})-\tilde{K}_{t+1}], \tag{3.6}$$

where the coefficients $\rho_y, \rho_k, \eta_y, \eta_k$ are again defined as in (1.7).

The calculation is explained in Woodford (2003, appendix, sec. D.3). Note that this equation is similar in form to (1.7), *except* that it is purely forward-looking; it determines the equilibrium size of the gap $\tilde{K}_{t+1}$ without any reference to predetermined state variables such as $\tilde{K}_t$.

Equations (1.8) – (1.10) similarly must hold in a flexible-price equilibrium, implying that the "gaps" must also satisfy equations

$$\tilde{I}_t = k\tilde{K}_{t+1}, \tag{3.7}$$

$$\tilde{\lambda}_t = -\sigma^{-1}(\tilde{Y}_t - \tilde{I}_t), \tag{3.8}$$

$$\tilde{r}_t = \tilde{\lambda}_t - (E_t\tilde{\lambda}_{t+1} + \eta_\lambda\tilde{K}_{t+1}). \tag{3.9}$$

Using equations (3.7) and (3.8) to eliminate $\tilde{\lambda}_t$ and $\tilde{K}_{t+1}$ from (3.6) and (3.9), one is left with a system of two equations that can be written in the form

$$E_t z_{t+1} = A\ z_t + a\ \tilde{r}_t, \tag{3.10}$$

for a certain matrix A and vector a of coefficients, where now

$$z_t \equiv \begin{bmatrix} \tilde{Y}_t \\ \tilde{I}_t \end{bmatrix}.$$

This pair of coupled difference equations generalizes the "gap" version (equation (1.12) of Woodford, 2003, chap. 4) of the IS relation of the basic neo-Wicksellian model.

Let me now close the model by specifying monetary policy in terms of an interest-rate feedback rule of the form[21]

$$\hat{\imath}_t = \hat{r}_t^n + \bar{\pi} + \phi_\pi(\pi_t - \bar{\pi}) + \phi_x(x_t - \bar{x})/4, \tag{3.11}$$

where ν_t is an exogenous disturbance term (possibly reflecting time variation in the inflation target relative to the baseline value $\bar{\pi}$). I now use the notation $x_t \equiv \tilde{Y}_t$ for the output gap, as in the analysis in Woodford (2003, chap. 4) of equilibrium determination under similar rules in the basic neo-Wicksellian model. (As in that treatment, $\bar{x}$ is the steady-state output gap corresponding to the steady-state inflation rate $\bar{\pi}$.) The specification (3.11) differs from the notation used there by the inclusion of an intercept that varies with variations in the natural rate of interest. In the basic neo-Wicksellian model, the natural rate of interest is a function of the exogenous disturbances alone, so that there is no need to distinguish between this term and the exogenous disturbance term ν_t; but this is no longer true in the present model, because of the dependence of the natural rate of interest on the (endogenous) aggregate capital stock. Failures of the intercept term to correctly track the variations in the natural rate of interest can still be considered, because of the presence of the term ν_t.[22] With policy specified by a "Taylor rule" of this kind, the

[21] Similar conclusions can be obtained in the case of policy rules that incorporate policy inertia, as in the treatment in Woodford (2003, chap. 4) for the basic model, but here I economize on algebra by treating only the case of a purely contemporaneous Taylor rule. Note that the disturbance term ν_t is not assumed to be serially uncorrelated.

[22] Implementation of a rule of the form (3.11) requires that the central bank know the current value of the natural rate of interest. However, this is not obviously a more onerous requirement than requiring that it know the current natural rate of output, which is also required for implementation of such a rule (if $\phi_x \neq 0$). Rules that can be implemented in the absence of *either* of these pieces of information correspond to cases in which ν_t is not exogenous, and require a more complex analysis.

interest-rate gap is given by

$$\tilde{r}_t = \nu_t - E_t(\pi_{t+1} - \bar{\pi}) + \phi_\pi(\pi_t - \bar{\pi}) + \phi_x(x_t - \bar{x})/4. \tag{3.12}$$

Note that in this last relation, the only endogenous variables are "gap" variables.

A complete system of equilibrium conditions for the determination of the variables $\{\tilde{Y}_t, \tilde{I}_t, \tilde{r}_t, \hat{s}_t, \pi_t\}$ is then given by (2.25), (2.3), (3.10), and (3.12). The system of equations may furthermore be written in the form

$$E_t \hat{z}_{t+1} = \hat{A}\ \hat{z}_t + \hat{a}\ \nu_t, \tag{3.13}$$

where now

$$\hat{z}_t \equiv \begin{bmatrix} \tilde{Y}_t - \bar{x} \\ \tilde{I}_t - \bar{I} \\ \pi_t - \bar{\pi} \end{bmatrix},$$

is a vector with 3 elements, $\bar{I}$ is the steady-state value of $\tilde{I}_t$ corresponding to steady inflation at the rate $\bar{\pi}$, and $\hat{A}$ and $\hat{a}$ are again a matrix and vector of coefficients. We obtain this system as follows. The first two rows are obtained by substituting for $\tilde{r}_t$ in (3.10) using (3.12).[23] The third row is obtained by solving (2.25) for $E_t\pi_{t+1}$, and then substituting for $\hat{s}_t$ using (2.3).

Because the system (3.13) is purely forward-looking (*i.e.*, there are no predetermined endogenous state variables), a policy rule of the kind defined by (3.11) results in determinate equilibrium dynamics for inflation and the output gap (among other variables) if and only if the matrix $\hat{A}$ has all three eigenvalues outside the unit circle. When this is true, the system can be "solved forward" in the usual way to obtain a unique bounded solution. The solutions for inflation and the output gap, and the implied solution for the nominal interest rate, are of the form

$$\pi_t \quad = \quad \bar{\pi} + \sum_{j=0}^{\infty} \psi_j^\pi E_t \nu_{t+j}, \tag{3.14}$$

[23] Note that all of these equations continue to be valid when we replace variables by the difference of those variables from their steady-state values. We choose to express the equations in this form in (3.13) because the policy rule (3.11) has already been expressed in this form.

$$x_t = \bar{x} + \sum_{j=0}^{\infty} \psi_j^x E_t \nu_{t+j}, \tag{3.15}$$

$$\hat{\imath}_t = \hat{r}_t^n + \bar{\pi} + \sum_{j=0}^{\infty} \psi_j^i E_t \nu_{t+j}, \tag{3.16}$$

just as in the case of the basic neo-Wicksellian model (Woodford, 2003, chap. 4, sec. 2.4).[24] However, the numerical values of the coefficients $\{\psi_j^\pi, \psi_j^x, \psi_j^i\}$ in these expressions will be different.

[MORE TO BE ADDED]

An immediate consequence is that once again a possible approach to the goal of inflation stabilization is to commit to a policy rule of the form (3.11) such that (i) the coefficients ϕ_π, ϕ_x are chosen so as to imply a determinate equilibrium, and (ii) the intercept tracks variations in the natural rate of interest, *i.e.*, $\nu_t = 0$ at all times. If it is possible to satisfy this condition with sufficient accuracy, then inflation can in principle be completely stabilized with finite response coefficients. Thus the requirement of tracking variations in the natural rate of interest continues to be as important to the pursuit of price stability as in the analysis of the basic neo-Wicksellian model presented in Woodford (2003, chap. 4).

[24]In the expressions given in Woodford (2003, chap. 4), the forcing process consists of expectations of the form $E_t(\hat{r}_{t+j}^n - \bar{\imath}_{t+j} + \bar{\pi})$, where $\bar{\imath}_t$ is the Taylor rule intercept in period t (an exogenous process in the earlier treatment). In specification (3.11), $\bar{\imath}_t$ has been replaced by $\hat{r}_t^n + \bar{\pi} + \nu_t$.

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