Single-Crossing Differences on Distributions

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Introduction (1)

- Single Crossing Differences is central to MCS

$$\forall a, a' \in A : v(a,\theta) - v(a',\theta) \text{ is single crossing in } \theta$$

$$\iff \text{choices are } \underbrace{\text{monotonic}}_{\text{strong set order}} \text{ in type } \forall A' \subseteq A$$

- Agent may be faced with lotteries over A
  - directly or indirectly (e.g., in a game)
  - e.g., Crawford and Sobel '82: what if S does not know R's prefs?

- For vNM agent, Single Crossing Expectational Differences

$$\forall P, Q \in \Delta A : \mathbb{E}_P[v(a,\theta)] - \mathbb{E}_Q[v(a,\theta)] \text{ is SC in } \theta$$

- Not assured by SCD over A

Introduction (2)

Our results:

1. Characterize $v(a, \theta)$ that have SCED

A Takeaway

SCED $\underbrace{\Longleftrightarrow}_{\text{often}}$ $v(a, \theta) \sim u(a) + f(\theta)w(a)$, with f monotonic

2. Establish SCED $\iff$ MCS on ΔA

3. Applications

In achieving (1):

- Characterize sets of functions whose linear combinations are SC
- A characterization of MLRP (known, but apparently not well)

Literature

More related (elaborate later):

- Kushnir and Liu 2017
- Quah and Strulovici ECMA 2012, Choi and Smith JET 2016
- Karlin 1968 book

- Milgrom and Shannon ECMA 1994

Less related:

- Milgrom RAND 1981
- Athey QJE 2002

Main Results

Setting

- A is some space (outcomes/allocations)
  - talk as if A finite; avoiding technical details
  - ΔA is set of all prob. measures
- $(\Theta, \leq)$ is a partially-ordered space (types)
  - $\leq$ is reflexive, transitive, antisymmetric
  - contains upper and lower bounds for all pairs
  - some results are trivial when $|\Theta| \leq 2$
- $v : A \times \Theta \to \mathbb{R}$ (payoff fn)
- Expected Utility: $V(P, \theta) \equiv \int_A v(a, \theta) \mathrm{d}P$
- Expectational Difference: $D_{P,Q}(\theta) \equiv V(P, \theta) - V(Q, \theta)$

Single Crossing

Definition

$f : \Theta \to \mathbb{R}$ is

1. **single crossing from below** if

$$(\forall \theta_l < \theta_h) \quad f(\theta_l) \geq (>) 0 \implies f(\theta_h) \geq (>) 0.$$

2. **single crossing from above** if

$$(\forall \theta_l < \theta_h) \quad f(\theta_l) \leq (<) 0 \implies f(\theta_h) \leq (<) 0.$$

3. single crossing if it is SC from below or from above.

E.g., $f(\cdot) > 0$ is SC from below *and* above.

SC Expectational Differences

Definition

Let X be arbitrary.

$f : X \times \Theta \to \mathbb{R}$ has SC Differences (SCD) if

$$\forall x, x' \in X : \ f(x,\theta) - f(x',\theta) \text{ is single crossing in } \theta.$$

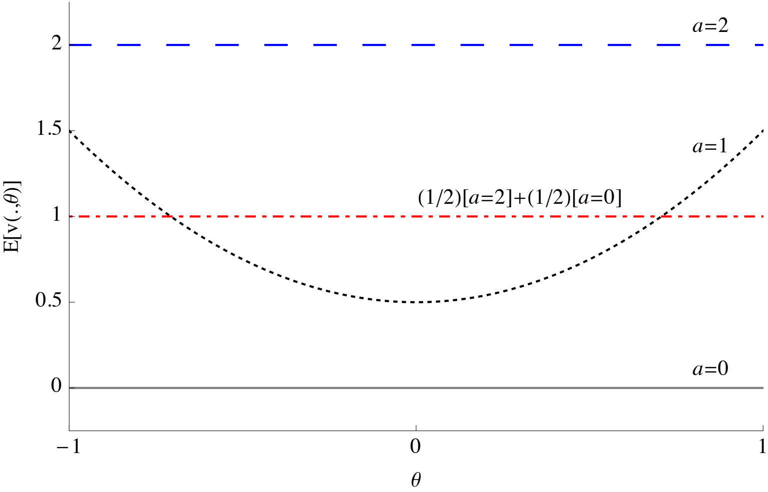
- Not quite the usual definition; X need not be ordered

Definition

v has SC Expectational Differences (SCED) if $V : \Delta A \times \Theta \to \mathbb{R}$ has SCD.

- $D_{P,Q}(\theta)$ is SC for all lotteries P, Q
- SCED is an ordinal property of prefs over ΔA
- When $|A| = 2$, equiv. to v having SCD

SCD $\not\Rightarrow$ SCED



Main Result

Theorem

v has SCED if and only if

$$v(a,\theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + c(\theta), \tag{1}$$

with f_1, f_2 each SC and ratio ordered.

- If $f_1, f_2 > 0$, then RO $\iff$ f_1/f_2 monotonic; and SC trivial
- Then interpret as: two prefs s.t. each θ's pref is a convex combination, with weight shifting monotonically in θ
- But f_1, f_2 need not be positive (nor single-signed)

$$(1) \implies D_{P,Q}(\theta) = \alpha_1 f_1(\theta) + \alpha_2 f_2(\theta) \text{ for some } \alpha \in \mathbb{R}^2$$

Is such $D_{P,Q}$ single crossing?

Ratio Ordering

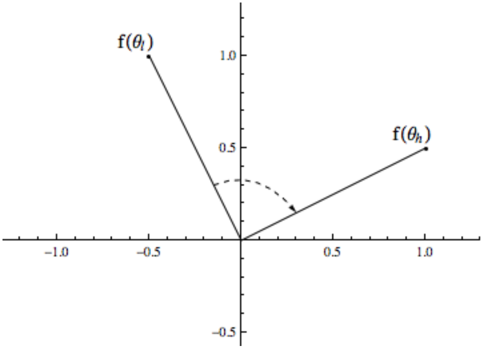
Definition

Let $f_1, f_2 : \Theta \to \mathbb{R}$ each be SC.

1. f_1 ratio dominates f_2 if
   (i) $(\forall \theta_l \leq \theta_h) \quad f_1(\theta_l) f_2(\theta_h) \leq f_1(\theta_h) f_2(\theta_l)$,
   (ii) omitted nuances. ▸ details
2. f_1 and f_2 are ratio ordered if f_1 ratio dominates f_2 or vice-versa.

- If both are (str. $+$) densities, simply likelihood ratio ordering
- Defn does not assume either f_i has constant sign
  - $(\forall f)$ f and $-f$ are ratio ordered

Geometric Interpretation



- f_1 RD $f_2 \implies (\forall \theta_l < \theta_h)\ f(\theta_l)$ rotates clockwise $(\leq 180^\circ)$ to $f(\theta_h)$

$$(f(\theta'), 0) \times (f(\theta''), 0) = \|f(\theta')\|\|f(\theta'')\| \sin(r) e_3 = \left(f_1(\theta')f_2(\theta'') - f_1(\theta'')f_2(\theta')\right) e_3$$

- Ratio ordering $\implies f(\theta)$ rotates monotonically $(\leq 180^\circ)$

$\impliedby$ modulo nuances

▸ point (ii)

Linear Combinations Lemma

Lemma

Let $f_1, f_2 : \Theta \to \mathbb{R}$ *each be SC.*

$\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ *is SC* $\forall \alpha \in \mathbb{R}^2 \iff f_1, f_2$ *are ratio ordered.*

- A characterization of LR ordering (for str. + densities) ▸ Strict
- Coeffs of opp signs are key
- f_1 and f_2 need not be SC in the same direction (e.g., $f_1 = -f_2$)

Linear Combinations Lemma

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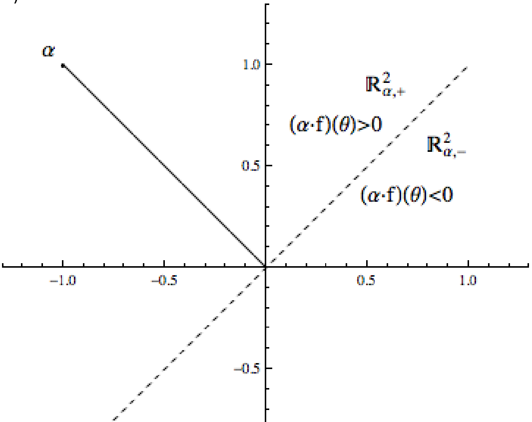
Intuition: ($\Longleftarrow$)



Linear Combinations Lemma

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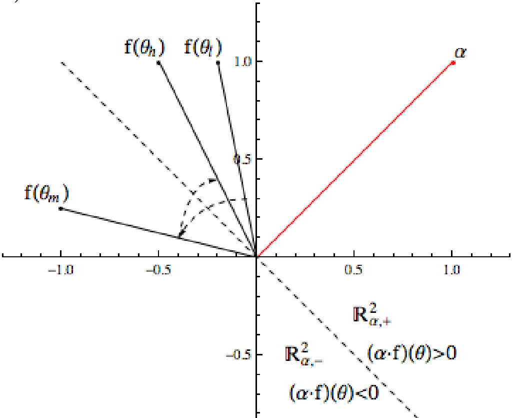
Intuition: $(\implies)$



Linear Combinations of Multiple Functions

- Necess. direction of Thm requires aggregating many SC functions

Proposition

Consider $\{f_i\}_{i=1}^n$, where each $f_i : \Theta \to \mathbb{R}$ is SC.

$\sum_i \alpha_i f(x_i, \theta)$ is SC $\forall \alpha \in \mathbb{R}^n$ if and only if $\exists i, j$ s.t.

1. Ratio Ordering: f_i and f_j are ratio ordered;
2. Spanning: $(\forall k)\ f_k(\cdot) = \lambda_k f_i(\cdot) + \gamma_k f_j(\cdot)$.

intuition

Main Result: SCED Characterization

Theorem

v *has SCED if and only if*

$$v(a,\theta) = g_1(a)f_1(\theta) + g_2(a)f_2(\theta) + c(\theta),$$

with f_1, f_2 *each SC and ratio ordered.*

- Sufficiency follows from Linear Combinations Lemma:
  $D_{P,Q}(\theta) = \left[\int g_1 \mathrm{d}P - \int g_1 \mathrm{d}Q\right] f_1(\theta) + \left[\int g_2 \mathrm{d}P - \int g_2 \mathrm{d}Q\right] f_2(\theta)$

Main Result: SCED Characterization

Theorem

v has SCED if and only if

$$v(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + c(\theta),$$

with f_1, f_2 each SC and ratio ordered.

Idea underlying necessity:

- Consider $A = \{a_0, \ldots, a_n\}$ and $v(a_0, \cdot) = 0$.
- SCED $\implies$ $(\forall a)\ v(a, \theta)$ is SC $\quad (\because \delta_a$ and $\delta_{a_0})$
- $\forall \lambda \in \mathbb{R}^n$, $\sum_i \lambda_i v(a_i, \theta) \propto \sum_i (p(a_i) - q(a_i)) v(a_i, \theta)$, where p, q are PMFs
- SCED $\implies$ every such linear combination is SC
- Linear Combinations Prop $\implies$ $\exists i, j$:

  $(\forall a)\ v(a, \cdot) = g_1(a) v(a_i, \cdot) + g_2(a) v(a_j, \cdot)$, with RO (and SC)

Main Result: SCED Characterization

Theorem

v has SCED if and only if

$$v(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + c(\theta),$$

with f_1, f_2 each SC and ratio ordered.

While SCED is restrictive, it is satisfied in some familiar cases

- **screening/mech design**: $v((q, t), \theta) = g_1(q) f(\theta) - g_2(t)$, f monotonic
  - unless g_1 is constant, $f(\cdot)$ must be monotonic
- **voting/communication**: $v(a, \theta) = -(a - \theta)^2 = -a^2 + 2a\theta - \theta^2$
  - for $v(a, \theta) = -|a - \theta|^d$ with $d > 0$, only $d = 2$ satisfies SCED
- **signaling**: $v((w, e), \theta) = w - e/\theta$ (usually $e, \theta > 0$)
- in all these cases, one $f_i(\cdot) = 1$

Main Result: SCED Characterization

Theorem

v has SCED if and only if

$$v(a,\theta) = g_1(a)f_1(\theta) + g_2(a)f_2(\theta) + c(\theta),$$

with f_1, f_2 each SC and ratio ordered.

Theorem

Assume some agreement: $(\exists P, Q)$ $(\forall \theta)$ $V(P,\theta) > V(Q,\theta)$.

v has SCED if and only if prefs have a representation

$$\tilde{v}(a,\theta) = g_1(a)f_1(\theta) + g_2(a),$$

with f_1 monotonic.

An MCS Characterization

Let $f : X \times \Theta \to \mathbb{R}$ with $(X, \succeq)$ an ordered set and $(\Theta, \leq)$ a directed set

- Assume X is minimal wrt f: $(\forall x \neq x')(\exists \theta)\ f(x,\theta) \neq f(x',\theta)$

Definition

f has Monotone Comparative Statics on $(X, \succeq)$ if

$$(\forall S \subseteq X, \theta \leq \theta')\ \arg\max_{x \in S} f(x,\theta') \succeq_{SSO} \arg\max_{x \in S} f(x,\theta).$$

- $Y \succeq_{SSO} Z$ if $(\forall y \in Y, z \in Z)\ (y \vee z \in Y, y \wedge z \in Z)$

- Cf. MS '94: X need not be lattice;
  monotonicity only in θ but $\forall S \subseteq X$ (not only all sublattices)

An MCS Characterization

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$$(\forall S \subseteq X, \theta \leq \theta')\ \arg\max_{x \in S} f(x,\theta') \succeq_{SSO} \arg\max_{x \in S} f(x,\theta).$$

- Define a reflexive relation $\succeq_{SCD}$ on X:

  $x \succ_{SCD} x'$ if $f(x,\theta) - f(x',\theta)$ is SC from *only* below

- If f has SCD, $\succeq_{SCD}$ is an order

Proposition

f has MCS on $(X, \succeq)$ $\iff$ f has SCD and $\succeq$ refines $\succeq_{SCD}$.

SCED and MCS

Apply MCS result to our setting; recall $D_{P,Q}(\theta) \equiv V(P,\theta) - V(Q,\theta)$

Definition

$P \succ_{SCED} Q$ if $D_{P,Q}(\cdot)$ is SC from only below;

$P \sim_{SCED} Q$ if $D_{P,Q}(\cdot) = 0$.

Let $\widetilde{\Delta} A$ be the quotient space defined by $\sim_{SCED}$

Corollary

V has MCS on $(\widetilde{\Delta} A, \succeq) \iff v$ has SCED and $\succeq$ refines $\succeq_{SCED}$.

A strict version of SCED yields a monotone selection result ▸ SSCED

Applications

Cheap Talk

- Sender with type $\theta \in \Theta$ chooses cheap-talk message $m \in M$
- Receiver with type ψ observes m and takes action $a \in A$
- vNM payoffs $v(a, \theta)$ for S and $u(a, \theta, \psi)$ for R
- θ and ψ are independently drawn, private info
- E.g.: $v(\cdot) = -(a - \theta)^2$, and $u(\cdot) = -(a - \psi_1 - \psi_2\theta)^2$

What assures "interval cheap talk"? In CS, concavity of u and SCD of v.

Focus on Bayesian Nash equilibria in which:

- S plays a pure strategy, $\mu : \Theta \to M$
- (Minimality.) If m, m' are on path, then $(\exists\theta)\ m \nsim_\theta m'$

SSCED

Claim

If v has **strict SCED**, then every eqm has interval cheap talk. If v **strictly violates SCED**, then $\exists$ params under which $\exists$ a non-interval "strict" eqm.

Collective Choice (1)

- Finite group, $\{1, 2, \ldots, N\}$, must choose from $\mathcal{A} \subseteq \Delta A$
- For simplicity, N odd and A finite; let $M \equiv (N+1)/2$
- Each i has vNM utility $v(a, \theta_i)$, where $\theta_i \in \Theta \subset \mathbb{R}$, $\theta_1 \leq \cdots \leq \theta_N$
- Majority preference relation:

$$P \succeq_{maj} Q \text{ if } |\{i : V(P, \theta_i) \geq V(Q, \theta_i)\}| \geq M$$

Is this transitive (i.e., would majority rule yield "rational choices")?

Claim

If v has strict SCED, then $\succeq_{maj}$ is transitive and rep. by. $V(\cdot, \theta_M)$

- Characterization of SSCED + Gans and Smart (1996)

Collective Choice (2)

Claim

If v has strict SCED, then $\succeq_{maj}$ is transitive and rep. by. $V(\cdot, \theta_M)$.

- Let $\{\theta_M\} = \text{argmax}_{a \in A}\, v(a, \theta_M)$
- Two office-seeking politicians can offer lotteries from ΔA
- Voters vote "sincerely"

Corollary

If v has strict SCED, political competition with lotteries has a unique Nash equilibrium: convergence to $a = \theta_M$.

- Compatible with voters being risk loving on subsets of policy space
- There is a sense in which SCED is necessary

Literature Connections

Literature Connections (1)

Definition

$v : A \times \Theta \to \mathbb{R}$ has Monotonic Expectational Differences if

$$(\forall P, Q \in \Delta A) \;\; D_{P,Q}(\theta) \text{ is monotonic in } \theta.$$

- Equiv., $V : \Delta A \times \Theta \to \mathbb{R}$ has Monotonic Differences, not just SCD

Proposition

v has MED if and only $v(a, \theta) = g_1(a) f_1(\theta) + g_2(a) + c(\theta)$, with $f_1 : \Theta \to \mathbb{R}$ monotonic.

- SCED characterization but with $(\forall \theta) f_2(\theta) = 1$
- SCED is strictly more general than MED
  - Paper characterizes when SCED prefs have MED representation
  - sufficient if $\exists P, Q \in \Delta A$ over which all types share same strict pref
- Kushnir and Liu (2016), for a subset of environments

Literature Connections (2)

Definition (Quah and Strulovici 2012)

f_1 and f_2 are **signed ratio monotonic** if for each $i, j \in \{1, 2\}$,

$$(\forall \theta_l \leq \theta_h) \quad f_j(\theta_l) < 0 < f_i(\theta_l) \implies f_i(\theta_h) f_j(\theta_l) \leq f_i(\theta_l) f_j(\theta_h).$$

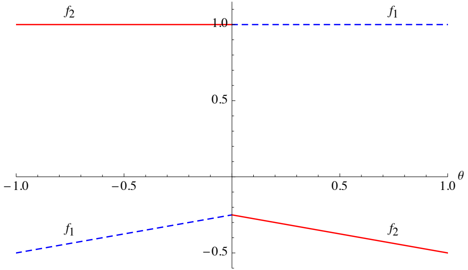
Proposition (Quah and Strulovici 2012)

Let f_1, f_2 both be SC from below (resp., above).

$\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ is SC from below (resp., above) $\forall \alpha \in \mathbb{R}^2_+$
$\iff$ f_1 and f_2 (resp., $-f_1$ and $-f_2$) are signed ratio monotonic.

Literature Connections (2)

- f_1 and f_2 could be SC from below and ratio ordered, yet $f_1 + f_2$ could be SC from **only above**! (Only if f_1 and f_2 are not SRM)
  - E.g.: $\Theta = [0,1]$, $f_1(\theta) = 1$, $f_2(\theta) = -1 - \theta$
- Ratio ordering $\not\Rightarrow$ (f_1, f_2) or $(-f_1, -f_2)$ are SRM



  - we allow the pair of SC functions to cross in opposite directions
- If f_1 and f_2 are both SC in same direction, ratio ordering is stronger than (f_1, f_2) or $(-f_1, -f_2)$ are SRM
  - we get / require all linear combinations to be SC

Recap

1. Characterized when set of SC fns. preserves SC $\forall$ linear combinations

2. Given $v : A \times \Theta \to \mathbb{R}$ with exp utility $V : \Delta A \times \Theta \to \mathbb{R}$,

   $V(P,\theta) - V(Q,\theta)$ is SC in θ $(\forall P, Q \in \Delta A)$

   $$\iff v(a,\theta) = g_1(a)f_1(\theta) + g_2(a)f_2(\theta) + c(\theta),$$

   with f_1, f_2 SC and ratio ordered

   - Necessary and sufficient for a form of MCS on ΔA

3. Useful for applications

Ratio Ordering

Definition

Let $f_1, f_2 : \Theta \to \mathbb{R}$ each be SC.

1. f_1 ratio dominates f_2 if

   (i) $(\forall \theta_l \leq \theta_h) \quad f_1(\theta_l) f_2(\theta_h) \leq f_1(\theta_h) f_2(\theta_l)$,

   (ii) $(\forall \theta_l \leq \theta_m \leq \theta_h)$

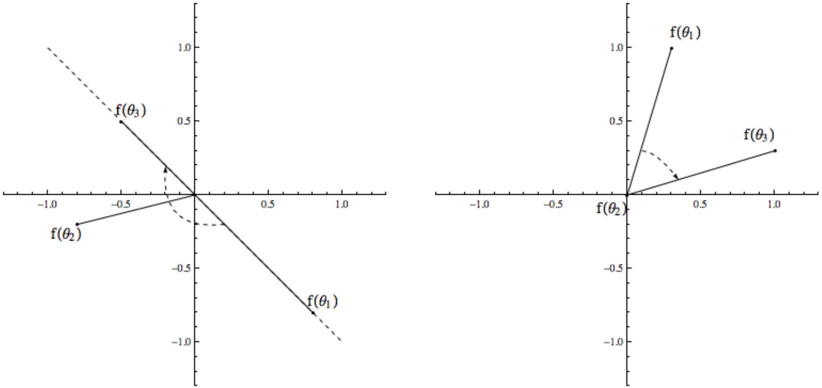
   $$f_1(\theta_l) f_2(\theta_h) = f_1(\theta_h) f_2(\theta_l) \iff \begin{cases} f_1(\theta_l) f_2(\theta_m) = f_1(\theta_m) f_2(\theta_l) \\ f_1(\theta_m) f_2(\theta_h) = f_1(\theta_h) f_2(\theta_m) \end{cases}$$

2. f_1 and f_2 are ratio ordered if f_1 ratio dominates f_2 or vice-versa.

return

Point (ii) of ratio ordering

$$(\forall \theta_l \leq \theta_m \leq \theta_h)\ f_1(\theta_l)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_l) \iff \begin{cases} f_1(\theta_l)f_2(\theta_m) = f_1(\theta_m)f_2(\theta_l) \\ f_1(\theta_m)f_2(\theta_h) = f_1(\theta_h)f_2(\theta_m) \end{cases}$$



(a) Failure of $\Longrightarrow$ (b) Failure of $\Longleftarrow$

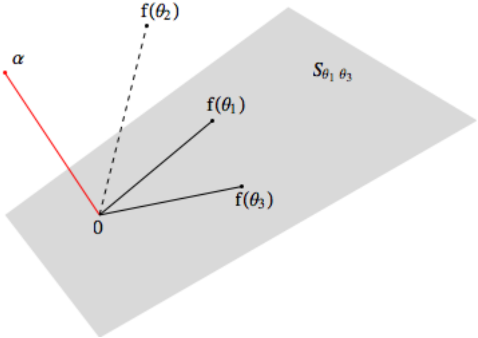
return

Intuition for Necessity

return

- Consider completely ordered Θ
- If $\{f_1(\cdot), f_2(\cdot), f_3(\cdot)\}$ are linearly independent,

$$(\exists\theta_1 < \theta_2 < \theta_3) \quad \{f(\theta_1), f(\theta_2), f(\theta_3)\} \text{ spans } \mathbb{R}^3.$$



- $(\alpha \cdot f)(\theta_1) = (\alpha \cdot f)(\theta_3) = 0 \neq (\alpha \cdot f)(\theta_2) \implies \alpha \cdot f$ is not SC

Variation of Lemma

Definition

$f : \Theta \to \mathbb{R}$ is **strictly SC** if either

1. $(\forall \theta < \theta')\ f(\theta) \geq 0 \implies f(\theta') > 0$; or
2. $(\forall \theta < \theta')\ f(\theta) \leq 0 \implies f(\theta') < 0$.

Definition

$f_1 : \Theta \to \mathbb{R}$ strictly ratio dominates $f_2 : \Theta \to \mathbb{R}$ if

$$(\forall \theta_l < \theta_h) \quad f_1(\theta_l) f_2(\theta_h) < f_1(\theta_h) f_2(\theta_l).$$

f_1 and f_2 are strictly ratio ordered if f_1 strictly RD f_2 or vice-versa.

Lemma (Strict Version)

$\alpha_1 f_1(\theta) + \alpha_2 f_2(\theta)$ *is strictly SC* $\forall \alpha \in \mathbb{R}^2 \backslash \{0\} \iff f_1, f_2$ *are strictly RO.*

- Strict RO $\implies$ each function is strictly SC
- New characterization of strict MLRP $\forall$ densities

◂ SC Lemma

Strict SCED

◂ Cheap Talk

Definition

$v : A \times \Theta \to \mathbb{R}$ has Strict SCED if

$$(\forall P, Q \in \Delta A)\ D_{P,Q} \text{ is a zero function or strictly SC.}$$

Theorem (Strict Version)

$v : A \times \Theta \to \mathbb{R}$ has Strict SCED if and only if

$$v(a, \theta) = g_1(a) f_1(\theta) + g_2(a) f_2(\theta) + c(\theta),$$

with f_1, f_2 strictly ratio ordered.