Bargaining Across Issues in Legislatures

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Abstract: Suppose a legislature votes on many issues, one dimension at a time, deciding each on its merits as a one-shot decision. The result is a dimension-by-dimension median voter outcome. This paper shows that after policy is decided on a large number of dimensions in this manner (where the number of dimensions equals the number of voters, or sometimes even less), the net result is Pareto inferior. That is, after voting the preference of each issue's median voter's over many issues, there exist other policy combinations that every legislator would have preferred to the median-by-median decision. The implications are discussed.

1. Introduction

What accounts for the organizational structure of legislatures? By the "distributive" model, legislator organize their institutions to enhance "legislative exchange," whereby legislators trade votes and otherwise bargain, explicitly and implicitly, across issues. See Shepsle and Weingast (1995) for the classic statement. Others, notably Gilligan and Krehbiel (1995), argue against the feasibility of the distributive model, largely because bargains can readily be thwarted by the majority acting in its short-term interest.[1] This paper addresses the question of the desirability of organizing legislators for redistributive purposes. It asks a seemingly naïve question: assuming that legislatures can enact safeguards to protect their bargains, does legislative exchange enhance the legislature's collective welfare?[2]

I set the stage as follow. In the absence of legislative exchange, legislators vote on each issue as it arrives in the legislative hopper for a one-shot decision in isolation from those made earlier and those expected to come later. Assuming that each issue represents a single dimension, the default outcome is the preference of each issue's median legislator.[3] With different legislators at the median on different issues, the net result is a multi-dimensional intersection of medians, which we can label the "median vector." Given this setup, the question is whether legislators have an incentive to coordinate in order to improve upon the median vector as the legislative outcome.

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[1] For instance, the floor median can overrule the committee median.

[2] In recent years, the term "legislative bargaining" is often associated with the rich literature spawned by Baron and Ferejohn's (1989) divide-the-dollar game. The present paper bears no direct relationship to that rich literature.

[3] On each issue, the median voter position is the decisive Condorcet-winner of the tournament of choices, defeating all other alternatives in a straight pair-wise vote (Black, 1948).

Voting separately on each dimension presents a policy solution that, at least at first glance, is not only stable but also attractive. Voting one dimension at a time guarantees a predictable solution as the median on each dimension—a structurally induced equilibrium (Shepsle, 1978). In addition to being stable, the intersection of medians is attractive in the sense that it must lie in the vicinity of the center of the policy space defined by the distribution of preferences.

However, the prediction of median-by-median outcomes when voting one dimension at a time is dependent on the assumption that each dimension is voted upon in isolation—that the legislators are unwilling or unable to coordinate on issues that arrive on the agenda at differing times. Suppose instead that legislators are capable of making binding agreements over issues decided at different times. Then, they no longer have the incentive to vote sincerely on individual dimensions. Instead, they can exchange favors on different issues in a series of over-time transactions. For instance, a minority with intense preferences on a current issue can maneuver the outcome away from the median voter's position by offering others the promise of future legislative favors in return for going along. One could imagine flurries of IOUs offered and later paid off, with the participants gaining from the net exchange.

In extreme, legislative exchange is modeled as a full-blown political market, with complex transactions of current payoffs in exchange for future favors (Coleman, 1966; Koford, 1993; see also Philipson and Snyder, 1996; Groseclose and Snyder, 1996). Other models posit institutional equivalents of markets while avoiding many of the complications that an actual political market would involve (Weingast and Marshall, 1988). One mechanism is a committee system where legislators self-select onto committees with jurisdiction over the issues one which they have the strongest interest (Shepsle and Weingast, 1987). In other models, party leaders broker complicated deals among members of the dominant party (Mayhew, 1966; Koford, 1992; Cox and McCubbins, 1993, 2005).

While the parties to a political exchange presumably benefit, we can ask whether the legislature in the aggregate also benefits. When in the economic rather than the political marketplace, trade is "good" in the sense that all traders gain from the exchange, while (presumably) nobody is hurt. But with political exchange, the trade's beneficiaries gain at the expense of non-traders, who in turn are motivated to respond with their own counter-proposals. This problem of "negative externalities" contributes to the inherent unpredictability and instability in the modeling of political trades. It also presents a normative ambiguity regarding whether vote trading is, on balance, good. When trading involves negative externalities, helping some but hurting others, it becomes an empty debate whether such exchanges produce a net increase in aggregate welfare. There can be no convincing verdict because one cannot make interpersonal comparison of utilities.

One way out of this dilemma is to focus attention on legislative trades that offer "Pareto improvements"—where some gain and none lose from the transaction. By the rules of evaluating social welfare, "Pareto improvement" provides a decisive criterion. If all voters prefer *a* to *b* (or are at least indifferent), "society" prefers *a* to *b*. For "society" to select *b* over *a* would be irrational (Arrow, 1951). By extension, for a legislature to resist a trade that is universally preferred to the outcome without trading would be irrational. Put positively, Pareto optimal trades unambiguously expand legislative (and presumably societal) welfare. They are good.

We started with the potential policy outcome of the median position on each dimension—the "median vector." The median vector is the policy outcome when each policy issue is one-dimensional with no bargaining across dimensions. Could it be possible for the legislature to engage in a complex political exchange that leads to *everybody* being better off than with the intersection of medians? The median vector now serves as our reversion point. We can ask, under what conditions could legislative exchange result in a "Pareto improvement" over the median vector? That is, under what conditions would alternatives exist in policy space that would "Pareto dominate''—be universally preferred to—the median vector? Put still differently, under what conditions is the median vector part of the "Pareto set"—that is, cannot be changed without making some actors worse off?

The difficulty with Pareto improvement as a criterion is that finding policies that make none worse off would seem to be a daunting challenge, especially in a legislative setting. The search is for instances when *no* voters would prefer the median vector to the proposed alternative. Under what circumstances would there be one or more multidimensional outcomes that all voters would prefer to the median voter position on each dimension?

This paper argues that under very general conditions, the median vector is Pareto inferior. That is, under very general conditions, there exists a set of policy alternatives that all legislators would find preferable to the median vector that would arise from dimension-by dimension decision-making. It follows that if legislatures could enforce institutional rules to create stable outcomes that are Pareto-superior to the median vector, they have a universal incentive to do so.

What are these very general conditions under which the median vector is Pareto-inferior? Whenever the number of issue dimensions (*K*) exceeds the number of legislative voters (*N*), the median vector is (except for rare knife-edge conditions) Pareto-inferior. There are even circumstances where the median vector is Pareto inferior even when $N>K$. And the result is not an artifice of the convenient assumption that isolated legislative decisions are single-dimensional. As we will see, if single issues are sufficiently complicated to involve multiple dimensions, the resultant collection of one-shot decisions will become Pareto-inferior even before the number of issues equals the number of voters.

With sufficient multi-dimensional complexity when policies are summed across issues, there exist locations in policy space that all voters in the legislature would prefer to the intersection of medians. After voting separately on many separate issue dimensions, the legislators would look back with universal regret at their collective choice. If they are able to anticipate this result, their unanimous choice at the outset would be a coordinated move from the median vector to a unanimously preferable alternative location in issue space.

The result that median-by-median outcomes over multiple dimensions are Pareto inferior may appear to be counterintuitive. Yet voting with "many" dimensions has received little attention compared to models of legislative voting in one or two dimensions. Several explanations can be offered for the general avoidance of models with more than two dimensions. First, it is widely thought that such an extension is of little value—that the key distinction is between the order of one dimension and the chaos of many (2 or more). Second, models in three or more dimensions do not allow the easy geometric depiction that models in one or two dimensions allow. Third, students of congressional voting find that one or two dimensions is a sufficient number to describe the general structure of roll call voting (Poole and Rosenthal, 1997, 2007; Groseclose, Levitt, and Snyder, 1999).[4]

2. The Model

In the model, the legislature decides a series of one-dimensional issues, one at a time. Each issue represents a unique dimension in the sense that the array of ideal points differs (even if slightly) from one issue to another.[5] After policy is decided on an issue, the decision cannot be reviewed; instead, the legislature decides on the next issue brought forth for decision. The sequence of issues follows an agenda determined by nature. The legislators can observe this sequence and deduce expectations of future sequences accordingly, but they cannot anticipate the menu of future issues with certainty.

From the usual assumption of weighted Euclidean preferences over separable[6] policy dimensions, utility is defined in terms of quadratic loss:

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[4] I am aware of only one previous discussion of the Pareto efficiency as the number of dimensions expands. In his interesting chapter on "pencil exercises," Tullock (1967) shows by example a curious pattern regarding dimensionality and the Pareto set: As the dimensionality of the space increases, the Pareto set grows at a slower rate than the potential issue space. The result is that the Pareto set shrinks as a proportion of the total space. As Tullock shows, this shrinkage is fastest at the point when the number of dimensions expands to equal the number of voters. The implications for deciding policy on the dimensions separately versus collectively, however, is not explored.

[5] This definition implies no expectation of orthogonality or statistical independence of preferences on different dimensions as is common in the factor analysis literature.

[6] With separable preferences, the median on each dimension is the equilibrium outcome with voting one dimension at a time. Uncomplicatedly, the equilibrium is achieved by sincere voting. With the combination of non-separable preferences plus weighted Euclidean distance, dimension-by-dimension voting becomes sophisticated, requiring anticipation of future votes which condition the current choice.. For an informative literature review, see Krehbiel, 1988.

(Equation 1) $$U_i = -\sum_{k=1}^{K} w_{ik}(Q_k - Z_{ik})^2$$

where U_i = voter i's utility from policy outcomes on K dimensions, Z_{ik} = voter i's position on dimension k and Q_k = the policy outcome on dimension k, and w_{ik} = voter i's intra-personal salience weight to dimension k. To aid the presentation, the zero-point on each dimension is set to the dimension's median. In other words, the zero origin in issue space is set to the median vector.

The salience weights capture the idea that legislators differ in the intensity of their preferences across different issues. In vote trading models, intensity is typically part of the discussion, with voters conceding on some issues to gain on issues they care more about. Although gains-from-trade are enhanced when legislators vary in how they weight specific issues, the mapping of when median outcomes are Pareto-optimal does not depend on differential salience weights (although the severity of the possible distortion does). Thus, for most of the presentation below, the simplifying assumption is invoked that the w_{ik} salience weights are constant for all k for all N voters.[7]

As issues are decided one dimension at a time, they cumulate to form a multi-dimensional verdict. At any moment, the N-member legislature has voted on K separate issue dimensions, yielding as the cumulative result a K-dimensional policy outcome, $\mathbf{Q_K}$. With the single (one-dimensional) issues voted upon one at a time as a series of one-shot decisions, the multi-dimensional outcome is in equilibrium at the median vector $\mathbf{M_K}$, the intersection of K median voter positions ($M_1, M_2 \ldots M_K$) on K dimensions.

[7] Thus, for example, the assumption is that in two dimensions all indifference curves are circular.

To summarize, the model has legislators repeatedly casting votes and deciding one new single-dimensional issue after another, always moving to the next issue without revisiting those that have gone before. Each dimension is different from its predecessors by the modest requirement of some novelty (perhaps minor) in the relative positions of the legislative bliss points. Issue space is Euclidean (quadratic loss) with uniform salience weights across issue dimensions. The dimensions are scaled so that the median vector, the outcome of interest, is at the origin.

The assumptions of one-dimensional issues and uniform salience weights may appear to be confining. However, these restrictions facilitate the discussion which follows. They are relaxed later in the paper, as the model generalizes in interesting ways to multi-dimensional issues and is enhanced rather than hindered by the introduction of variable salience weights.

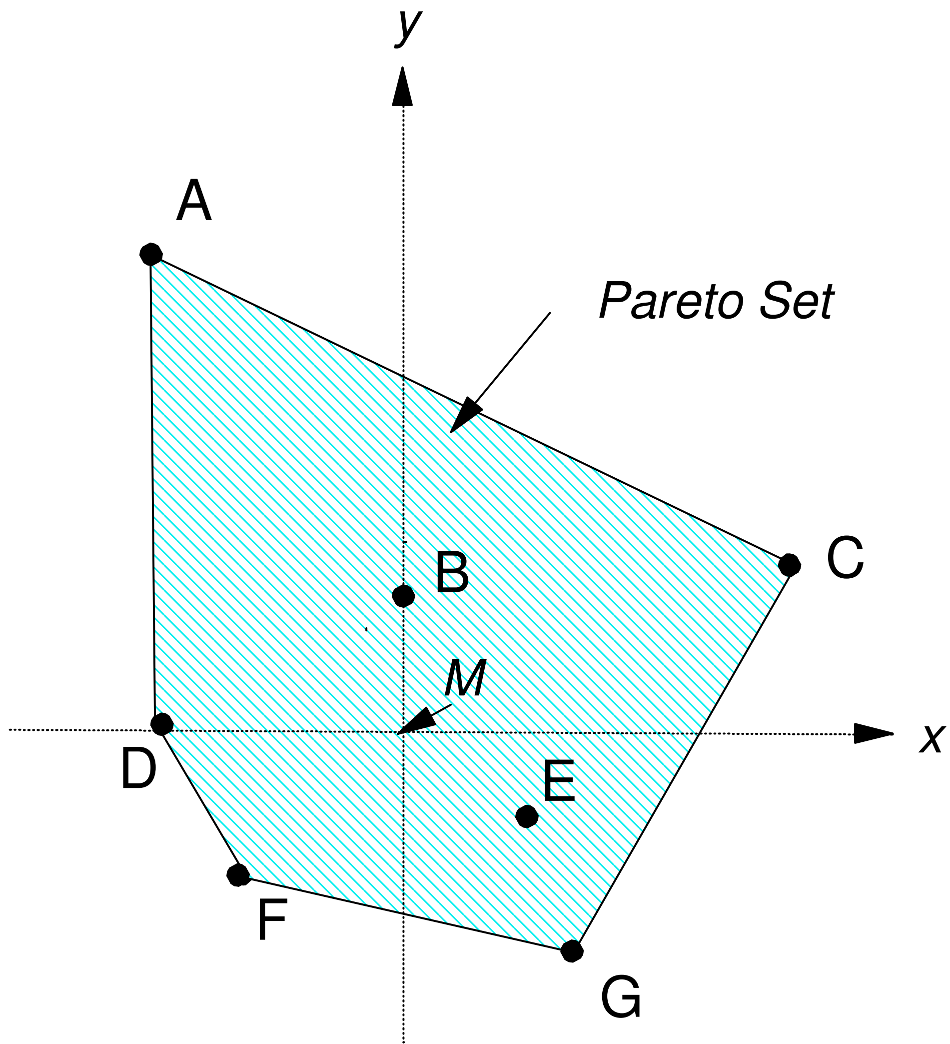

Figure 1. Seven voters, two dimensions.

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3. The Setup: Voting in Two Dimensions

The typical graphic presentation of spatial models of legislative voting depicts some small odd number of voters' ideal points in two dimensions with circular indifference curves.. Figure 1 presents such a graph, with seven arbitrary ideal points for seven voters. The polygon connecting the outer ideal points comprises the convex hull which defines the Pareto set.[8] For any point *g* outside the Pareto set, the seven voters can find a set of points within the Pareto set that they all prefer to *g*. The intersection of the two medians (*M*) represents one point well toward the center of the Pareto set.[9]

One might think that this depiction in Figure 1 must generalize to all multi-dimensional arrays of ideal points. The Pareto set contains so vast a space that one might think it to be implausible that the median vector could lie outside the Pareto set. But that surmise is incorrect. To set up an illustration, we start with a new example—in two dimensions but with only 3 voters.

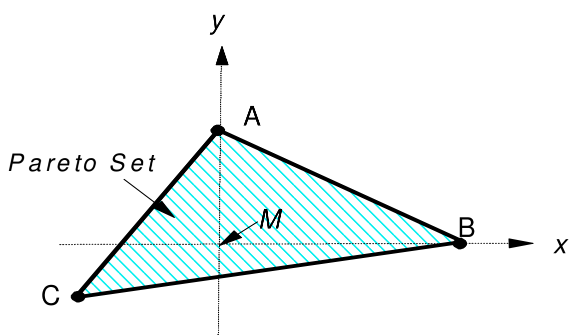

Figure 2. Three voters, two dimensions.

Figure 2 presents an arbitrary 3-voter issue space in two dimensions, where *M* = the intersection of medians on the *x* and *y* axes. The convex hull defining the Pareto set is simply the triangle connecting the ideal points of the three voters, *A, B,* and *C.* As before the intersection of medians is within the confines of the Pareto set.[10]

[8] On the Pareto set being bound by the convex hull, see (for instance), Schofield (1995). The convex hull is the smallest boundary that can encompass all the data (ideal points). See also the discussion below.
[9] And, although vulnerable to defeat like any other position in two-dimensional space, the median vector is reasonably close to the center of the electorate. This would also be true for alternative versions of the median vector produced by rotating the axes. In short, with a few voters distributed reasonably in two-dimensional space, the intersection of medians is not only within the Pareto set but also a reasonably centrist outcome.
[10] Note, however, that with three voters, if the same voter is the median on both dimensions, then the median vector is at a corner of the Pareto set.

Figure 2 portrays a familiar story about the intersection of the medians. Following voting that takes place one dimension at a time, the median vector is the equilibrium outcome. We cannot claim that the median vector would necessarily be the policy choice if voters were to vote on both dimensions at once. But it would be a plausible choice, and passes the minimal standard that it lies within the Pareto space—there exists no place in policy space that all three voters would agree is preferable. But the three voters of Figure 2 decide only two issues. When they also vote on a third dimension, everything changes.

4. Three voters and three dimensions

Next we introduce a third dimension. Figure 3 presents a 3-dimensional variation on the 2-dimensional structure of Figure 2. In addition to the horizontal (x) and vertical (y) dimensions, we add a third (z) dimension going inward from (or behind) and outward from (or forward of) the xy plane. The triangle *ABC* replicates the Pareto set of Figure 2 in the x and y dimensions. We assign voter positions on the third dimension and observe what happens. Since voter *A* is the median on the vertical y dimension and voter *B* is the median voter on the horizontal x dimension, let us make voter *C* the median on the in-out dimension. This is visualized in the graph by moving voter *A* inward away from the viewer (to *A'*) and voter *B* outward toward the viewer (to *B'*), while leaving voter *C* on the original plane, as the median voter on dimension z. As this example is constructed, the location of the median vector *M* is unchanged from Figure 2—the intersection of the three medians is identical to the original intersection of medians in two dimensions.

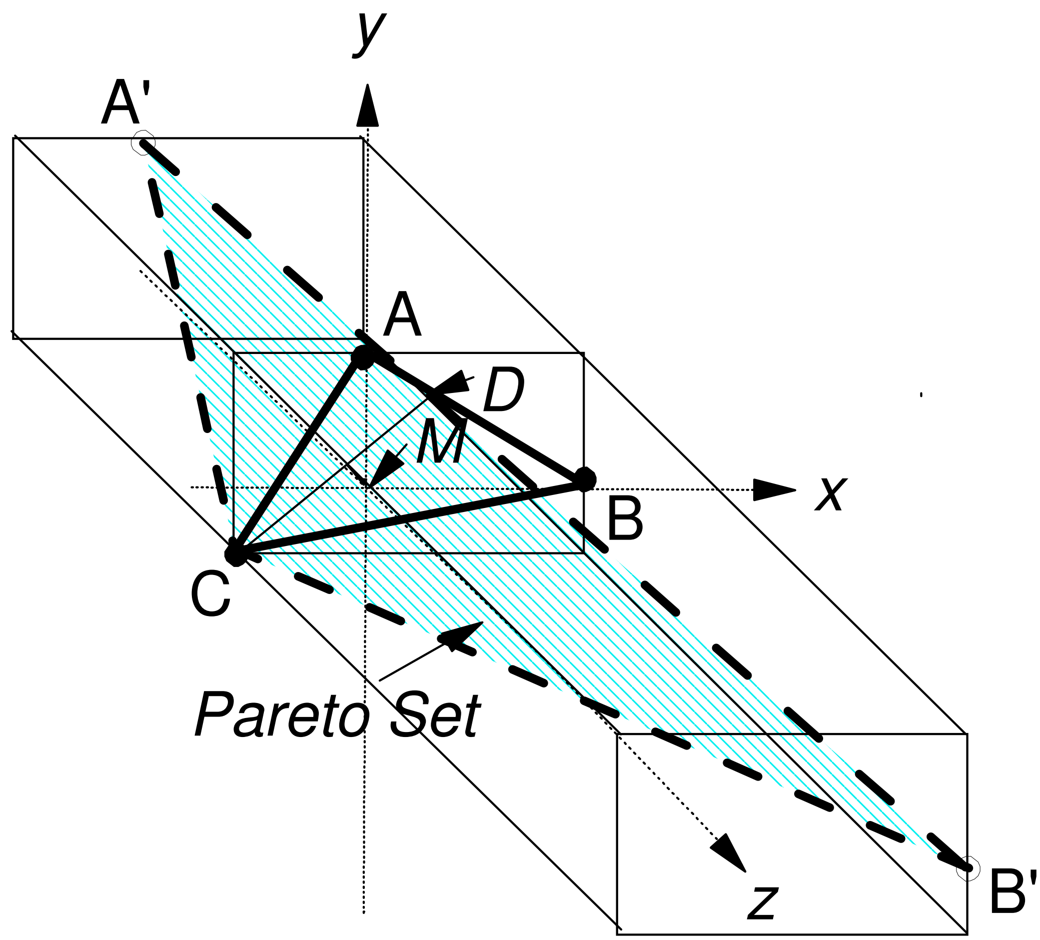

Figure 3. Three voters, three dimensions.

The new Pareto set is the convex hull connecting the three revised ideal points now at locations *A', B',* and original *C*. The space defined by the convex hull again is a two dimensional plane, but now one that cuts across the three original dimensions. As the Pareto set is transformed from the triangle *ABC* to the triangle *A'B'C,* the median vector *M* no longer is part of the Pareto set.

To get a fix on the 3-dimensional argument as depicted in the limited 2-dimensional medium of Figure 3, note the line *CD*, where the triangles *ABC* (on the plane of the page) and *A'B'C* (passing through) intersect. For the portion of *ABC* within the triangle *ACD*, triangle *A'B'C* passes underneath (i.e., lower *z*); for the portion of *ABC* within the triangle *BCD*, triangle *A'B'C* passes overhead (i.e., higher *z*). Since the median vector *M* is within *BCD*, there is an area of *A'B'C* that is overhead of the median vector *M*. This is the neighborhood of the Pareto set *A'B'C* that all three voters prefer to the median vector.

Figures 4 and 5 depict the same example from different perspectives. Figure 4 presents triangle *A'B'C* on the two dimensional plane of the paper, as the original *x, y*, and *z* axes are rotated to two new axes *x'* and *y'*. The median vector *M* is underneath the page, not shown, on dimension *z'* outside the Pareto set. The point *L* on the plane is the end-point of a line connecting *M* to the Pareto set that is perpendicular to the Pareto set. (*M* is directly underneath *L*.) Passing through *M*, each voter has a circular indifference *sphere*. These three spheres overlap. The indifference sphere of the three voters overlap not only in pairs (indicative that *M* is not a Condorcet winner) but also overlap as a group of three, consistent with *M* being outside the Pareto set. Although Figure 4 does not show the median vector *M* and its three indifference spheres, it does display the three indifference curves traced by the three indifference spheres which touch *M* as they cross the Pareto set plane. For each voter, the radius of the circular indifference curve is the square root of the sum of the squared distance from the voter's preference to *L* plus the square of the distance between *L* and *M*. Figure 5 shows the geometry from a different perspective, with the z' axis up down and the x'y' plane as a cross-section for one voter, voter C. Like voters A and B, C is closer to *L* than to *M,* the median vector.

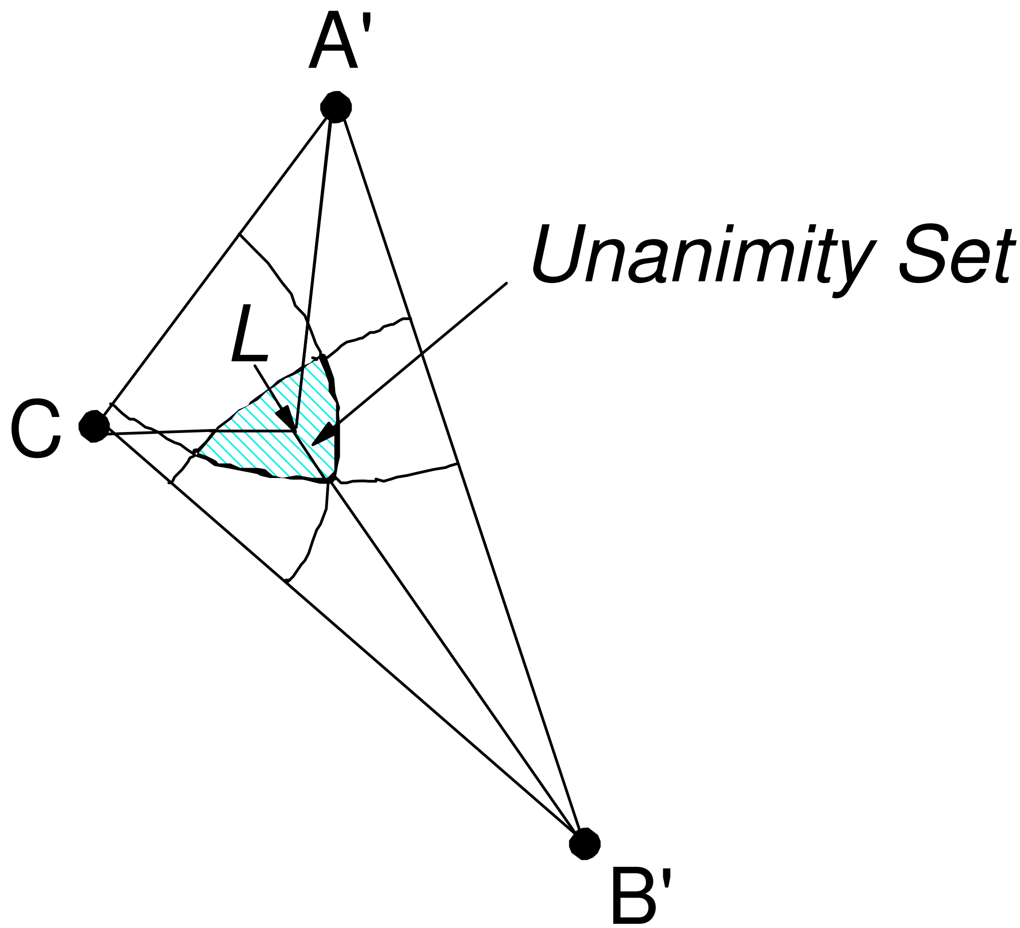

Figure 4. The Pareto Set A'B'C (from Figure 3).

In Figure 4, the interior bounded by the three indifference curves comprise a "unanimity set." All three voters would prefer to be anywhere within this set than at the intersection of the three medians, below the triangle on the page.. Figure 4 also shows a wider win-set (the unanimity set plus the three areas between two indifferent curves) where a majority would prefer to *M* as the policy location.[11] Much (but not all) of the Pareto set is within the win-set.

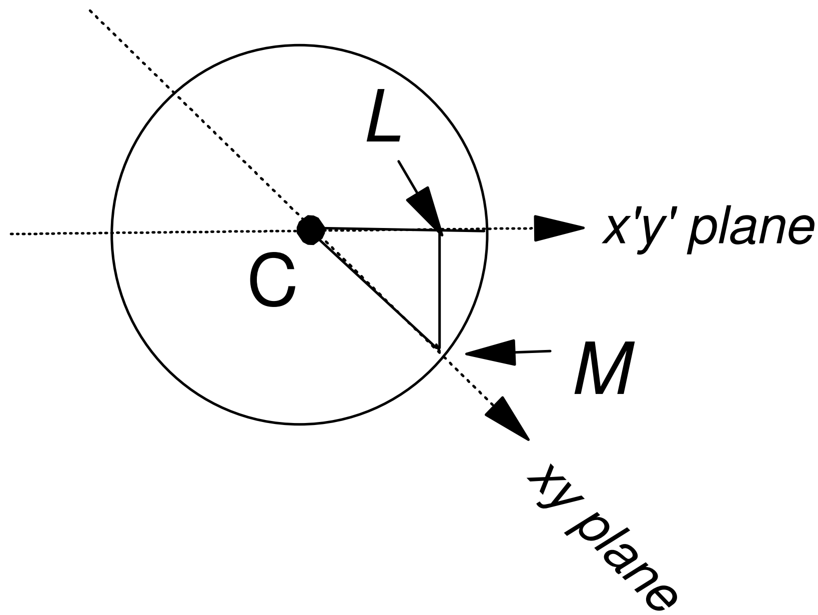

Figure 5. A cross-section (from Figure 3).

Below, we will see that the example of Figure 3 generalizes. For three voters and three dimensions, the median vector almost always lies outside the Pareto set. For intuition, consider whether the illustration in Figure 3 could be a mere contrivance in choosing the coordinates on the third (*z*) axis. Start with the original coordinates for *A*,*B*, and *C* on the *x* and *y* axes and then fix coordinates on the *z* axis for two voters, perhaps keeping C at zero and arbitrarily moving *B* to *B'* as shown. Now, select a hypothetical location on *z* for voter *A*. It should be obvious that of all the possible coordinates on the *z* axis, only one would allow the *ABC* and *A'B'C* planes to intersect

[11] The unanimity set and the win set each encompass a larger three-dimensional volume than the portions shown in the two-dimensional Pareto set.

through the two-dimensional median vector, thereby placing the three-dimensional median vector precisely on the *A'B'C'* Pareto set. We discuss the mathematics of this below.

5. The General Case: When is the Median Vector Outside the Pareto Set?

This section discusses the conditions under which the median vector is within the Pareto set. The presentation here covers the simple case without salience weights (i.e. when all intrapersonal salience weights equal 1.0). A more formal proof, covering the general case with interpersonal salience weights, is presented in the appendix.

Consider the following mental experiment. A researcher is given a "gold mine" of data, consisting of the actual ideal points of Congress members on each of K single-peaked issue dimensions. Our researcher's natural inclination might be to first conduct some sort of factor analysis on the data. Most likely, the researcher would identify two major factors, akin to Poole and Rosenthal's two dimensions, that define the data along with a member-specific error term. Suppose, however, that for some reason the researcher were to continue the extraction of factors until all error disappears and the data were entirely "explained." The researcher would find an absolute ceiling at N-1 factors. This is because mathematically, the number of factors (or explanatory dimensions) cannot be as large as the number of observations (legislators).

This is the same limit that applies to the Pareto set. The number of dimensions comprising the Pareto set (J) cannot be greater than N-1. This is no problem if the number of issue dimensions (K) is less than the number of voters (N). But when K equals N or greater, the medians on the K dimensions are in a larger K-space than the J-space of the Pareto set. Thus, with N or more dimensions, the J-space describing the Pareto set is slimmer than the K-space where the median vector resides, with the consequence that the median vector defined in K dimensions is unlikely to be within the Pareto set.

A more formal argument follows. In the general model, N legislators vote on K one-dimensional issues, where the distribution of ideal points on each dimension is unique and not a scalar product of the ideal points on another issue.[12] On each of K dimensions, voters decide policy positions ($Q_1, Q_2 \ldots Q_K$), which form the column vector $\mathbf{Q_K}$ as the grand policy decision over K dimensions. The feasible set where voters could choose $\mathbf{Q_K}$ *is* anywhere in *K-dimensional* space; i.e., $\mathbf{Q_K} \in \Re^K$. When voting takes place one issue at a time as a series of one-shot games, the predicted net outcome is the median vector, $\mathbf{Q_K}=\mathbf{M_K}$. An important assumption is that we rule out the trivial case where one voter is at the median on all K dimensions, .so that the median vector will not equal any one voter's composite K-dimensional ideal point.[13]

The Pareto set is bound by the convex hull of the ideal points—the minimum boundary needed to cover the ideal points—as if with K-dimensional Saran Wrap. The convex hull (Pareto set) includes all the outcomes definable as a weighted sum of the N multi-dimensional ideal points, for all possible combinations of weights for the N voters, where each voter is assigned a π_i weight.

[12] Allowance must be made for repeated appearance of issues with the identical one dimension. Repeated issues can be treated as one. For instance, consider ideal points on issue b as a linear function of ideal points on issue a: $b=ca$ where c= a scalar constant, scaled so $c>0$. Scores on the combined issue dimension k take the form $Z_k=a(1+c)^{1/2}$

[13] From the previous footnote, one-dimensional distributions of preferences that repeat themselves are treated as single dimensions. Treating such instances as one single policy dimension involves the implicit assumption that every iteration of the same preferences on the dimension reproduces the identical policy outcome, whether the median or some other position. In other words, the feasible set of outcomes is assumed to be constant for each repetition. As a purely technical matter, we could instead treat each repetition as a separate dimension. This would widen the number of dimensions required to define the feasible set while complicating the algebra.

$$Q_k = \frac{\sum_{i=1}^{N} \pi_i Z_{ik}}{\sum_{i=1}^{N} \pi_n}$$

for all Q_k in $\mathbf{Q_K}$. If a policy outcome is not definable as a weighted sum of ideal points, it is not within the Pareto set.[14] The appendix presents the circumstances when the median vector can and cannot be definable as a π-weighted average of preferences, and thus part of the Pareto set. Here, we offer a simpler discussion.

When *N*=*K*. When $N=K$, the ideal points on any one k dimension are linear dependent on the ideal points on every other dimension. With this redundancy, the number of analytical (Pareto set) dimensions (J) must always be one less than the number of voting dimensions (K). It takes only N-1 dimensions to fully describe the coordinates of N data points.

To illustrate, refer again to Figure 3, where with $K=N=3$, $J=2$. Even though Figure 3 depicts three voting dimensions, the Pareto set connecting the observations is a two-dimensional plane. In general, with K dimensions and N observations, it takes K dimensions to connect the observations, but only up to a maximum of N-1.

For the three-dimensional world of Figure 3, two dimensions are sufficient to describe the three data points because the coordinates in any one dimension (e.g., the z axis) are linearly dependent on the coordinates in the other two (e.g., the x and y axes). To see this, let us first define the three coordinates (Z_{ik}) describing the three-dimensional ideal points as x_i, y_i, and z_i, where each axis is calibrated so that the origin is at the median.

[14] For this mathematical definition of the convex hull, see (for example), Sydsaeter, Strom, and Berck (2000), p. 79.

(Equation 2) $$z_i = \lambda_0 + \lambda_1 x_i + \lambda_2 y_i$$

With three observations, a "regression" of z on x and y must yield perfect prediction. More generally, with $N \le K$ observations, regressing ideal points on one dimension upon the scores on the other dimensions must yield an "R squared" of 1.0, no matter what their values. (There are no remaining degrees of freedom.) But the median vector has no such restriction. While the Pareto set is restricted to an N-1 dimensional surface, the median vector is free to locate in K-dimensional space. Under these circumstances, the chance is remote that the median vector would intersect the Pareto set.

A remote chance of course is not an impossible chance. We should inquire, what are the conditions that produce the unlikely outcome where the median vector can be within the Pareto set even when $N=K$? As we will see, when issue-space is calibrated with the median vector at the origin, with $N=K$ the linear equation defining one dimension in terms of the others must have an intercept (constant term) equal to zero.

In the 3-voter, 3-dimension case, the 2-dimensional J-space describing the Pareto set is a special rotation of the 3-dimensional axes of K-space. Consider again equation 2. The axes of x and y can be rotated so that the coordinates of the three ideal points become identified in terms of only two dimensions, x' and y', where $x_i^{'} = \sqrt{1+\lambda_1} x_i$, $y_i^{'} = \sqrt{1+\lambda_2} y_i$. This is the plane defining the Pareto set. Ideal points are now constant on the third dimension, as $z_i^{'} = \lambda_0$. We immediately see the rare condition for the median vector to be within the Pareto set: with the zero point on each dimension set to its median, the only way that $z_i^{'}$ can be on the plane of the Pareto set is when λ_0=0.[15]

Thus, the requirement for the median vector $\mathbf{M_3}$ to lie within the Pareto set is that the "intercept" of the regression of *z* on *y* and *x* must take the specific value of λ_0=0. The results readily generalize to more than three issue dimensions. For the general case where *K*=*N*, equation 2 becomes:

(Equation 3) $$Z_{iK} = Z_{iN} = \lambda_0 + \sum_{k=1}^{N-1} \lambda_k Z_{ik}$$

where the ideal points on the *K*th (*N*th) dimension are linear dependent on the ideal points on dimensions 1 through *K*-1. The Pareto set includes the convex hull defined by the coordinates $\sum_{k=1}^{K-1} Z_{ik}^{'}$ where $Z_{ik}^{'} = \sqrt{1+\lambda_k Z_{ik}^2}$ and λ_0 is constant for all voters. Only if λ_0=0 will the median vector $\mathbf{M_K}$ be in the Pareto set.

When *K*>*N*. As we have just seen, when the *N*th issue dimension is reached (*K*=*N*), the chance of a median vector within the Pareto set becomes remote. As still additional issue dimension are incorporated (*K*=*N*+1, *N*+2, etc.) the possibility becomes even more remote. With each new issue, the equation for $Z_{i,N+m}$ takes the form of Equation 3, above:

[15] The methodology of this section involves nothing more than applying the Pythagorean theorem to standard problems of solid geometry.

$$Z_{i,N+1} = \lambda_{0,N+1} + \sum_{k=1}^{N-1} \lambda_{k,N+1} Z_{ik},$$

$$Z_{i,N+2} = \lambda_{0,N+2} + \sum_{k=1}^{N-1} \lambda_{k,N+2} Z_{ik}$$

(Equation set 4)

$$\cdot$$

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$$Z_{i,N+m} = \lambda_{0,N+m} + \sum_{k=1}^{N-1} \lambda_{k,N+m} Z_{ik}$$

where the λ's are subscripted twice, for past issues 1 through N-1 and for the sequence of the new issue, N+1 through $N+m$. For the median vector to be within the Pareto set, the data must run the gauntlet whereby the intercept λ_{0k} must be zero for each issue from N to $N+m$.[16] Then for issue $N+m+1$ and each one thereafter, the requirement occurs again. While it is theoretically possible for the median vector to in the Pareto set, the likelihood is extremely negligible—roughly the same as the miniscule likelihood that the next set of multi-dimensional set of ideal points will have the core solution of a median in all directions.[17]

Moreover, with each new issue, the evolving median vector outcome moves farther and farther from the Pareto set. We have seen that for N dimensions, the shortest Euclidean distance from $\mathbf{M_N}$ to the Pareto set is λ_{0N} (using the notation of equation set 4). For $N+m$ decisions, the difference expands to $\sqrt{\lambda_{0,N}^2 + \lambda_{0,N+1}^2 + \lambda_{0,N+2}^2 + \ldots \lambda_{0,N+m}^2}$.

[16] For this exercise, the order of issues is arbitrary. One could scramble the order from the first to the $N+m$th issues and the equations predicting the final $N+m$ issues from the first N-1 must all have zero intercepts.

[17] The challenge can be appreciated the following way. With $K=N$, finding the median vector within the Pareto set is equivalent to a 2-dimensional plane cutting through 3-dimensional space and hitting a particular point in this space head on. With $K=N+1$, the equivalence is for a 1 dimensional line cutting through 3-dimensional space hitting a particular point head on.

***When* N > K *but* J<K.** An objection to this analysis might be that while legislatures vote on many issues, the various issue-specific dimensions can be decomposed into only a small set of analytical dimensions. For instance, suppose that legislative ideal points could be fully explained with no error by only two Poole-Rosenthal type dimensions. How would this affect our theoretical result? The answer is that when the data can be explained with fewer than N-1 dimensions (but of course more than one), the median vector lies outside the Pareto set even when $K<N$; that is, even when there are fewer issue dimensions than voters.

The median vector is likely to lie outset the Pareto set when K>J+1. Normally J=N-1 because that the data (ideal points) can be fully explained by N-1 dimensions. Conceivably, however, J could be less than N-1 as in our hypothetical example where two Poole-Rosenthal type-dimensions predict perfectly. However, if all ideal points can be "explained" by a mere J dimensions where J is less than $N>K$ but $J<K$, then the median would depart from the Pareto set after only J+1 dimensions. The result could be that that the median is unlikely to be part of the Pareto set even when a large legislature votes on only three issues!

To see this, first consider the following data generating process. Suppose that ideal preferences on each of K dimensions are determined as:

(Equation 5) $$Z_{ik} = \lambda_{1k} F_{i1} + \lambda_{2k} F_{i2} + u_{ik}$$

where (in terms of deviations from the mean rather than median vector) F_1 and F_2 are two common ideological factors and u_{ik} is a normally distributed disturbance. Here, preferences on the single issue dimension k derive as linear functions of the two factors $F1$ and $F2$ (with the linear weights varying from issue to issue) plus idiosyncratic individual variation Defined as the coordinates of Z_i over K issue dimensions, the K dimensions will be non-redundant as long as $K<(N-1)$. But now suppose that preferences are drawn as various linear functions of F_1 and F_2 as above, but without error. That is, preferences vary across issues as a function of the relative weighting of F_1 and F_2, but that is all:

(Equation 6) $$Z_{ik} = \lambda_{1k} F_{i1} + \lambda_{2k} F_{i2}$$

If equation 6 is true, preferences on any two k dimensions would completely predict the preferences on each of the remaining K-2 dimensions. Thus, two dimensions would be sufficient to describe the distribution of ideal points (the J-space). Yet, with N=3 or greater, they would generally be insufficient to describe the coordinates of the median vector (in K-space).

Two visualize this, consider a simple three-issue example, where ideal points for a large legislature are identified as positions on x, y, and z where $z=\lambda+x+y$. Ideal points on z equal the sum of x and y in terms of deviations from the mean., Calibrated with the median vector at the origin, z has a λ component to account for the fact that the median is not necessarily the mean. The Pareto set would be a plane parallel to the x,y plane but with a deviation of λ up or down.[18] With the ideal points clearly defined in two dimensions, one might think that the median on the x and y dimensions would comprise an attractive solution. Yet, there would be certain positions above or below the x,y median vector that all members would prefer to the median vector.

[18] Picture the example in three dimensions, x, y, and z, where z is constrained to be $\lambda+x+y$. All observations are on the plane where x'=2x and y'=2y, where the perpendicular distance to the origin (the median vector) is. λ.

Of course the gap between the mean and median can be trivial. The point, however, is the existence of the paradox that when the issues dimensions are fully described by a small number of underlying analytical dimensions or "factors" (but more than 1), as soon as the number of issues exceed the number of factors (J), the median vector will probably lie outside the Pareto set. Thus, the more orderly the data, the sooner the median vector fails as a plausible outcome.

6. Intensity of Preferences

So far our analysis has been restricted to examples where the N voters all weigh the K dimensions the same, in effect giving each dimension a weight of 1.0. This restriction has kept the examples simple. The requirement of concave loss functions is sufficient to drive the result.[19] In the lore of politics, however, it is varying intensities of preference rather than concave loss functions that motivates discussions of vote trading, logrolling, horse trading, and the like. The standard imagery is of legislators trading their votes on issues upon which they care little for better outcomes on issues about which they care a lot. The net result is a majority coalition of indefinite size voting into law a net policy reflecting the preferred positions of intense minorities on each issue.

Differential intra-personal weighting of issue dimensions affects the Pareto set. Instead of a J-dimensional hyper-plane, the Pareto set is a curved J-dimensional hyper-surface. For three voting dimensions and J=2, the shape is a bowl-like curved surface rather than a flat plane. The curvature of this bowl depends on the distribution of issue

[19] As an alternative to quadratic utility loss, one can consider utility loss as the sum across dimensions of the absolute distance from one's ideal point. In two dimensions, this is the city-block or taxi-cab model.

salience relative to ideal points.[20] If voters' ideal points tend toward the median on their most salient dimensions, the crown of the curved surface of the Pareto set is pulled from the plane of the convex hull in the direction of the median vector, shrinking the distance between the median vector and the Pareto set. But if voters' ideal points tend to be more extreme on their most salient dimensions, the crown of the curved surface is pulled from the plane of the convex hull in the direction further away from the median vector, thus accentuating the distance between the median and the Pareto set.[21]

Thus, when legislators care the most about issues upon which they hold relatively extreme positions, the cost of a median vector outcome increases beyond the cost when legislators weigh all issues equally. Put positively, caring most about extreme positions enhances the value of gains from trade over and above the reversion point of the median vector.

7. A variation: Multi-dimensional decisions, one at a time

So far, we have enforced the assumption that issues arrive well-behaved, with single-peaked preferences along one dimension, a contrivance that induces the median as the one-shot equilibrium outcome. In this section we consider what happens if issues arrive instead in multi-dimensional form as m-dimensional decisions where $1<m$. With more than one dimension, the intersection of medians has no special cache as a focal point for the decision. According to the rules of social choice, a one-shot decision in m

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With utility loss as the sum of absolute distances, for the median vector to lie outside the Pareto set requires a sufficient degree of differential weighting of the different issue dimensions.

[20] Keep in mind that the Pareto-set is the surface of the "bowl" and not its interior.

[21]. With two dimensions and three or more voters with variation in intrapersonal weighting of the dimensions, the Pareto set is a set of curves rather than straight lines connecting the points of the convex hull. .

dimensions is inherently indeterminate; the one prediction would be an outcome within the local Pareto set for the *m* dimensions voted on at the time. The legislature would choose some set of coordinates $Q_1, Q_2 \ldots Q_m$ to represent policy on each set of *m* dimensions, which would not necessarily represent the median vector.

With multiple dimensions per decision, the relevant question is whether solutions within the local *m*-dimensional Pareto sets are still Pareto optimal when cumulated over multiple issues. We have seen that when voting one dimension at a time, the intersection of medians is in the Pareto set only as long as $J>K$. Similarly, the combination of *any* set of multi-dimensional outcomes will be Pareto optimal only as long as $J>K$. This can be seen by simply rescaling the zero points on the *K* dimensions to whatever outcomes are chosen on the individual dimensions, $Q_1, Q_2 \ldots Q_K$. instead of the medians. With this rescaling, the arguments follow directly in the same way as when the policies chosen are the dimension medians. See also, the appendix.

It can now be seen that a series of one-shot resolutions of multidimensional issues is likely to result in a net outcome outside the Pareto set whenever the number of dimensions *K* reaches the number of voters *N*. This is precisely the resolution of one-shot *one*-dimensional decisions, but with the following exception: with multidimensional decisions it takes fewer decisions (adding to *K* dimensions) before the limit is reached where $K=N$. If decisions are bundled as multi-dimensional choices, *N* voters vote on *N* dimensions with fewer than *N* votes.[22]

Thus the implications extend beyond merely the efficiency of the median voter position as a decision rule. Whenever the outcomes of multiple one-shot decisions (each

[22] A disadvantage of modeling cumulative decisions in multiple dimensions, however, is the loss of the median as the one-shot equilibrium and therefore as a reversion point for legislative bargaining.

involving one or more dimensions) are cumulated so that K, the number of dimensions exceeds the number of voters N (or more generally, when $J<K$), the cumulative net outcome $\mathbf{Q_K}$ will not necessarily be within the Pareto set. Suppose, then, our legislature has the capacity to "solve" its inefficiency problem by postponing issue resolution until the cumulation of disagreements over $N+m$ dimensions. It can then achieve its short-term solution by passing an omnibus bill over $K=N+m$ dimensions with an outcome that is within the Pareto set (and if it chooses, in the unanimity set where outcomes for all voters are preferred to the median vector).

But then, the inefficiency problem arises once again when the legislature must vote on issues still further in the future. Say the legislature votes into policy a Pareto optimal decision over dimensions 1 through $N+m$ and then again a separate Pareto optimal decision over dimensions $m+1$ through $m+h$. The pooled outcome, while possibly in the unanimity set, will probably not be within the Pareto set for dimensions 1 through $N+m+h$. That is, after $N+m+h$ dimensions, there are other net outcomes that all voters would prefer to the combination of the two omnibus decision. This is because with $K>N$, the outcome cannot be in the Pareto set unless the net decision is a weighted mean of a set of interpersonal π-weights that is constant for every dimension. (See the appendix.)[23] The practical implication is that it is difficult for a legislature to approximate Pareto-efficiency over time unless it is able to make decisions that incorporate information about the future.

[23] One "solution" for restoring Pareto optimality would be to keep adding voters when K approaches N. Consider the case where $K=N$ so that the median vector is likely to be out of the Pareto set. Suppose this legislature had been expanded by adding two new voters, who always paired so that they were on opposite sides of the median. The net policy would be unchanged at the median vector , but this policy would now be in the Pareto set! However, this is a hollow victory because what changed is the loss of the potential for Pareto improvement.

8. Discussion

This paper has modeled legislative decisions as choices on a series of one-dimensional policy issues. If each issue is considered as a one-shot collective decision, the policy result is the median voter's preference on each issue dimension—the median vector. After many issues are decided however, the median vector is an inefficient outcome—no longer within the Pareto set. The implication is the existence of alternative outcomes that *every* legislator would prefer to the median vector outcome generated from voting one dimension at a time. The severity of the inefficiency of the median vector solution becomes magnified when legislators typically weigh most heavily the dimension upon which they hold extreme positions.

If the model presented here approximates legislative reality, what should one conclude? The most obvious implication is the advantage of combining issues over many dimensions in such a way that everybody gains and nobody loses. A universal gain in utility from bundling decisions over many dimensions trumps the certainty of median voter equilibria from voting one dimension at a time. Crucially, bundling over many issues presents a special challenge when agenda items arrive on the legislative doorstep one dimension—or perhaps a few dimensions—at a time. The challenge is how to achieve efficient outcomes when taking into account the choices that await in a future that is at least partially unknown.

A pessimistic conclusion would be that politics must indeed be a dismal business, with democratic voting invariably leading only to undesirable policies. Whereas legislators might recognize the value of moving from the median vector toward the

Pareto set, they would have no way to get there. For example, any attempted system whereby current concessions are bartered for future favors would face the risk of defeat in an atmosphere of distrust, deception, ignorance, and short-sightedness. Without mechanisms for enforcing bargains across time, the legislators are trapped in a series of suboptimal one-shot games with outcomes governed by the short-term preferences of the median voter of the moment.

A more optimistic view is that the need for bargaining across time simply defines the legislators' collective action problem. Furthermore, we can posit that legislators possess certain political capabilities that help them reach a solution. For example, legislators' preferences can be assumed to be transparent so that as issues arise, legislators recognize each others' interests without posturing or deception. Second, legislators can be assumed to be capable of learning—able to infer future issue alignments from the patterns of preferences arising in the past. More specifically, legislators can be assumed capable of holding rational expectations about future issue alignments. Although they can know the future only up to a limit, with inevitable error or uncertainty; what they know they know without bias.

The findings of this paper reinforce the importance of designing legislative institutions. Based on the logic of this paper, legislative majorities have the incentive to thwart rather than facilitate the median voter of the moment.[24] Of course just because

[24]One institutional example is a strong committee system where enforced norms allow outlier committees to substitute the committee mediansfor the floor medians. This arrangement could push policy toward the Pareto set while imposing the stability of a structurally induced equilibrium (Schepsle, 1979).. The benefit would depend on the differential intensity of preferences across issues. Consider a toy version of this model where a legislature divides into three equal groups, with each in control of the committee deciding its preferred issue.. Say each had a preference of 1 for its preferred policy, -1 for the preferred policy of other groups, and zero for the status quo. Each group weighs its preferred dimension as w relative to 1.0 for the otheres. Thus, each group member derives utility of $-w(1-P)^2$ for policy on its most salient issue but $-P^2$ on the other two issues. As long as w is 2 or greater, the members are better off from the arrangement,

there exist potential policy outcomes that all legislators would prefer to the median vector position, it does not follow that the institutional arrangement must be the result of a universal bargain. When the median vector is Pareto inferior, *any* majority coalition could find a net outcome that its members prefer to the median vector. Anticipating this result and the uncertainty it entails, legislators would be attracted to a universalistic bargain among a coalition of all members. Following the logic of Weingast 1979), a universalist coalition is preferable to the expectation when one does not know the membership of the winning coalition.

Suppose, however, that a small majority coalition commits to a strategy favoring its members. For instance, a specific majority could commit as a political party to set the agenda in its interests. If so, the interests of the party would be not only to improve on the "floor" median vector but also the median vector within the party. That is, the majority party could search for multi-dimensional outcomes that its members prefer to the collection of median preferences of the legislature but also to the median preferences of its own members.[25]

9. Conclusion

Discussions of the dimensionality of voting issues typically contrast the equilibrium of the median voter solution given only one dimension with the disequilibrium and potential chaos of many dimensions. Channeling policy-making into a series of one-dimensional policy decisions leads to predictable outcomes as the median on each dimension. The

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as they Pareto-improve over the median of no action. The legislature would be best off with norms allowing some concession to the floor median, however. Weighting each legislator's utility equally in the toy example, the Pareto set would be reached when policy on each dimension equals the weighted mean, $w/(w+2)$ on the 0-1 continuum..

[25] When a coalition of majority party members determines policy outcomes within the Pareto set for the legislative subset who are party members, the outcomes are also within the Pareto set for the legislature as a

presumption usually is that predictable equilibria are in the interests of the actors. This paper takes the reverse approach. It starts with unidimensional decisions and asks whether, disregarding the inherent unpredictability of the outcome, it is in the actors interests to bundle issues over many dimensions. The general answer, we have seen is yes: with many dimensions, the intersection of medians is Pareto inferior. The legislature is better off moving to an outcome in the Pareto set uniformly preferred to the intersection of medians. The challenge is figuring out both how to get there and stay there.

Appendix: The Proof

The argument is that under very general conditions, the intersection of medians *M* is outside the Pareto set. For the proof we make use of the fact that the Pareto set represents the *inter*personally weighted average of the voters' *intra*personally weighted positions on the *K* dimensions. The *K*-dimensional outcome $\mathbf{Q_K}$ represents the intersection of Q_k decisions on dimensions 1 through *K*:

$$\mathbf{Q_K} = \begin{bmatrix} Q_1 \\ Q_2 \\ . \\ . \\ Q_K \end{bmatrix}$$

$\mathbf{Q_k}$ will be in the Pareto set if and only if it the K Q_k components can all be described as a weighed average of voter preferences,

whole. Outcomes within the Pareto set are a weighted average of preferences of all members. With a dominant majority party coalition, the weights for minority party members might approach zero.

$$Q_k = \frac{\sum_{i=1}^{N} \pi_i w_{ik} Z_{ik}}{\sum_{i=1}^{N} \pi_i}$$

where the π_i weights assigned to the individual voters (π_is) take on positive values. For the median vector *M* to be within the Pareto set, there must be at least one set of π_is that include *M* within the Pareto set.

We represent the Pareto set by a series of equations describing the weighted mean over a number of dimensions up to and beyond $K=N$ dimensions. Our question is under what conditions will find an outcome that is both in the Pareto set and the median vector. For convenience, we apply two normalizing conventions:

(1) voter ideal points are normalized by setting the median on each dimension to equal zero.

(2) the individual π_is are normalized by setting the weight of voter *N* (π_N) to equal 1.

The Pareto set includes all weighted averages of the voters' ideal points. Thus, if the median vector is within the Pareto set, it must be possible to represent it as a weighted average of ideal points. With the zero point set to equal the median on each dimension, a dimension's weighted mean will equal the dimension's median only when the weighted mean also equals zero. Thus, we set both the weighted mean and the median at zero and look for contradictions.

If the median vector is part of the Pareto set when $K=N$, there must be a set of weights $\pi_1......\pi_{N-1}$ where $\pi_N = 1.0$ so that the following conditions are satisfied:

(Equation Set A1)

$$(k=1)\quad \pi_1 w_{11} Z_{11} \ +\pi_2 w_{21} Z_{21} \ +\pi_3 w_{31} Z_{31} \ +\pi_4 w_{41} Z_{41} \ \ ......+w_{N1} Z_{N1}=0$$
$$(k=2)\quad \pi_1 w_{12} Z_{12} \ +\pi_2 w_{22} Z_{22} \ +\pi_3 w_{32} Z_{32} \ +\pi_4 w_{42} Z_{42} \ \ ......+w_{N2} Z_{N2}=0$$
$$(k=3)\quad \pi_1 w_{13} Z_{13} \ +\pi_2 w_{23} Z_{23} \ +\pi_3 w_{33} Z_{33} \ +\pi_4 w_{43} Z_{43} \ \ ......+w_{N3} Z_{N3}=0$$
$$(k=4)\quad \pi_1 w_{14} Z_{14} \ +\pi_2 w_{24} Z_{24} \ +\pi_3 w_{34} Z_{34} \ +\pi_4 w_{44} Z_{44} \ \ ......+w_{N4} Z_{N4}=0$$
$$\vdots$$
$$(k=N)\quad \pi_1 w_{1N} Z_{1N} +\pi_2 w_{2N} Z_{2N} +\pi_3 w_{3N} Z_{3N} +\pi_4 w_{4N} Z_{4N} ......+w_{NN} Z_{NN}=0$$

where each equation represents the sum of the N voters' weighted preferences on the kth issue dimension and

$Z_{ik} = Voter\ i$'s ideal point on dimension k,
w_{ik} = $Voter\ i$'s intra-personal relative weight of dimension k in $Voter\ i$'s quadratic loss function, and
$\pi_i =$ the weight for $Voter\ i$ that potentially maps the grand median M onto the Pareto set.
and $\pi_N = 1$ and the median on each dimension k=0.

The w_{ik} salience weights are included to incorporate the general case where voters weight the dimensions differently.. Where each voter weighs all dimensions equally (e.g., circular indifference curves), all w_{ik} can be set to 1.0 and ignored.

Let us begin with a system of the first *N-m* equations where the dimensionality *K*=*N-m* is at least two less than *N*, the total number of voters. With *K*<*N*-1, there are more unknowns (the π's) than equations. Under these circumstances, the intersection of medians can coincide with many different sets of π weighting schemes, and therefore lies within the Pareto set. The two-dimensions seven voter illustration of Figure 1 is an example. The median vector is within the Pareto set and without requiring one specific weighting of preferences to get this result.

Next consider the case where the number of dimensions is one less than the number of voter, so that the number of equations (*K*) precisely equals *N*-1. If we try to solve for the π's using only the equations from *k*=1 through *k*=*K*=*N*-1, we find exactly the same number of equations as unknowns, so that (assuming no multicolinearity among the *N*-1 dimensions—see below) the median vector is exactly identified. Under these circumstances, the median vector outcome is within the Pareto set and implies one specific set of π's. For instance, in Figure 2 with two dimensions and three voters, only one set of π's can translate the median vector into the intersection of weighted means.

Next, consider the case where the number of dimensions and equations equals the number of voters (K=N). With the number of equations equaling the number of voters, the voters' w-weighted preferences on dimension k=K=N must be an exact linear function of their w-weighted preferences on dimensions 1 through N-1. The question is, how likely is it that the π-weighted mean on dimension N would be at zero, the median. Since the π weights are exactly determined via equations 1 through N-1, a mean of zero for equation N is possible only as an unlikely knife-edge result. In general, the left-hand side of the equation for dimension N would not equal zero, and the median preference on dimension N will be outside the Pareto set. Thus, with at least N voting dimensions, the median vector is almost certainly outside the Pareto set. With still additional voting dimensions (N+1, N+2, etc.), the unlikelihood of preferences falling within the Pareto set only compounds.

We can describe further the unique circumstances for which the median vector will be part of the Pareto set when N=K. With N=K, the ith voter's (salience-weighted) preferences on dimension N must be an exact linear function of the ith voter's (salience-weighted) preferences on the first N-1 dimensions:

$$w_{iN} Z_{iN} = \lambda_0 + \sum_{k=1}^{N-1} \lambda_k w_{ik} Z_{ik}$$

where $\lambda_0 \ldots \lambda_{N-1}$ are the parameters of the linear equation. Aggregating across voters, the equation for k=N in equation set (A1) can be re-stated as:

(Equation A2) $$\sum_{i=1}^{N} \pi_i (\lambda_0 + \sum_{k=1}^{N-1} \lambda_k w_{ik} Z_{ik}) = 0$$

Rearranging again,

(Equation A3) $$\lambda_0 \sum_{i=1}^{N} \pi_i + \sum_{i=1}^{N} \pi_i \sum_{k=1}^{N-1} \lambda_k w_{ik} Z_{ik} = 0$$

We look for conditions under which equation (A3) can be true when equations (k=1) through (k=N-1) of equation set (A1) are true. Note that the second component on the left-hand side of equation (A3) must equal zero. This is because it equals the sum of equations (k=1) through (k=N-1) where each equation is multiplied by λ_k. With the second left-hand component and the right hand component both equaling zero, the first left-hand component must equal zero also. This requires that λ_0, the intercept of the equation predicting the positions on one dimension from the position on all others must equal zero.

We are left with the following: Equation (A3) can be true only if λ_0=0. Thus we learn that when K=N, the median vector can be within the Pareto set only if the intercept is zero for the equation predicting the (salience-weighted) preferences on dimension N from the salience weighted dimensions 1 through N-1.

The knife-edged nature of the necessary conditions for the N-dimensional grand median to lie within the Pareto set can be illustrated with a simple 3-voter 3-dimension example. Following is the three-dimensional version where all the w's equal 1 and the medians are identified by their zero values. (Each voter is the median voter on one of the three dimensions.) In this example, the requirement that the intersection of three medians is in the Pareto set becomes:

(Equation set A4)

$$\begin{array}{lllll} (k=1) & \pi_1 Z_{11} + \pi_2 Z_{21} + & 0 & = & 0 \\ (k=2) & \pi_1 Z_{12} + \quad 0 & + Z_{32} & = & 0 \\ (k=3) & \quad 0 \quad + \pi_2 Z_{23} & + Z_{33} & = & 0 \end{array}$$

For three voters and two dimension (1 and 2 only), we can solve for the π's by using the k=1 and k=2 equations while ignoring the k=3 equation. We obtain

$$\pi_1 = -\frac{Z_{32}}{Z_{12}}$$

and

$$\pi_2 = \frac{Z_{32} Z_{11}}{Z_{12} Z_{21}}$$

Substituting π_2 into the k=3 equation, we find the condition required for the intersection of medians to lie within the Pareto set. The ratio of the two non-median ideal points on the third dimension must satisfy the following equation:

$$\frac{Z_{33}}{Z_{23}} = -\frac{Z_{32}Z_{11}}{Z_{12}Z_{21}}$$

Only if this equation is satisfied, will the intersection of medians be within the Pareto set. (By a bit of tedious algebra, this equation can be shown to be equivalent to the requirement of a zero intercept when Z_{i3} is defined as a linear function of Z_{i1} and Z_{i2}.)

When **N>K:**

Having a number of independent issue dimensions equal to the number of voters is a necessary condition for it to be improbable that the median intersection would be within the Pareto set. However it is not sufficient. The median vector lying outside the Pareto set is also improbable with fewer dimensions than voters, if some of the equations are redundant. Consider three dimensions with N>3 voters, where the third dimension is redundant, identifiable an exact function of the other two:

(Equation Set A5)

$$\begin{array}{lllll}
(k=1) & \pi_1 w_{11} Z_{11} & +\pi_2 w_{21} Z_{21} & ......+w_{N1}Z_{N1} & =0 \\
(k=2) & \pi_1 w_{12} Z_{12} & +\pi_2 w_{22} Z_{22} & ......+w_{N2}Z_{N2} & =0 \\
(k=3) & \pi_1 w_{13}(\lambda_0+\lambda_1 Z_{11}+\lambda_2 Z_{12}) & +\pi_2 w_{23}(\lambda_0+\lambda_1 Z_{21}+\lambda_2 Z_{22}) & ......\quad +w_{N3}(\lambda_0+\lambda_1 Z_{N1}+\lambda_2 Z_{N2}) & =0
\end{array}$$

where, as usual, the median ideal point on each dimension is set to zero, and we start by assuming that the median is also the weighted mean, the condition that allows the median vector to be within the Pareto set. We will see that given the k=1 and k=2 equations, the k=3 equation is unlikely. The left-hand side will not sum to zero except under knife-edge conditions.

Here, there is no scarcity of the number of π's to identify relative to the number of equations. There is a surplus of π's and degrees of freedom—N-1 π's for 3 equations. Still, the Z_{i3}s of the equation for k=3 are redundant with the Z_{i1}s and Z_{i2}s of equations k=1 and k=2 in a manner analogous to when the number of issues (equations) equals the number of voters. We can rewrite equation for k=3 of equation set (A5) as:

(Equation A6) $$\lambda_0 \sum_{i=1}^{N} \pi_i + \lambda_1 \sum_{i=1}^{N} \pi_i w_{i1} Z_{i1} + \lambda_2 \sum_{i=1}^{N} \pi_i w_{i2} Z_{i2} = 0$$

The component $\lambda_1 \sum_{i=1}^{N} \pi_i w_{i1} Z_{i1} + \lambda_2 \sum_{i=1}^{N} \pi_i w_{i2} Z_{i2}$ equals the sum of the equations for k=1 and k=2, multiplied by λ_1 and λ_2 respectively. They must therefore sum to zero. This means that for the left hand side of equation (A6) to sum to zero, the intercept λ_0 must equal zero.

When voting decisions are multidimensional:

When voting decisions are made on two or more dimensions at a time, there is no reason to expect the legislature to select the median on the various dimensions to be the outcome. Consider a set of m-dimensional one-shot decisions where $1<m<K$. For any m-dimensional one-shot decision, the only expectation is that the legislature will choose a policy outcome that is within the Pareto set for the m dimensions. That is, each m-dimensional outcome can be interpreted as the π-weighted average on the m dimensions, where each π_i is constant for all m dimensions

The m-dimensional decisions cumulate to a grand outcome $\mathbf{Q_K}$, the intersection of the policy decisions (Q_k) on each of K dimensions. At this point it is helpful to rescale the ideal points so that the origin is now the generic grand outcome $\mathbf{Q_K}$ rather than the median vector $\mathbf{M_K}$. (That is, each ideal point Z_{ik} is rescaled as a deviation from Q_k=0.). Then, assuming $m<N$, the analysis can proceed directly as for Equation set A1. The first N-1 equations define the unique values of the π weights that comprise the Pareto set for N-1 dimensions. The equations for $K=N$, $K=N+1$, $K=N+2$ etc. will not be in the Pareto set except under the unique conditions where the decisions on these dimensions can be described as the π-weighted mean using the same π weights used for equations 1 through N-1.

Summary

In summary, the intersection of medians (the median vector) is certain to be in the Pareto set only when voter positions on the issue dimensions are independent of each other in the sense that none are an exact function of others. This set of circumstances normally arises when the number of voters exceeds the number of issue dimensions. If the number of independent issue dimensions equals or exceeds the number of voters, positions on one or more issue dimension must be a linear function of positions on other dimensions. As a consequence, the median vector almost certainly lies outside the Pareto set. The median vector lies outside the Pareto set whenever positions on some dimensions are an exact function of positions on others—as when a set of "factor scores" perfectly define positions on seemingly separate issue dimensions that are voted on separately. Thus, for the number of voters to exceed the number of dimensions is a necessary but not sufficient condition for the median vector to lie within the Pareto set.

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