

# Indicator Variables for Optimal Policy\*

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## Abstract

The optimal weights on indicators in models with partial information about the state of the economy and forward-looking variables are derived and interpreted, both for equilibria under discretion and under commitment. An example of optimal monetary policy with a partially observable potential output and a forward-looking indicator is examined. The optimal response to the *optimal estimate* of potential output displays certainty equivalence, whereas the optimal response to the *imperfect observation* of output depends on the noise in this observation.

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# 1 Introduction

It is a truism that monetary policy operates under considerable uncertainty about the state of the economy and the size and nature of the disturbances that hit the economy. This is a particular problem for a procedure such as inflation-forecast targeting, under which a central bank, in order to set its interest-rate instrument, needs to construct conditional forecasts of future inflation, conditional on alternative interest-rate paths and the bank's best estimate of the current state of the economy and the likely future development of important exogenous variables.<sup>1</sup> Often, different indicators provide conflicting information on developments in the economy. In order to be successful, a central bank then needs to put the appropriate weights on different information and draw the most efficient inference. In the case of a purely backward-looking model (both of the evolution of the bank's target variables and of the indicators), the principles for efficient estimation and signal extraction are well known. But in the more realistic case where important indicator variables are forward-looking variables, the problem of efficient signal-extraction is inherently more complicated. The purpose of this paper is to clarify the principles for determining the optimal weights on different indicators in such an environment.

In the case where there are no forward-looking variables, it is well known that a linear model with a quadratic loss function and a partially observable state of the economy (partial information) is characterized by *certainty equivalence*. That is, the optimal policy is the same as if the state of the economy were fully observable (full information), except that one responds to an efficient estimate of the state vector rather than to its actual value. Furthermore, a *separation principle* applies, according to which the selection of the optimal policy (the optimization problem) and the estimation of the current state of the economy (the estimation or signal-extraction problem) can be treated as separate problems. In particular, the observable variables will be predetermined and the innovations in the observable variables (the difference between the current realization and previous prediction of each of the observable variables) contain all new information. The optimal weights to be placed on the innovations in the various observable variables in one's estimate of the state vector at each point in time are provided by a standard Kalman filter (see, for instance, Chow (1975), Kalchenbrenner and Tinsley (1975) and LeRoy and Waud (1977)).<sup>2</sup>

The case without forward-looking variables is, however, very restrictive. In the real world, many important indicator variables for central banks are forward-looking variables, variables that depend on private-sector expectations of the future developments in the economy and

future policy. Central banks routinely watch variables that are inherently forward-looking, like exchange rates, bond rates and other asset prices, as well as measures of private-sector inflation expectations, industry order-flows, confidence measures, and the like. Forward-looking variables complicate the estimation or signal-extraction problem significantly. They depend, by definition, on private-sector expectations of future endogenous variables and of current and future policy actions. However, these expectations in turn depend on an estimate of the current state of the economy, and that estimate in turn depends, to some extent, on observations of the current forward-looking variables. This circularity presents a considerable challenge for the estimation problem in the presence of forward-looking variables.

It is well known that forward-looking variables also complicate the optimization problem. For example, optimal policy under commitment ceases in general to coincide with the outcome of discretionary optimization, as shown by Kydland and Prescott (1980); the point is developed in the context of a general linear model with quadratic objectives of the kind that we assume here in Backus and Driffill (1986) and Currie and Levine (1993). With regard to the estimation problem, Pearlman, Currie and Levin (1986) showed in a linear (non-optimizing) model with forward-looking variables and partial symmetric information that the solution can be expressed in terms of a Kalman filter, although the solution is much more complex than in the purely backward-looking case. Pearlman (1992) later used this solution in an optimizing model to demonstrate that certainty equivalence, and hence the separation principle, applies under both discretion and commitment, in the presence of forward-looking variables and symmetric partial information.

The present paper extends this previous work on partial information with forward-looking variables by providing simpler derivations of the optimal weights on the observable variables, and clarifying how the updating equations can be modified to handle the circularity mentioned above. We also provide a simple example, in a now-standard model of monetary policy with a forward-looking aggregate supply relation and a forward-looking “expectational IS” relation. We believe this example clarifies several issues raised by Orphanides (2000). He argues, for instance, with reference to real-time U.S. data from the 1970s, that it is better that monetary policy disregards uncertain data about the output gap and responds to current inflation only. Our findings are different and in line with the conventional wisdom. First, we find that the monetary-policy response to the optimal *estimates* of the current output gap is the same as under certainty, that is, that certainty equivalence applies. Second, the optimal weights put on

the noisy observations, the *indicators*, used in constructing the optimal estimate of the output gap depends on the degree of uncertainty. For instance, when the degree of noise in an indicator of potential output is large, the optimal weight on that indicator becomes small.<sup>3</sup>

Section 2 presents a relatively general linear model of an aggregate private sector and a policy-maker, called the central bank, with a quadratic loss function. It then characterizes optimizing policy under discretion, demonstrates certainty equivalence, and derives the corresponding updating equation in the Kalman filter for the estimation problem. Section 3 does the same for the optimal policy with commitment.<sup>4</sup> Throughout the paper, we maintain the assumption of symmetric information between the private-sector and the central bank; a case with asymmetric information (for which certainty equivalence holds, but not the separation principle) is treated in Svensson and Woodford (2002c).

Section 4 discusses the interpretation of the Kalman filter. It shows how the Kalman filter can be modified to handle the simultaneity and circularity referred to above, and that the current estimate of the state of the economy can be expressed as a distributed lag of current and past observable variables, with the Kalman gain matrix providing the optimal weights on the observable variables.

Section 5 presents an example of optimal monetary policy in a simple forward-looking model, where inflation is forward-looking and depends on expectations of future inflation, on a partially observable output gap (the difference between observable output and a partially unobservable potential output), and on an unobservable “cost-push” shock. Since the observable rate of inflation both affects and depends on the current estimates of potential output and the cost-push shock, this example illustrates the gist of the estimation problem with forward-looking variables. Finally, section 6 presents some conclusions, while appendices A–B report some technical details.

## 2 Optimization under discretion

We consider a linear model of an economy with two agents, an (aggregate) private sector and a policymaker, called the central bank.<sup>5</sup>

The model is given by

$$\begin{bmatrix} X_{t+1} \\ \Omega x_{t+1|t} \end{bmatrix} = A^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + A^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + B i_t + \begin{bmatrix} u_{t+1} \\ 0 \end{bmatrix}, \quad (1)$$

where  $X_t$  is a vector of  $n_X$  *predetermined variables* in period  $t$ ,  $x_t$  is a vector of  $n_x$  *forward-looking variables*,  $i_t$  is (a vector of) the central bank's  $n_i$  *policy instrument(s)*,  $u_t$  is a vector of  $n_X$  iid shocks with mean zero and covariance matrix  $\Sigma_{uu}$ , and  $A^1$ ,  $A^2$ ,  $B$  and  $\Omega$  are matrices of appropriate dimension. The  $n_x \times n_x$  matrix  $\Omega$  (which should not be confused with the expectations operator  $E[\cdot]$ ) may be singular (this is a slight generalization of usual formulations when  $\Omega$  is the identity matrix). For any variable  $z_t$ ,  $z_{\tau|t}$  denotes  $E[z_{\tau}|I_t]$ , the rational expectation (the best estimate) of  $z_{\tau}$  given the information  $I_t$ , the information available in period  $t$  to the central bank. The information is further specified below. Let  $Y_t$  denote a vector of  $n_Y$  *target variables* given by

$$Y_t = C^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + C^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + C_i i_t, \quad (2)$$

where  $C^1$ ,  $C^2$  and  $C_i$  are matrices of appropriate dimension. Let the quadratic form

$$L_t = Y_t' W Y_t \quad (3)$$

be the central bank's period loss function, where  $W$  is a positive-semidefinite weight matrix.

Let the vector of  $n_Z$  *observable variables*,  $Z_t$ , be given by

$$Z_t = D^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + D^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + v_t, \quad (4)$$

where  $v_t$ , the vector of noise, is iid with mean zero and covariance matrix  $\Sigma_{vv}$ . The information  $I_t$  in period  $t$  is given by

$$I_t = \{Z_{\tau}, \tau \leq t; A^1, A^2, B, C^1, C^2, C_i, D^1, D^2, \Omega, W, \delta, \Sigma_{uu}, \Sigma_{vv}\}, \quad (5)$$

where  $\delta$  ( $0 < \delta < 1$ ) is a discount factor (to be introduced below). This incorporates the case when some or all of the predetermined and forward-looking variables are observable.<sup>6</sup>

Note that (1) assumes that the expectations  $x_{t+1|t}$  in the second block of equations are conditional on the information  $I_t$ . This corresponds to the case when the private sector and the central bank has the same information  $I_t$ , so information is assumed to be symmetric. The case of asymmetric information when these expectations are replaced by a private sector expectations  $E[x_{t+1}|I_t^P]$ , where the private-sector information  $I_t^P$  differs from  $I_t$ , is treated in Svensson and Woodford (2002c).

Assume first, that there is no commitment mechanism, so the central bank acts under discretion. Assume that central bank each period, conditional on the information  $I_t$ , minimizes the

expected discounted current and future values of the intertemporal loss function,

$$\mathbb{E}\left[\sum_{\tau=0}^{\infty} \delta^{\tau} L_{t+\tau} | I_t\right]. \quad (6)$$

As shown in Pearlman (1992) and in appendix A, *certainty equivalence* applies when the central bank and the private sector has the same information. Certainty equivalence means that the *estimation* of the partially observed state of the economy can be separated from the *optimization*, the setting of the instrument so as to minimize the intertemporal loss function.

The equilibrium under discretion will be characterized by the instrument being a linear function of the current estimate of the predetermined variables,

$$i_t = F X_{t|t}. \quad (7)$$

Furthermore, the estimate of the forward-looking variables will fulfill

$$x_{t|t} = G X_{t|t}, \quad (8)$$

where the matrix  $G$  by appendix A fulfills

$$G = (A_{22} - \Omega G A_{12})^{-1} [-A_{21} + \Omega G A_{11} + (\Omega G B_1 - B_2)F], \quad (9)$$

where

$$A \equiv A^1 + A^2, \quad (10)$$

the matrices  $A$ ,  $A^j$  ( $j = 1, 2$ ) and  $B$  are decomposed according to  $X_t$  and  $x_t$ ,

$$A^j = \begin{bmatrix} A_{11}^j & A_{12}^j \\ A_{21}^j & A_{22}^j \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

and we assume that the matrix  $A_{22} - \Omega G A_{12}$  is invertible. The matrices  $F$  and  $G$  depend on  $A$ ,  $B$ ,  $C \equiv C^1 + C^2$ ,  $C_i$ ,  $\Omega$ ,  $W$  and  $\delta$ , but (corresponding to the certainty equivalence referred to above) *not* on  $D^1$ ,  $D^2$ ,  $\Sigma_{uu}$  and  $\Sigma_{vv}$ .

Now, the lower block of (1) implies

$$A_{21}^1(X_t - X_{t|t}) + A_{22}^1(x_t - x_{t|t}) = 0. \quad (11)$$

Combining this with (8) and assuming that  $A_{22}^1$  is invertible gives

$$x_t = G^1 X_t + G^2 X_{t|t}, \quad (12)$$

where  $G^1$  and  $G^2$  fulfill

$$G^1 = -(A_{22}^1)^{-1}A_{21}^1, \quad (13)$$

$$G^2 = G - G^1. \quad (14)$$

The matrices  $G^1$  and  $G^2$  depend on  $G$  and  $A^1$ , hence also on  $B$ ,  $C \equiv C^1 + C^2$ ,  $C_i$ ,  $\Omega$ ,  $W$  and  $\delta$ , but (because of the certainty equivalence) they are independent of  $D^1$ ,  $D^2$ ,  $\Sigma_{uu}$  and  $\Sigma_{vv}$ .

It follows from (7) and (12) that the dynamics for  $X_t$  and  $Z_t$  follows

$$X_{t+1} = HX_t + JX_{t|t} + u_{t+1}, \quad (15)$$

$$Z_t = LX_t + MX_{t|t} + v_t, \quad (16)$$

where

$$H \equiv A_{11}^1 + A_{12}^1G^1, \quad (17)$$

$$J \equiv B_1F + A_{12}^1G^2 + A_{11}^2 + A_{12}^2G, \quad (18)$$

$$L \equiv D_1^1 + D_2^1G^1, \quad (19)$$

$$M \equiv D_2^1G^2 + D_1^2 + D_2^2G, \quad (20)$$

where  $D^j = [D_1^j \ D_2^j]$  ( $j = 1, 2$ ) is decomposed according to  $X_t$  and  $x_t$ . (Note that the matrix  $L$  in (19) should not be confused with the period loss function  $L_t$  in (3).)

We note that the problem of estimating the predetermined variables has been transformed to a problem without forward-looking variables, (15) and (16). This means that the estimation problem becomes a simpler variant of the estimation problem with forward-looking variables that is solved in Pearlman, Currie and Levine (1986). The derivations below is hence a simplification of that in that paper.<sup>7</sup>

## 2.1 Optimal filtering

Assume that the optimal prediction of  $X_t$  will be given by a Kalman filter,

$$X_{t|t} = X_{t|t-1} + K(Z_t - LX_{t|t-1} - MX_{t|t}), \quad (21)$$

where the Kalman gain matrix  $K$  remains to be determined. We can rationalize (21) by observing that  $Z_t - MX_{t|t} = LX_t + v_t$ , hence,

$$Z_t - LX_{t|t-1} - MX_{t|t} = L(X_t - X_{t|t-1}) + v_t,$$

so (21) can be written in the conventional form

$$X_{t|t} = X_{t|t-1} + K[L(X_t - X_{t|t-1}) + v_t], \quad (22)$$

which allows us to identify  $K$  as (one form of) the Kalman gain matrix.<sup>8</sup> From (15) we get

$$X_{t+1|t} = (H + J)X_{t|t}, \quad (23)$$

and the dynamics of the model are given by (15), (12), (22) and (23).

It remains to find an expression for  $K$ . Appendix B shows, by expressing the problem in terms of the prediction errors  $X_t - X_{t|t-1}$  and  $Z_t - Z_{t|t-1}$ , that  $K$  is given by

$$K = PL'(LPL' + \Sigma_{vv})^{-1}, \quad (24)$$

where the matrix  $P \equiv \text{Cov}[X_t - X_{t|t-1}]$  is the covariance matrix for the prediction errors  $X_t - X_{t|t-1}$  and fulfills

$$P = H[P - PL'(LPL' + \Sigma_{vv})^{-1}LP]H' + \Sigma_{uu}. \quad (25)$$

Thus  $P$  can be solved from (25), either numerically or analytically, depending upon the complexity of the matrices  $H$ ,  $L$  and  $\Sigma_{uu}$ . Then  $K$  is given by (24).

Note that (24) and (25) imply that  $K$  only depends on  $A^1$ ,  $D^1$ ,  $\Sigma_{uu}$  and  $\Sigma_{vv}$ , and hence is independent of  $C^1$ ,  $C^2$ ,  $C_i$ ,  $W$  and  $\delta$ . Thus,  $K$  is independent of the policy chosen. This demonstrates that the determination of the optimal policy given an estimate of the state of the economy and the estimation of the state of the economy can be treated as separate problems, as in the case without forward-looking variables treated in Chow (1975), Kalchenbrenner and Tinsley (1975) and LeRoy and Waud (1977). This is no longer true under asymmetric information, as demonstrated in Svensson and Woodford (2002c).

### 3 Optimal policy with commitment

Consider again the model described by equations (1)–(4), but suppose instead that the central bank commits itself in an initial *ex ante* state (prior to the realization of any period zero random variables) to a state-contingent plan for the indefinite future that minimizes the expected discounted losses

$$\mathbb{E} \left[ \sum_{t=t_0}^{\infty} \delta^t L_t \right].$$



Here  $E[\cdot]$  indicates the expectation with respect to information in the initial state in period  $t_0$ , in which the commitment is made. It is important to consider optimal commitment from such an *ex ante* perspective, because, in the case of partial information, the information that the central bank possesses in any given state depends upon the way that it has committed itself to behave in other states that might have occurred instead.

As shown in Pearlman (1992) for a slightly less general case, certainty equivalence applies in this case as well. A more intuitive proof of certainty equivalence is supplied in Svensson and Woodford (2002d). Svensson and Woodford (2002d) show that the optimal policy under commitment satisfies

$$i_t = FX_{t|t} + \Phi \Xi_{t-1}, \quad (26)$$

$$x_{t|t} = GX_{t|t} + \Gamma \Xi_{t-1}, \quad (27)$$

$$\Xi_t = SX_{t|t} + \Sigma \Xi_{t-1}, \quad (28)$$

for  $t \geq t_0$ , where  $F$ ,  $G$ ,  $S$ ,  $\Phi$ ,  $\Gamma$  and  $\Sigma$  are matrices of appropriate dimension, and  $\Xi_t$  is the vector of (the central bank's estimate of) the  $n_x$  Lagrange multiplier of the lower block of (1), the equations corresponding to the forward-looking variables. Furthermore,  $\Xi_{t_0-1} = 0$ . (The matrices  $F$  and  $G$  are normally different from the corresponding matrices under discretion.)

Woodford (1999b) and Svensson and Woodford (2002a) discuss a socially optimal equilibrium in a “timeless perspective,” which involves a stationary equilibrium corresponding to a commitment made far in the past, corresponding to  $t_0 \rightarrow -\infty$ . Then, (26)–(28) apply for all  $t > -\infty$ . Here, we consider this stationary equilibrium.

Note that (28) can then be solved backward to yield

$$\Xi_{t-1} = \sum_{\tau=0}^{\infty} \Sigma^\tau SX_{t-1-\tau|t-1-\tau}.$$

Thus, a fundamental difference with respect to the discretion case is that, under the optimal commitment,  $x_{t|t}$  is no longer a linear function of the current estimate of the predetermined variable alone,  $X_{t|t}$ , but instead depends upon past estimates  $X_{t-\tau|t-\tau}$  as well. The inertial character of optimal policy that this can result in is illustrated in Woodford (1999a) and (1999b) and in Svensson and Woodford (2002a).

Svensson and Woodford (2002a) also show that the socially optimal equilibrium can be achieved under discretion, if the intertemporal loss function in period  $t$  is modified to equal

$$E_t \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau} + \Xi_{t-1}(x_t - x_{t|t-1}). \quad (29)$$

That is, the central bank internalizes the cost of letting the forward-looking variables,  $x_t$ , deviate from previous expectations,  $x_{t|t-1}$ , using the Lagrange multiplier  $\Xi_{t-1}$  for (41) in period  $t-1$ , thus determined in the previous period, as a measure of that cost.<sup>9</sup>

As explained in detail in Svensson and Woodford (2002d), the matrices  $F$ ,  $G$ ,  $S$ ,  $\Phi$ ,  $\Gamma$  and  $\Sigma$  depend on  $A, B, C, C_i, W$  and  $\delta$ , but that they are independent of  $\Sigma_{uu}$ . Thus, these coefficients are the same as in the optimal plan under *certainty*. This is the *certainty equivalence* result for the case of partial information.

Using the same reasoning as in the derivation of (12) and substituting in (27) for  $x_{t|t}$ , we obtain

$$x_t = G^1 X_t + G^2 X_{t|t} + \Gamma \Xi_{t-1}, \quad (30)$$

where  $G^1$  and  $G^2$  again are given by (13) and (14). Again, the matrices  $G^1$  and  $G^2$ , like the others, are independent of the specifications of  $D$ ,  $\Sigma_{uu}$ , and  $\Sigma_{vv}$ .

Substitution of (26), (27) and (30) into the first row of (1) furthermore yields

$$X_{t+1} = H X_t + J X_{t|t} + \Psi \Xi_{t-1} + u_{t+1}, \quad (31)$$

where  $H$  and  $J$  are again given by (17) and (18), and

$$\Psi \equiv A_{12}\Gamma + B_1\Phi. \quad (32)$$

Equations (28) and (30)–(31) then describe the evolution of the predetermined and forward-looking variables,  $X_t$  and  $x_t$ , once we determine the evolution of the estimates  $X_{t|t}$  of the predetermined variables.

### 3.1 Optimal filtering

Substituting (30) into (4), we obtain

$$Z_t = L X_t + M X_{t|t} + \Lambda \Xi_{t-1} + v_t, \quad (33)$$

where  $L$  and  $M$  are again given by (19) and (20), and

$$\Lambda \equiv D_2\Gamma. \quad (34)$$

Equations (31) and (33) are then the transition and measurement equations for an optimal filtering problem. Again the transformation into a problem without forward-looking variables

allows us to derive the estimation equations in a manner that is simpler than that used in Pearlman, Currie and Levine (1986).

The optimal linear prediction of  $X_t$  is again given by a Kalman filter,

$$X_{t|t} = X_{t|t-1} + K(Z_t - LX_{t|t-1} - MX_{t|t} - \Lambda \Xi_{t-1}), \quad (35)$$

analogously to (21). From (31) we get

$$X_{t+1|t} = (H + J)X_{t|t} + \Psi \Xi_{t-1}, \quad (36)$$

and a complete system of dynamic equations for the model is then given by (28), (30), (31), (35) and (36).

It remains to find an expression for the Kalman gain matrix  $K$ . Again, as in appendix B, it is practical to work in terms of the prediction errors  $X_t - X_{t|t-1}$  and  $Z_t - Z_{t|t-1}$ , and equations (87)–(99) and (24)–(25) continue to apply, exactly as in the discretion case. Note that this implies that the Kalman gain matrix  $K$  is exactly the same matrix as in the discretion equilibrium; in fact, it depends only upon the matrices  $A^1$ ,  $\Sigma_{uu}$ ,  $D^1$  and  $\Sigma_{vv}$ .

## 4 Optimal weights on indicators: General remarks

In this section, we offer some general conclusions about the way in which the vector of observed variables  $Z_t$ , the indicators, is used to estimate the current state of the economy. As in sections 2 and 3, we assume that the central bank and the private sector have the same information, but our comments apply both to the discretion equilibrium and the commitment equilibrium. In either case, the observed variables matter only insofar as they affect the central bank's estimate  $X_{t|t}$  of the predetermined states.

Let us restate (4) and (33),

$$\begin{aligned} Z_t &= D_1^1 X_t + D_2^1 x_t + D_1^2 X_{t|t} + D_2^2 x_{t|t} + v_t \\ &= LX_t + MX_{t|t} + \Lambda \Xi_{t-1} + v_t, \end{aligned}$$

where we note that the second equation applies also in the discretion case, if we set  $\Lambda \equiv 0$  in that case. When  $D_2^1 \neq 0$ , the observable variables include or depend on the forward-looking variables. Then there is a contemporaneous effect of  $X_{t|t}$  on  $Z_t$ , due to the effect of  $X_{t|t}$  on both expectations  $x_{t+1|t}$  and the equilibrium choice of the instrument  $i_t$ . If  $D_1^2 \neq 0$ , there is a direct effect of  $X_{t|t}$  on the observable variables; if  $D_2^2 \neq 0$ , there is an effect of  $X_{t|t}$  on the observable

variables via  $x_{t|t}$ . In the commitment case, if  $\Lambda \neq 0$ , there is also a lagged effect, through the effect on  $\Xi_{t-1}$  of  $X_{t|t-j}$  on for  $j \geq 1$  (due to (28)), which in turn affects  $Z_t$  through its effect upon  $i_t$  and  $x_{t|t}$  (due to (26) and (27)).

In order to estimate  $X_t$  using a Kalman filter, we would like to find an indicator with the property that its innovation is a linear function of the forecast error,  $X_t - X_{t|t-1}$ , plus noise. The contemporaneous effect on  $Z_t$  means that its innovation does not meet this condition, since

$$Z_t - Z_{t|t-1} = L(X_t - X_{t|t-1}) + M(X_{t|t} - X_{t|t-1}) + v_t,$$

which also includes the terms  $M(X_{t|t} - X_{t|t-1})$  (we have used that  $\Xi_{t-1} = \Xi_{t-1|t-1}$ ). Thus, the contemporaneous effect enters via  $MX_{t|t}$ . In order to eliminate these effects of the estimated state upon the indicators, we might consider the vector of “ideal” indicators  $\bar{Z}_t$ , defined by the condition

$$\bar{Z}_t \equiv Z_t - MX_{t|t} - \Lambda \Xi_{t-1}, \quad (37)$$

where the contemporaneous effect is subtracted (the redundant component  $\Lambda \Xi_{t-1}$  is also subtracted to get a more parsimonious indicator). These ideal indicators then have the desired property that their innovation is a linear function of the forecast error of the predetermined variables plus noise,

$$\begin{aligned} \bar{Z}_t &= LX_t + v_t, \\ \bar{Z}_t - \bar{Z}_{t|t-1} &= L(X_t - X_{t|t-1}) + v_t. \end{aligned}$$

However, these ideal indicators do not provide an *operational* way of eliminating the contemporaneous influence. Indeed, (37) is only an *implicit* definition, in the sense that the estimates  $X_{t|t}$  that depend on the observable variables still enters into the identity and is assumed to be known. The ideal indicators can nonetheless provide a useful representation of the filtering problem for computational purposes, as we illustrate in the next section.

To get a recursive updating equation that is operational, we instead need one that only has current observable variables and previous estimates on the right side. We can use the prediction equation (35) ((21) in the discretion case) and solve for  $X_{t|t}$  to get

$$X_{t|t} = (I + KM)^{-1}[(I - KL)X_{t|t-1} - K\Lambda \Xi_{t-1} + KZ_t], \quad (38)$$

where the matrix  $I + KM$  must be invertible. We can then use (36) and (28) (where  $\Xi_{t-1} \equiv 0$

in the discretion case) to express the dynamic equation for  $X_{t|t}$  in terms of  $X_{t-1|t-1}$  and  $\Xi_{t-2}$ ,

$$\begin{aligned} X_{t|t} &= (I + KM)^{-1} \{ (I - KL) [(H + J)X_{t-1|t-1} + \Psi \Xi_{t-2}] - K\Lambda (SX_{t-1|t-1} + \Sigma \Xi_{t-2}) + KZ_t \} \\ &= (I + KM)^{-1} \{ [(I - KL)(H + J) - K\Lambda S]X_{t-1|t-1} + [(I - KL)\Psi - K\Lambda \Sigma]\Xi_{t-2} + KZ_t \}. \end{aligned} \quad (39)$$

Solving the system consisting of this equation and (28) backwards, we can express  $X_{t|t}$  as the weighted sum of current and past observable variables,

$$X_{t|t} = \sum_{\tau=0}^{\infty} Q_{\tau} K Z_{t-\tau}, \quad (40)$$

where the matrix  $Q_{\tau}$  is  $[(I + KM)^{-1}(I - KL)(H + J)]^{\tau}$  in the discretion case and the upper left submatrix of the matrix

$$\begin{bmatrix} (I + KM)^{-1}[(I - KL)(H + J) - K\Lambda S] & (I + KM)^{-1}[(I - KL)\Psi - K\Lambda \Sigma] \\ S & \Sigma \end{bmatrix}^{\tau}$$

in the commitment case. The consequence of the contemporaneous effect via the matrix  $M$  only shows up in the premultiplication of the matrix  $(I + KM)^{-1}$  above.

Thus, the evolution over time of the central bank's estimate of the predetermined states, and of the Lagrange multipliers needed to determine its action under the commitment equilibrium, can be expressed as a function of the observable variables. Furthermore, the Kalman gain matrix  $K$  gives the optimal weights on the vector of observable variables.. Row  $j$  of  $K$  gives the optimal weights in updating of element  $j$  of  $X_t$ . Column  $l$  of  $K$  gives the weights a particular observable variable  $Z_{lt}$  receives in updating the elements of  $X_t$ .

Since the estimate is a distributed lag of the observable variables, the estimate is updated only gradually. Thus, even under discretion, the observed policy will display considerable inertia, the more the noisier the current observables and the less the weight on current observations relative to previous estimates.

The elements of the Kalman gain matrix  $K$  depend upon the information structure (by (24) and (25) they depend on  $L$ , which by (19) depends on  $D^1$ , and on the covariance matrix  $\Sigma_{vv}$ ). They also depend on part of the dynamics of the predetermined variables (by (25), they depend on  $H$ , which by (17) and (13) depends only on  $A^1$ , and on the covariance matrix  $\Sigma_{uu}$ ). However, the elements of  $K$  are independent of the central-bank's objective, described by the matrices  $C^1$ ,  $C^2$ ,  $C_i$ ,  $W$  and the discount factor  $\delta$ , or, alternatively, of the central bank's reaction function

$(F, \Phi)$  in (26) (where  $\Phi = 0$  in the discretion case). This again illustrates the separation of the estimation problem from the optimization problem that arises under certainty equivalence.

Suppose that, in row  $j$  of  $L$ , only one element is nonzero, say element  $(j, j)$ . Then

$$Z_{jt} = X_{jt} + M_{j\cdot} X_{t|t} + \Lambda_{j\cdot} \Xi_{t-1} + v_{jt}$$

corresponds to an observation of  $X_{jt}$  with measurement error  $v_{jt}$  (we let  $j\cdot$  denote row  $j$  of a matrix, and we assume that element  $(j, j)$  of  $M$ ,  $m_{jj}$ , fulfills  $m_{jj} \neq -1$ ; this is now a necessary condition for the matrix  $I + KM$  to be invertible). Suppose the variance of the measurement error approaches zero. Then the elements of row  $j$  in the Kalman gain matrix will approach zero, except the element  $(j, j)$  which approaches unity. This corresponds to  $X_{jt}$  being fully observable, resulting in  $X_{jt|t} = X_{jt}$ . Suppose instead the variance of  $v_{jt}$  becomes unboundedly large. Then  $Z_{jt}$  is a useless indicator, and the Kalman gain matrix will assign a zero weight to this indicator; that is, all the elements in column  $j$  of  $K$  will be zero.

## 5 Example: Optimal monetary policy with unobservable potential output

Consider the following simple model, a variant of the model used, for example, in Clarida, Galí and Gertler (1999), Woodford (1999a,b) and Svensson and Woodford (2002a). The model equations are

$$\pi_t = \delta \pi_{t+1|t} + \kappa(y_t - \bar{y}_t) + \nu_t, \quad (41)$$

$$y_t = y_{t+1|t} - \sigma(i_t - \pi_{t+1|t}), \quad (42)$$

$$\bar{y}_{t+1} = \gamma \bar{y}_t + \eta_{t+1}, \quad (43)$$

$$\nu_{t+1} = \rho \nu_t + \varepsilon_{t+1}, \quad (44)$$

where  $\pi_t$  is inflation,  $y_t$  is (log) output,  $\bar{y}_t$  is (log) potential output (the natural rate of output),  $\nu_t$  is a serially correlated “cost-push” shock, and  $i_t$  is a one-period nominal interest rate (the central bank’s monetary-policy instrument). In our specification of the exogenous disturbance processes, the shocks  $\eta_t$  and  $\varepsilon_t$  are iid with means zero and variances  $\sigma_\eta^2$  and  $\sigma_\varepsilon^2$ , and the autoregressive coefficients  $\gamma$  and  $\rho$  satisfy  $0 \leq \gamma, \rho < 1$ . In our structural equations, the coefficient  $\delta$  satisfying  $0 < \delta < 1$  is also the discount factor for the central bank’s loss function, and the coefficients  $\kappa$  and  $\sigma$  are positive.<sup>10</sup>

We assume a period loss function of the kind associated with flexible inflation targeting with a zero inflation target,<sup>11</sup>

$$L_t = \frac{1}{2}[\pi_t^2 + \lambda(y_t - \bar{y}_t)^2], \quad (45)$$

where  $\lambda > 0$ . We assume that there is an imperfect observation,  $\tilde{y}_t$ , of potential output,

$$\tilde{y}_t = \bar{y}_t + \theta_t, \quad (46)$$

where the measurement error  $\theta_t$  is iid with zero mean and variance  $\sigma_\theta^2$ . We also assume that inflation is directly observable. Then the vector of observables is

$$Z_t = \begin{bmatrix} \bar{y}_t + \theta_t \\ \pi_t \end{bmatrix}. \quad (47)$$

Since we assume that there are no unobservable shocks in the aggregate-demand equation, (42), in equilibrium output will be perfectly controllable. Then, we can consider a simplified variant of your model, with output as the control variable and consisting of the equations (41), (43) and (44). For the resulting equilibrium stochastic processes for  $y_t$ ,  $y_{t+1|t}$  and  $\pi_{t+1|t}$ , we can then use the aggregate-demand equation to infer the corresponding interest rates according to

$$i_t = \pi_{t+1|t} + \frac{1}{\sigma}(y_{t+1|t} - y_t). \quad (48)$$

We can now rewrite the model (41), (43) and (44) in the form (1):

$$\begin{bmatrix} \bar{y}_{t+1} \\ \nu_{t+1} \\ \pi_{t+1|t} \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \rho & 0 \\ \kappa/\delta & -1/\delta & 1/\delta \end{bmatrix} \begin{bmatrix} \bar{y}_t \\ \nu_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\kappa/\delta \end{bmatrix} y_t + \begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \\ 0 \end{bmatrix}. \quad (49)$$

Here  $X_t$  is the vector  $[\bar{y}_t \ \nu_t]'$ ,  $x_t$  is just the scalar  $\pi_t$ , and we let thin lines denote the decomposition of  $A^1$  and  $B$  into its submatrices. We note that  $\Omega = 1$  and  $A^2 = 0$ . We can write the equation for the observables (4) as

$$Z_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_t \\ \nu_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \theta_t \\ 0 \end{bmatrix},$$

which allows us to identify  $D^1$  and  $v_t$ . Once again the thin lines denote the decomposition of  $D^1$  into  $D_1^1$  and  $D_2^1$ . We observe that  $D^2 = 0$ .

In this model, the central bank needs to form an estimate of the current potential output and cost-push shock,  $\bar{y}_{t|t}$  and  $\nu_{t|t}$ , in order to set policy, the output level  $y_t$ . It observes an

imperfect measure of potential output,  $\tilde{y}_t$ , and inflation,  $\pi_t$ , exactly. Since potential output is predetermined and independent both of current expectations and of the current instrument setting, noisy observation of it does not raise any special problems. In contrast, the observed inflation is here a forward-looking variable, which depends both on current expectations of future inflation and the current instrument setting. Current expectations and the instrument setting, furthermore, depend on the estimates of both current potential output and the current cost-push shock. These depend on the observation of inflation, completing the circle. Thus the central bank must sort through this simultaneity problem. Consequently our special case, in spite of its simplicity, incorporates the gist of the signal-extraction problem with forward-looking variables.

### 5.1 Equilibrium under discretionary optimization and under an optimal commitment

Due to the certainty equivalence, in order to find the optimal policy, we can directly apply the solution of the full-information version of this model in Clarida, Galí and Gertler (1999) and Svensson and Woodford (2002a). Under discretionary optimization, the solution is<sup>12</sup>

$$\begin{aligned} y_t &= \bar{y}_{t|t} - \frac{\kappa}{\kappa^2 + \lambda(1 - \delta\rho)} \nu_{t|t}, \\ \pi_t &= \frac{\lambda}{\kappa^2 + \lambda(1 - \delta\rho)} \nu_{t|t} \end{aligned}$$

(where  $\pi_t = \pi_{t|t}$  since inflation by assumption is directly observable). Under an optimal commitment, the solution is<sup>13</sup>

$$y_t = \bar{y}_{t|t} - \frac{\kappa}{\lambda} \frac{\mu}{1 - \delta\rho\mu} \nu_{t|t} - \frac{\kappa}{\lambda} \mu \Xi_{t-1}, \quad (50)$$

$$\pi_{t|t} = \frac{\mu}{1 - \delta\rho\mu} \nu_{t|t} - (1 - \mu) \Xi_{t-1}, \quad (51)$$

$$\Xi_t = \frac{\mu}{1 - \delta\rho\mu} \nu_{t|t} + \mu \Xi_{t-1}. \quad (52)$$

In the commitment case,  $\Xi_t$  is the Lagrange multiplier of the constraint corresponding to (41) (the last row of (49)), and  $\mu$  ( $0 < \mu < 1$ ) is a root of the characteristic equation of the difference equation for  $\Xi_t$  that results from substitution of the first-order conditions into (41).

### 5.2 An optimal targeting rule

The above characterization of the optimal commitment allows us to derive a simple *targeting rule*, a rule for the central bank's target variables  $\pi_t$  and  $y_t$ , which represents one practical



approach to the implementation of optimal policy, as discussed in Svensson and Woodford (2002a). By (50) and (52), we have

$$y_t - \bar{y}_{t|t} = -\frac{\kappa}{\lambda} \Xi_t, \quad (53)$$

and by (51) and (52), we have

$$\pi_t = \Xi_t - \Xi_{t-1}. \quad (54)$$

These are just the first-order conditions under commitment, the combination of which with the dynamic equations (41), (43) and (44) then result in (50)–(52). We can furthermore eliminate the Lagrange multipliers from (53) and (54) and get a consolidated first-order condition,

$$\pi_t = -\frac{\lambda}{\kappa} [(y_t - \bar{y}_{t|t}) - (y_{t-1} - \bar{y}_{t-1|t-1})]. \quad (55)$$

In the full-information case,  $\bar{y}_t$  and  $\bar{y}_{t-1}$  would be substituted for  $\bar{y}_{t|t}$  and  $\bar{y}_{t-1|t-1}$  in (55). As discussed in detail in Svensson and Woodford (2002a), the full-information analogue of (55) can be interpreted as a targeting rule, which if followed by the central bank will result in the full social optimum under commitment (when the intertemporal loss function with the period loss function (45) is interpreted as the social loss function). Thus, inflation should be adjusted to equal the negative change in the output gap, multiplied by the factor  $\lambda/\kappa$ .

This targeting rule is remarkable in that it only depends on the relative weight on output-gap stabilization in the loss function,  $\lambda$ , and the slope of the short-run Phillips curve,  $\kappa$ . In particular, the targeting rule is robust to the number and stochastic properties of additive shocks to the aggregate-supply equation (as witnessed by the lack of dependence on the AR(1) coefficient of the cost-push shock,  $\rho$ , and the variances of the iid shock,  $\sigma_\varepsilon^2$ ) and (as long as the interest rate does not enter the loss function) completely independent of the aggregate-demand equation (42).

An alternative formulation of the targeting rule is in terms of a target for the price level, rather than the inflation rate. We observe that (55) implies that

$$p_t - p^* = -\frac{\lambda}{\kappa} (y_t - \bar{y}_{t|t}), \quad (56)$$

where  $p_t$  is the (log) price level ( $\pi_t \equiv p_t - p_{t-1}$ ) and  $p^*$  is a constant that can be interpreted as an implicit price-level target. Similarly, (56) implies (55), so these are equivalent targeting rules, each equally consistent with the optimal commitment. (It is worth noting that under our informational assumptions,  $p_t$  is also public information at date  $t$ .) This illustrates the close relation between inflation targeting under commitment and price-level targeting, further

discussed in Vestin (2000), Svensson (1999a,d) and Woodford (1999a,b). We also note that under the optimal commitment, the Lagrange multipliers satisfy

$$\Xi_t = p_t - p^*. \quad (57)$$

This is useful below as an empirical proxy for variation in the Lagrange multipliers.

An interesting feature of both of these characterizations of optimal policy is that, under partial information, the targeting rule has *exactly* the same form as under full information, except that the *estimated* output gap,  $y_t - y_{t|t}$ , is consistently substituted for the *actual* output gap,  $y_t - \bar{y}_t$ . Thus, policy should respond to exactly the same extent to the estimated output gap under partial information as to the actual output gap under full information. This is a clear illustration of the certainty equivalence result demonstrated earlier in the paper.

However, it is important to note that the targeting rules (55) and (56) are written in terms of the *optimal estimate* of the output gap,  $y_t - \bar{y}_{t|t}$ , *not* in terms of the output-gap measure  $y_t - \tilde{y}_t$  implied by the *imperfect observation* of potential output,  $\tilde{y}_t$ . As we shall see, the optimal degree of response to an imperfect observation of the output gap does indeed depend on the degree of noise in the observation.

### 5.3 An optimal instrument rule (in terms of the optimal estimate of the predetermined variables)

We now consider instead the nature of an optimal *instrument rule*, specifying how the central bank's instrument, the nominal interest rate  $i_t$ , should be set each period, both as a function of the optimal estimate of the predetermined variables (corresponding to (7) or (26)) and as a function of the observations of the observable variables up through the current period (corresponding to (7) or (26) when (40) are substituted for  $X_{t|t}$ ). We shall give particular attention to the question of how the coefficients of such a rule are affected by the presence of measurement error in the observable measure of the output gap,  $\tilde{y}_t$ .

We first must compute the evolution of the nominal interest rate under the optimal commitment characterized above. We recall that output and inflation evolve according to equations of the form

$$y_t = \bar{y}_{t|t} + f\nu_{t|t} + \Phi\Xi_{t-1}, \quad (58)$$

$$\pi_t = g\nu_{t|t} + \Gamma\Xi_{t-1}, \quad (59)$$

where the coefficients  $f, g, \Phi$  and  $\Gamma$  are identified in (50) and (51). Substituting these solutions for output and inflation into (48), and using (43) and (44) to forecast  $\bar{y}_{t+1|t}$  and  $\nu_{t+1|t}$  as multiples of  $\bar{y}_{t|t}$  and  $\nu_{t|t}$ , we obtain

$$i_t = -\frac{1}{\sigma}(1-\gamma)\bar{y}_{t|t} + [\rho g - (1-\rho)f]\nu_{t|t} + [\Gamma + \frac{1}{\sigma}\Phi]\Xi_t - \frac{1}{\sigma}\Phi\Xi_{t-1}. \quad (60)$$

It remains to express the variables on the right-hand side of (60) in terms of observables.

We next recall that in the optimal equilibrium, the values of  $\Xi_t$  and  $\nu_{t|t}$  can be inferred from the (observable) path of the price level, using (57) and inverting (59) to obtain

$$\begin{aligned} \nu_{t|t} &= \frac{1}{g}(\pi_t - \Gamma\Xi_{t-1}) \\ &= \frac{1}{g}(p_t - p^*) - \frac{1}{g}(1+\Gamma)(p_{t-1} - p^*). \end{aligned} \quad (61)$$

Substituting (57) and (61) into (60), we obtain

$$i_t = -\vartheta\bar{y}_{t|t} + \mu_0(p_t - p^*) + \mu_1(p_{t-1} - p^*), \quad (62)$$

where

$$\begin{aligned} \vartheta &\equiv \frac{1-\gamma}{\sigma} > 0, \\ \mu_0 &\equiv (\rho + \mu - 1) - (\rho + \frac{\mu}{\sigma} - 1)\frac{\kappa}{\lambda}, \\ \mu_1 &\equiv -\mu \left[ \rho - (\rho + \frac{\mu}{\sigma} - 1)\frac{\kappa}{\lambda} \right]. \end{aligned}$$

It still remains, however, to express  $\bar{y}_{t|t}$  as a function of the observables. This requires consideration of an optimal filtering problem.

Note that if we were content to derive an instrument rule in terms of the *optimal estimate* of potential output,  $\bar{y}_{t|t}$ , rather than the *noisy observation* of that variable,  $\tilde{y}_t$ , then (62) would serve. In this case, certainty equivalence applies once again; one observes that the coefficients  $\vartheta, \mu_0$ , and  $\mu_1$  are all independent of the degree of noise in the observation of potential output. However, the evolution of the optimal estimate  $\bar{y}_{t|t}$  as a function of the observables *does* depend upon the degree of noise in the observation  $\tilde{y}_t$ .

## 5.4 The filtering problem

We turn next to that filtering problem, which requires us to determine the law of motion for  $X_{t|t}$ , a problem treated in sections 3 and 4. Note that equations (50) and (51) have already allowed

us to identify the matrices  $F$ ,  $\Phi$ ,  $G$ ,  $\Gamma$ ,  $S$  and  $\Sigma$  in (26)–(28). We are then able to compute the matrices

$$\begin{aligned} G^1 &= \begin{bmatrix} -\kappa & 1 \end{bmatrix}, \quad G^2 = \begin{bmatrix} \kappa & g-1 \end{bmatrix}, \\ H &= \begin{bmatrix} \gamma & 0 \\ 0 & \rho \end{bmatrix}, \quad J = 0, \quad L = \begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix}, \end{aligned} \quad (63)$$

$$M = \begin{bmatrix} 0 & 0 \\ \kappa & g-1 \end{bmatrix}, \quad \Psi = 0, \quad \Lambda = \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}. \quad (64)$$

As discussed in section 4, the updating equation takes the simple form

$$\begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} = \begin{bmatrix} \bar{y}_{t|t-1} \\ \nu_{t|t-1} \end{bmatrix} + K [\bar{Z}_t - \bar{Z}_{t|t-1}]$$

in terms of the *ideal indicators*  $\bar{Z}_t$  given by

$$\bar{Z}_t \equiv \begin{bmatrix} \bar{y}_t + \theta_t \\ \pi_t \end{bmatrix} - M \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} - \Lambda(p_{t-1} - p^*) = \begin{bmatrix} \bar{y}_t + \theta_t \\ -\kappa\bar{y}_t + \nu_t \end{bmatrix}. \quad (65)$$

Thus, the filtering problem may be reduced to one of observing a noisy measure of potential output,  $\bar{y}_t + \theta_t$ , along with a linear combination of potential output and the cost-push shock,  $-\kappa\bar{y}_t + \nu_t$ . That observation of the forward-looking inflation rate implies the observability of this linear combination of the potential output and cost-push shock is quite intuitive. From the aggregate supply equation (41) we see that, in equilibrium, observability of  $\pi_t$ ,  $\pi_{t+1|t}$  and  $y_t$  implies that the remainder,  $-\kappa\bar{y}_t + \nu_t$ , must be observable as well.

The ideal indicators are not operational, as their definition above involves  $\bar{y}_{t|t}$  and  $\nu_{t|t}$ , which we seek to determine. However, consideration of the simple problem that would result if these indicators were available is useful as a way of determining the Kalman gain matrix  $K$ . This estimation problem consists of the simple transition equation,

$$\begin{bmatrix} \bar{y}_{t+1} \\ \nu_{t+1} \end{bmatrix} = H \begin{bmatrix} \bar{y}_t \\ \nu_t \end{bmatrix} + \begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \end{bmatrix}, \quad (66)$$

where  $H$  is given by (63), and the measurement equation (65). The transition equation is so simple in the present case because the predetermined variables  $\bar{y}_t$  and  $\nu_t$  are exogenous; that is,  $A_{12}^1 = 0$ ,  $A_{11}^2 = 0$ ,  $A_{12}^2 = 0$ ,  $B_1 = 0$ .

In appendix C of the working-paper version of this paper, Svensson and Woodford (2002b), we derive an analytical expression for the Kalman gain matrix,

$$K \equiv \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \quad (67)$$

as a function of the coefficients  $\kappa$ ,  $\gamma$ ,  $\rho$  and the variances  $\sigma_\eta^2$ ,  $\sigma_\varepsilon^2$  and  $\sigma_\theta^2$ . Svensson and Woodford (2002b, appendix C) furthermore shows that the elements of  $K$  satisfy

$$0 < k_{11} < k_{22} < 1, \quad (68)$$

$$k_{12} \equiv \kappa k_{11} < 0, \quad k_{21} \equiv \kappa k_{12} + 1 > 0. \quad (69)$$

Note that these elements depend on the variances of the shocks. In particular, we can examine how the response to the noisy measure of potential output  $\tilde{y}_t$  depends on the degree of noise in this measure, i.e., the variance of the measurement error  $\sigma_\theta^2$ . In Svensson and Woodford (2002b, appendix C), we show that, in the limit as  $\sigma_\theta^2 \rightarrow \infty$ ,

$$k_{11}, k_{21} \rightarrow 0, \quad k_{12} \rightarrow k_{12}^* < 0, \quad k_{22} \rightarrow k_{22}^* > 0. \quad (70)$$

Thus, the optimal weights on the measure of potential output go to zero when its information content goes to zero. This is an example of the Kalman filter assigning zero weight to useless indicators, mentioned in section 4. Again, this does not mean that the response to the optimal estimate of potential output,  $\bar{y}_{t|t}$ , changes. By certainty equivalence, it stays the same. It is only that the *measure* of potential output  $\tilde{y}_t$  is disregarded in the construction of the optimal estimate. Instead, in this case the central bank will rely only on the observed inflation rate in estimating potential output.

Having determined the Kalman gain matrix  $K$ , we may return to the consideration of an operational procedure for computing the optimal estimates of the underlying exogenous disturbances. For this the central bank can use the operational recursive updating equations (39), which in the present case may be written

$$\begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} = (I + KM)^{-1} \left( (I - KL)H \begin{bmatrix} \bar{y}_{t-1|t-1} \\ \nu_{t-1|t-1} \end{bmatrix} - K\Lambda(p_{t-1} - p^*) + KZ_t \right). \quad (71)$$

This last equation is simpler than (39), because in our example  $J = 0$  and  $\Psi = 0$ .

Furthermore, in writing the updating equation for  $\bar{y}_{t|t}$ , we can use the fact that we have already solved for  $\nu_{t|t}$  as a function of the history of the price level, in (61). The updating

equation for  $\bar{y}_{t|t}$  then takes the simple form

$$\bar{y}_{t|t} = \omega \bar{y}_{t-1|t-1} + \alpha_0 (p_t - p^*) + \alpha_1 (p_{t-1} - p^*) + \zeta \tilde{y}_t, \quad (72)$$

where the coefficients are defined in Svensson and Woodford (2002b, appendix D). It is shown there in particular that

$$\omega \equiv \gamma \frac{k_{22} - k_{11}}{k_{22}}, \quad \zeta \equiv \frac{k_{11}}{k_{22}}, \quad (73)$$

from which it follows (using (68)) that

$$0 < \omega < \gamma, \quad 0 < \zeta < 1.$$

## 5.5 Optimal instrument rules in terms of observable variables

Let us finally derive optimal instrument rules in terms of observations of the observable variables up through the current period. Solving (62) for  $\bar{y}_{t|t}$  as a function of  $i_t$  and the history of the price index, and substituting that expression for  $\bar{y}_{t|t}$  in (72), we obtain a law of motion for interest rates of the form

$$i_t = \omega i_{t-1} + \sum_{j=0}^2 \beta_j (p_{t-j} - p^*) - \xi \tilde{y}_t \quad (74)$$

where the coefficients are given in Svensson and Woodford (2002b, appendix D). This can be interpreted as an instrument rule for setting the nominal interest rate as a function of the observables, namely the price level and the potential output measure  $\tilde{y}_t$ .

As is explained further in Svensson and Woodford (2002b, appendix D), the coefficients of rule (74) depend upon the variances of the various disturbances, so that certainty equivalence does not apply to this particular characterization of optimal policy. In particular, it is shown that

$$\xi \equiv \vartheta \zeta > 0.$$

It then follows from (70) and (73) that in the limit as  $\sigma_\theta^2 \rightarrow \infty$ ,  $\zeta \rightarrow 0$ , from which it follows that  $\xi \rightarrow 0$  as well. On the other hand, it is shown in Svensson and Woodford (2002b, appendix D) that the  $\beta_j$  coefficients remain bounded away from zero. Thus in the limiting case of an extremely noisy measure of potential output, the optimal instrument rule makes the interest rate solely a function of the history of the price level.

The instrument rule (74) makes the interest rate a function solely of the history of the price level, and of the measure of potential output, as these have been assumed to be the only observables. However, we may consider the question of how the interest rate should depend on a

measure of the output *gap*, if we now assume instead that real output  $y_t$  is directly observable as well. This assumption implies no change in the information structure, since (42) already implies that  $y_t$  must belong to the period- $t$  information set (in the sense that its value can be inferred precisely from the variables that are directly observed, even if it is not itself directly observed). However, adding  $y_t$  to the list of period- $t$  observables does make possible a more flexible class of explicit instrument rules, as interest rates may now be made a function of the history of the price level, of the potential-output measure, or of the level of output.

In this case, there will no longer be a unique optimal instrument rule. The rule (74) would still count as *one* explicit rule that is consistent with the optimal equilibrium; but because of the presence of a redundant observable, there would no longer be a unique such rule, as rules consistent with the same equilibrium could also be constructed with any given degree of dependence upon the observation of  $y_t$ . One way of resolving this indeterminacy is to assume that the rule should depend only upon the history of prices and of the “direct” measure of the output gap, namely, the quantity  $y_t - \tilde{y}_t$ , rather than allowing for any independent response to either  $y_t$  or  $\tilde{y}_t$ .

It is shown in Svensson and Woodford (2002b, appendix D) that the unique optimal instrument rule of this kind is of the form

$$i_t = \gamma i_{t-1} + \sum_{j=0}^2 \tilde{\beta}_j (p_{t-j} - p^*) + \tilde{\xi} (y_t - \tilde{y}_t), \quad (75)$$

where the coefficients are defined in (2002a, appendix D). We note in particular that

$$\tilde{\xi} \equiv \lambda \frac{\zeta}{1 - \zeta} > 0.$$

Thus in the case that we do not allow separate responses to current output and potential output, it is optimal for the interest rate to be an increasing function of the current direct measure of the output gap.

Once again, one observes that in the limit as  $\sigma_\theta^2 \rightarrow \infty$ ,  $\zeta \rightarrow 0$ , so that the optimal coefficient  $\tilde{\xi} \rightarrow 0$ . Extreme noise in the measure of potential output implies that the measure of the output gap based on it is similarly noisy, and so it is optimal not to respond to such an indicator at all. Instead, in the limit it is again optimal to respond only to the history of the price level.

## 6 Conclusions

In this paper, we have restated the important result that, under symmetric partial information, certainty equivalence and the separation principle continue to hold in the case of linear rational-expectations models and a quadratic loss function. Then optimal policy as a function of the current estimate of the state of the economy is the same as if the state were observed.

However, policy as a function of the observable variables (and the actual, as distinct from the estimated, state of the economy) will display considerable inertia, since the current estimate will be a distributed lag of the current and past observable variables (and actual states of the economy). Thus, discretionary policy—which, as discussed in Woodford (1999a,b) and Svensson and Woodford (2002a), often lacks the history-dependence that characterizes optimal policy under commitment—will in this case display a certain inertial character as a consequence of partial information. It seems likely that this inertial character will be more pronounced the noisier the information in the observable variables, as this should lead to slower updating of the current estimate of the state of the economy. To what extent this may affect the welfare comparison between discretionary policy and the optimal policy under commitment (which represents the social optimum) is a topic for future research.

Even given certainty equivalence and the separation principle, the estimation problem with forward-looking observable variables presents a challenge, due to the circularity in the way that the observable variables both affect and depend on the current estimate. The optimal operational Kalman filter under these circumstances needs to be modified to circumvent that circularity, as we have shown.

Our results have been derived under the assumption of symmetric information between the central bank and the aggregate private sector, as a result of which certainty equivalence and the separation between optimization and estimation hold. This case seems to us to be of practical interest, since we believe that any informational advantage of central banks consists mainly of better information about their own intentions (as in the papers of Cukierman and Meltzer (1986) and Faust and Svensson (1999)). Any such private information is nowadays increasingly being eroded by the general tendency toward increased transparency in monetary policy, whether willingly adopted by the central banks or, in some cases, forced upon them by irresistible outside demands. Nevertheless, it is of interest to understand how these results are modified when there is asymmetric information (especially in the direction of central banks having less information than parts of the private sector); this topic is taken up in Svensson and Woodford (2002b).



We have illustrated our general results in terms of a forward-looking model of monetary policy with unobservable potential output and a partially observable cost-push shock, where the observable variables both affect and depend on the current estimates of potential output and the cost-push shock. This situation is obviously highly relevant for many central banks, including the recently established Eurosystem. We note that our analysis of optimal policy does imply an important role for an estimate of current potential output, and that the proper weight to be put on such an *estimate* under an optimal policy rule is unaffected by the degree of noise in available measures of potential output. Thus the lack of more accurate measures is not a reason for policy to respond less to optimally estimated fluctuations in the output gap (though inaccuracy of particular indicators can be a reason for a bank's estimate of the output gap to be less influenced by those indicators). This is an important qualification of Orphanides's (2000) suggestion that monetary policy should not respond to uncertain real-time output-gap data. If those data represent an optimal estimate of the output gap, then the optimal response should be the same as under certainty. If instead the data represent only a noisy observation of the true output gap, the optimal response will indeed depend on the degree of noise.<sup>14</sup>

Thus in the case of pure indicator variables—variables that are neither target variables (variables that enter the loss function) nor direct causal determinants of target variables, and that accordingly would not be responded to under an optimal policy in the case of full information—the degree to which monetary policy should take account of them is definitely dependent upon how closely they are in fact associated with the (causal) state variables that one seeks to estimate. This precept does not always play as large a role in current central banking practice as it might.

As an example, the Eurosystem has put special emphasis on one particular indicator, the growth of Euro-area M3 relative to a reference value of 4.5 percent per year, elevating this money-growth indicator to the status of one of two “pillars” of the Eurosystem monetary strategy (in addition to “a broadly-based assessment of the outlook for future price developments”).<sup>15</sup> Money growth in excess of the reference value is supposed to indicate “risks to price stability.” As discussed by commentators such as Svensson (1999c), Rudebusch and Svensson (1999) and Gerlach and Svensson (2002), it is difficult to find rational support for this prominence of the money-growth indicator. Instead, monetary aggregates would seem to be properly viewed as just one set of indicators among many others, the relative weight on which should exclusively depend on their performance in predicting the relevant aspects of the current state of the economy; more

specifically, how useful current money growth is as an input in conditional forecasts of inflation some two years ahead.

Under normal circumstances, the information content of money growth for inflation forecasts in the short and medium term seems to be quite low.<sup>16</sup> Only in the long run does a high correlation between money growth and inflation result. Under the special circumstances of the introduction of a new common currency, the demand for money is likely to be quite unpredictable and possibly very unstable, since important structural changes are likely to occur in financial markets and banking. Under such circumstances, the information content of money is likely on theoretical grounds to be even lower than under normal circumstances. Thus the uncertainty associated with the introduction of the new currency should provide an argument for relying less, rather than more, on monetary aggregates as indicators.<sup>17</sup>

## A Optimization under discretion and certainty equivalence

Consider the decision problem to choose  $i_t$  in period  $t$  to minimize (6) (with  $0 < \delta < 1$ ) under discretion, that is, subject to (1)–(5) and

$$i_{t+1} = F_{t+1}X_{t+1|t+1} \quad (76)$$

$$x_{t+1|t+1} = G_{t+1}X_{t+1|t+1}, \quad (77)$$

where  $F_{t+1}$  and  $G_{t+1}$  are determined by the decision problem in period  $t + 1$ .

For the full information case, Oudiz and Sachs (1985) have derived an algorithm for the discretionary equilibrium, which is further discussed in Backus and Driffill (1986) and Currie and Levin (1993).<sup>18</sup> Following Pearlman (1992), but with a more explicit proof, this appendix shows that this algorithm, appropriately adapted, is valid also for the partial-information case.

First, using (77), taking expectations in period  $t$  of the upper block of (1), and using (10), we get

$$x_{t+1|t} = G_{t+1}X_{t+1|t} = G_{t+1}(A_{11}X_{t|t} + A_{12}x_{t|t} + B_1i_t). \quad (78)$$

Taking the expectation in period  $t$  of the lower block of (1), we get

$$\Omega x_{t+1|t} = A_{21}X_{t|t} + A_{22}x_{t|t} + B_2i_t \quad (79)$$

(recall that  $\Omega$  is a matrix and not the expectations operator). Multiplying (78) by  $\Omega$ , setting the result equal to (79) and solving for  $x_{t|t}$  gives

$$x_{t|t} = \tilde{A}_t X_{t|t} + \tilde{B}_t i_t, \quad (80)$$

where

$$\begin{aligned}\tilde{A}_t &\equiv (A_{22} - \Omega G_{t+1} A_{12})^{-1}(\Omega G_{t+1} A_{11} - A_{21}), \\ \tilde{B}_t &\equiv (A_{22} - \Omega G_{t+1} A_{12})^{-1}(\Omega G_{t+1} B_1 - B_2)\end{aligned}$$

(we assume that  $A_{22} - \Omega G_{t+1} A_{12}$  is invertible). Using (80) in the expectation of the upper block of (1) then gives

$$X_{t+1|t} = A_t^* X_{t|t} + B_t^* i_t, \quad (81)$$

where

$$\begin{aligned}A_t^* &\equiv A_{11} + A_{12} \tilde{A}_t, \\ B_t^* &\equiv B_1 + A_{12} \tilde{B}_t.\end{aligned}$$

Second, by (2) and (3) we can write

$$L_{t|t} = \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix}' Q \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + 2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix}' U i_t + i_t' R i_t + l_t, \quad (82)$$

where

$$\begin{aligned}C &\equiv C^1 + C^2, \quad Q \equiv C' W C, \quad U \equiv C' W C_i, \quad R \equiv C_i' W C_i \\ l_t &\equiv E \left\{ \begin{bmatrix} X_t - X_{t|t} \\ x_t - x_{t|t} \end{bmatrix}' C^1 W C^1 \begin{bmatrix} X_t - X_{t|t} \\ x_t - x_{t|t} \end{bmatrix} \middle| I_t \right\}.\end{aligned} \quad (83)$$

Using (80) in (82) leads to

$$L_{t|t} = X_{t|t}' Q_t^* X_{t|t} + 2 X_{t|t}' U_t^* i_t + i_t' R_t^* i_t + l_t, \quad (84)$$

where

$$\begin{aligned}Q_t^* &\equiv Q_{11} + Q_{12} \tilde{A}_t + \tilde{A}_t' Q_{21} + \tilde{A}_t' Q_{22} \tilde{A}_t, \\ U_t^* &\equiv Q_{12} \tilde{B}_t + \tilde{A}_t' Q_{22} \tilde{B}_t + U_1 + \tilde{A}_t' U_2, \\ R_t^* &\equiv R + \tilde{B}_t' Q_{22} \tilde{B}_t + \tilde{B}_t' U_2 + U_2' \tilde{B}_t,\end{aligned}$$

and  $Q$  and  $U$  are decomposed according to  $X_{t|t}$  and  $x_{t|t}$ .

Third, since the loss function is quadratic and the constraints are linear, it follows that the optimal value of the problem will be quadratic. In period  $t+1$  the optimal value will depend on the estimate  $X_{t+1|t+1}$  and can hence be written  $X_{t+1|t+1}' V_{t+1} X_{t+1|t+1} + w_{t+1}$ , where  $V_{t+1}$  is

a positive semidefinite matrix and  $w_{t+1}$  is a scalar. Then the optimal value of the problem in period  $t$  is associated with the positive semidefinite matrix  $V_t$  and the scalar  $w_t$ , and fulfills the Bellman equation

$$X'_{t|t} V_t X_{t|t} + w_t \equiv \min_{i_t} \left\{ L_{t|t} + \delta E[X'_{t+1|t+1} V_{t+1} X_{t+1|t+1} + w_{t+1} | I_t] \right\}, \quad (85)$$

subject to (81) and (84). Indeed, the problem has been transformed to a standard linear regulator problem without forward-looking variables, albeit in terms of  $X_{t|t}$  and with time-varying parameters. The first-order condition is, by (84) and (85),

$$\begin{aligned} 0 &= X'_{t|t} U_t^* + i'_t R_t^* + \delta E[X'_{t+1|t+1} V_{t+1} B_t^* | I_t] \\ &= X'_{t|t} U_t^* + i'_t R_t^* + \delta (X'_{t|t} A_t^{*'} + i'_t B_t^{*'}) V_{t+1} B_t^*. \end{aligned}$$

Here we have assumed that  $l_t$  is independent of  $i_t$ , which assumption is verified below. The first-order condition can be solved for the reaction function

$$i_t = F_t X_{t|t}, \quad (86)$$

where

$$F_t \equiv - (R_t^* + \delta B_t^{*'} V_{t+1} B_t^*)^{-1} (U_t^{*'} + \delta B_t^{*'} V_{t+1} A_t^*)$$

(we assume that  $R_t^* + \delta B_t^{*'} V_{t+1} B_t^*$  is invertible). Using (86) in (80) gives

$$x_{t|t} = G_t X_{t|t},$$

where

$$G_t \equiv \tilde{A}_t + \tilde{B}_t F_t.$$

Furthermore, using (86) in (85) and identifying gives

$$V_t \equiv Q_t^* + U_t^* F_t + F_t' U_t^{*'} + F_t' R_t^* F_t + \delta (A_t^* + B_t^* F_t)' V_{t+1} (A_t^* + B_t^* F_t).$$

Finally, the above equations define a mapping from  $(F_{t+1}, G_{t+1}, V_{t+1})$  to  $(F_t, G_t, V_t)$ . The solution to the problem is a fixpoint  $(F, G, V)$  of the mapping. It is obtained as the limit of  $(F_t, G_t, V_t)$  when  $t \rightarrow -\infty$ . The solution thus fulfills the corresponding steady-state matrix equations. Thus, the instrument  $i_t$  and the estimate of the forward-looking variables  $x_{t|t}$  will be linear functions, (7) and (8) of the estimate of the predetermined variables  $X_{t|t}$ , where the corresponding  $F$  and  $G$  fulfill the corresponding steady-state equations. In particular,  $G$  will fulfill (9).

It also follows that  $F$ ,  $G$  and  $V$  only depend on  $A \equiv A^1 + A^2$ ,  $B$ ,  $C \equiv C^1 + C^2$ ,  $C_i$ ,  $\Omega$ ,  $W$  and  $\delta$  and are independent of  $D^1$ ,  $D^2$ ,  $\Sigma_{uu}$  and  $\Sigma_{vv}$ . This demonstrates the certainty equivalence of the discretionary equilibrium.

It remains to verify the assumption that  $l_t$  in (83) is independent of  $i_t$ . Since by (12)–(13),  $x_t - x_{t|t} = -(A_{22}^1)^{-1}A_{21}^1(X_t - X_{t|t})$ , it is sufficient to demonstrate that  $E[(X_t - X_{t|t})(X_t - X_{t|t})'|I_t]$  is independent of  $i_t$ . By (22),

$$X_t - X_{t|t} = X_t - X_{t|t-1} + K[L(X_t - X_{t|t-1}) + v_t] = (I + KL)(X_t - X_{t|t-1}) + Kv_t.$$

Since  $X_t$  and  $X_{t|t-1}$  are predetermined and  $v_t$  is exogenous, the assumption is true.

## B The Kalman gain matrix and the covariance of the forecast errors

It is practical to express the dynamics in terms of the prediction errors of  $X_t$  and  $Z_t$ , relative to period  $t - 1$  information,

$$\begin{aligned}\tilde{X}_t &\equiv X_t - X_{t|t-1}, \\ \tilde{Z}_t &\equiv Z_t - Z_{t|t-1} = Z_t - (L + M)X_{t|t-1},\end{aligned}$$

where we have used (16). Then the prediction equation can be written

$$X_{t|t} = X_{t|t-1} + K(L\tilde{X}_t + v_t). \quad (87)$$

First, (16) implies that

$$Z_{t|t-1} = (L + M)X_{t|t-1}$$

and hence that

$$\tilde{Z}_t = L\tilde{X}_t + M(X_{t|t} - X_{t|t-1}) + v_t$$

Substitution of (87) into this then yields

$$\tilde{Z}_t = (I + MK)(L\tilde{X}_t + v_t). \quad (88)$$

Thus we get the desired expression

$$\tilde{Z}_t = N\tilde{X}_t + \nu_t, \quad (89)$$

where

$$N \equiv (I + MK)L, \quad (90)$$

$$\nu_t \equiv (I + MK)v_t. \quad (91)$$

In order to find the dynamics for the prediction error  $\tilde{X}_t$ , we subtract (23) from (15) and use (87), which gives

$$\tilde{X}_{t+1} = H(X_t - X_{t|t}) + u_{t+1} = H\tilde{X}_t - HK(L\tilde{X}_t + v_t) + u_{t+1}.$$

Hence we get the desired expression

$$\tilde{X}_{t+1} = T\tilde{X}_t + \omega_{t+1}, \quad (92)$$

where

$$T \equiv H(I - KL), \quad (93)$$

$$\omega_{t+1} \equiv u_{t+1} - HKv_t. \quad (94)$$

Now, (92) and (89) can be seen as the transition and measurement equations, respectively, for a standard Kalman-filter problem for the unobservable variable  $\tilde{X}_t$  with  $\tilde{Z}_t$  being the observable variable. Consequently, the prediction equation for  $\tilde{X}_{t|t}$  can be written

$$\tilde{X}_{t|t} = PN'(NPN' + \Sigma_{\nu\nu})^{-1}(N\tilde{X}_t + \nu_t) \quad (95)$$

where  $'$  denotes transpose and where we have used  $\tilde{X}_{t|t-1} \equiv 0$  and  $P \equiv \text{Cov}[\tilde{X}_t - \tilde{X}_{t|t-1}] = \text{Cov}[\tilde{X}_t]$  is the covariance matrix for the prediction errors. By (92) we directly get

$$P = TPT' + \Sigma_{\omega\omega}. \quad (96)$$

We also have

$$\Sigma_{\nu\nu} = E[\nu_t\nu_t'] = (I + MK)\Sigma_{vv}(I + MK)', \quad (97)$$

$$\Sigma_{\omega\omega} = HK\Sigma_{vv}K'H' + \Sigma_{uu}. \quad (98)$$

We express  $X_{t|t}$  in terms of the prediction error  $\tilde{Z}_t$  by solving for  $X_{t|t}$  in (21), which gives

$$\begin{aligned} X_{t|t} &= (I + KM)^{-1}[X_{t|t-1} + K(Z_t - LX_{t|t-1})] \\ &= X_{t|t-1} + (I + KM)^{-1}K[Z_t - (L + M)X_{t|t-1}] \\ &= X_{t|t-1} + (I + KM)^{-1}K\tilde{Z}_t \\ &= X_{t|t-1} + K(I + MK)^{-1}\tilde{Z}_t, \end{aligned} \quad (99)$$

where we have used the convenient identities  $(I + KM)^{-1} \equiv I - (I + KM)^{-1}KM$  and  $(I + KM)^{-1}K \equiv K(I + MK)^{-1}$ .

Now, comparing (95) and (99), using (89) and  $\tilde{X}_{t|t} = X_{t|t} - X_{t|t-1}$ , we see that

$$K(I + MK)^{-1} = PN'(NP N' + \Sigma_{\nu\nu})^{-1}.$$

Substituting (90) for  $N$  and (97) for in the right side, we get the final expression for  $K$ , (24).

Substituting (24) for  $K$  in  $T$  in (93) and (96) then gives the final equation for  $P$ , (25).

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## Notes

<sup>1</sup>See Svensson (1997a), (1999b) and (2001) for discussion of inflation targeting and references to the literature.

<sup>2</sup>See Gerlach and Smets (1999), Peersman and Smets (1998) and Smets (1998) for recent applications to estimation of the output gap in purely backward-looking frameworks. Since the first version of this paper was written, Swanson (2000) has examined the monetary-policy consequences of output-gap uncertainty in a backward-looking model.

<sup>3</sup>Since the first version of this paper was written, we have received papers by Lansing (2000) and Tetlow (2000) which consider the consequences of partial observability of potential output for monetary policy in forward-looking models. Lansing considers learning about a shift in trend output. Tetlow examines the performance of alternative simple instrument rules. He uses a Kalman filter for the estimation of potential output, as we do, but as far as we can see without considering the complications for such estimation caused by the existence of forward-looking variables. Other recent papers that consider optimal monetary policy in the presence of imperfect information on the part of the central bank include Aoki (1998, 2000), Beaudry and Doyle (2000), and Dotsey and Hornstein (2000). All of these papers assume models of the monetary transmission mechanism that include at least some forward-looking elements.

<sup>4</sup>The demonstration of certainty equivalence under commitment raises some special difficulties which are treated in a separate paper, Svensson and Woodford (2002d).

<sup>5</sup>Here the linear (or log-linear) structural equations are simply taken as given. In practice, one will often be interested in the analysis of optimal policy in models where relations of this kind are obtained as linear approximations to the exact structural equations derived from first-order conditions for private-sector optimization. The conditions under which a linear-quadratic policy problem of the kind considered here gives a correct linear approximation to optimal policy are discussed in section 1 of Woodford (2002). (An alternative approach, illustrated for example by Khan, King and Wolman (2002), linearizes only after the first-order conditions characterizing optimal policy have been derived; under the somewhat restrictive conditions discussed in Woodford (2002), the two calculations yield identical linear approximations to optimal policy.) It should be noted that the values of the state variables around which one should linearize the exact structural equations, in order to obtain an accurate approximation to equilibrium dynamics in the case of small enough disturbances, are not necessarily the same in the cases of discretion and commitment. For the average values of state variables need not be the same under these two policies; for example, in many familiar models, there is an “inflation bias” associated with discretionary policy, meaning that the long-run average inflation rate under discretion is higher than under commitment. Nonetheless, we use the same notation for the coefficients of the linearized structural equations in both sections of our paper.

<sup>6</sup>Note that the predetermined and forward-looking variables can be interpreted as deviations from unconditional means and the target variables can be interpreted as deviations from constant target levels. More generally, constants, non-zero unconditional means and non-zero target levels can be incorporated by including unity among the predetermined variables, for instance, as the last element of  $X_t$ . The last row of the relevant matrices will then include the corresponding constants/means/target levels.

<sup>7</sup>Pearlman (1992) refers to the complex derivation of the Kalman filter in Pearlman, Currie and Levine (1986) but doesn’t report that the derivation is actually much easier than in Pearlman, Currie and Levine (1986).

<sup>8</sup>Harvey (1981) defines the Kalman gain matrix in this way, whereas Harvey (1989) defines it as the transition matrix (yet to be specified in our case) times  $K$ .

<sup>9</sup>Adding a linear term to the loss function is similar to the linear inflation contracts discussed in Walsh (1995) and Persson and Tabellini (1993). Indeed, the term added in (29) corresponds to a state-contingent linear inflation contract, which, as discussed in Svensson (1997b), can remedy both “stabilization bias” and “average inflation bias.”

<sup>10</sup>Note that  $y_t - \bar{y}_t$  and  $\nu_t$  here corresponds to  $x_t$  and  $u_t$ , respectively, in Svensson and Woodford (2002a). Furthermore, current inflation and output are here forward-looking variables, whereas they are predetermined one period in Svensson and Woodford (2002a). The assumption that inflation and output are predetermined is arguably more realistic, but in the present context would not allow us to present a simple example in which one of the observables is a forward-looking variable. A more elaborate example (for instance, along the lines of Svensson (2000)), that would be more realistic but less transparent in its analysis, would allow inflation and output to be predetermined, but introduce other forward-looking indicator variables, such as the exchange rate, a long bond rate, or other asset prices.

<sup>11</sup>See Woodford (2002) for a welfare-theoretic justification of this loss function, in the case of exactly the microeconomic foundations that justify structural equations (41)–(42).

<sup>12</sup>See section 3.2 of Svensson and Woodford (2002a). Recall that  $y_t - \bar{y}_t$  and  $\nu_t$  here corresponds to  $x_t$  and  $u_t$ , respectively, in Svensson and Woodford (2002a). Since the present model has an output target equal to potential output in the period loss function, (45), it corresponds to the case  $x^* = 0$  in Svensson and Woodford (2002a).

<sup>13</sup>See section 2.1 of Svensson and Woodford (2002a). Note that  $\Xi_{t-1}$  here corresponds to  $\varphi_{t-1}$  in Svensson and Woodford (2002a). Because the present model corresponds to the case  $x^* = 0$  in Svensson and Woodford (2002a),  $\Xi^* = 0$ .

<sup>14</sup>Orphanides (1998) discusses the application of the principle of certainty equivalence to this kind of problem, albeit in the context of a purely backward-looking model. However, the analysis there suggests that certainty equivalence applies to a representation of policy that is of less practical relevance than a simple rule that allows “policy to be formulated in terms of just current data”. The difference that we would stress, however, relates not to the complexity of the rule but to whether it is expressed as an explicit function of direct observables. Even in the case of the *optimal* explicit instrument rule, the optimal coefficients are not independent of the nature of the measurement error in the direct observation of potential output, as we illustrate above.

<sup>15</sup>See, for instance, European Central Bank (1999).

<sup>16</sup>See Estrella and Mishkin (1997) and Stock and Watson (1999); Gerlach and Svensson (2002) find, for reconstructed Euro-area data, information for future inflation in another monetary indicator, the “real money gap,” but little or no information in the Eurosystem’s money-growth indicator.

<sup>17</sup>Furthermore, the high long-run correlation between money and prices found in historical data may to some extent depend on the high money growth and inflation that have occurred in the past, dominating fluctuations in output and velocity. Under a low-inflation regime of the kind that the ECB is expected to maintain, even the long-run correlation may well be weaker.

<sup>18</sup>See Söderlind (1999) for a detailed presentation.