A Markov Chain Approximation to Choice Modeling

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Assortment planning is an important problem that arises in many industries such as retailing and airlines. One of the key challenges in an assortment planning problem is to identify the "right model" for the substitution behavior of customers from the data. Error in model selection can lead to highly sub-optimal decisions. In this paper, we present a new choice model that is a simultaneous approximation for all random utility based discrete choice models including the multinomial logit, the nested logit and mixtures of multinomial logit models. Our model is based on a new primitive for substitution behavior where substitution from one product to another is modeled as a state transition of a Markov chain. We show that the choice probabilities computed by our model are a good approximation to the true choice probabilities of a random utility discrete based choice model under mild conditions. Moreover, they are exact if the underlying model is a Multinomial logit model. We also show that the assortment optimization problem under our choice model can be solved efficiently in polynomial time. In addition to the theoretical bounds, we also conduct numerical experiments and observe that the average maximum relative error of the choice probabilities of our model with respect to the true probabilities for any offer set is less than 3% (the average being taken over different offer sets). Therefore, our model provides a tractable data-driven approach to choice modeling and assortment optimization that is robust to model selection errors. Moreover, the state transition primitive for substitution provides interesting insights to model the substitution behavior in many real-world applications.

Categories and Subject Descriptors: J.4 [**Social and Behavioral Sciences**]: Economics

General Terms: Algorithms, Economics, Management

Additional Key Words and Phrases: choice Modeling, data-driven Algorithms, revenue Management, assortment Optimization

ACM Reference Format:
Jose Blanchet, Guillermo Gallego, and Vineet Goyal, 2013. A Markov Chain Approximation to Choice Modeling. *ACM* 9, 4, Article 39 (June 2013), 20 pages.
DOI:http://dx.doi.org/10.1145/0000000.0000000

1. INTRODUCTION

Assortment optimization is an important problem that arises in many industries such as retailing and airlines where the decision maker needs to select an optimal subset of products to offer to maximize the expected rewards, which can be revenues or profit contributions. The demand and the revenue of any product depends on the complete set of offered products since customers potentially substitute to an available product if their most preferred product is not available. Such a substitution behavior is captured by a customer choice model that can be thought of as distribution over preference lists (or permutations of products). A customer with a particular preference list purchases the most preferable product that is available. It is possible that the no-purchase alternative is the most preferable among the offered products in which case the customer leaves without purchasing any product. Therefore, the choice model specifies the probability that a customer selects a particular product for every offer set. One of the key challenges of any assortment planning problem is to find the "right choice

J. Blanchet is supported by NSF Grants CMMI-0846816 and CMMI-1069064. V. Goyal is supported by the NSF grant CMMI-1201116.
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DOI:http://dx.doi.org/10.1145/0000000.0000000

model" to describe the substitution behavior when we only observe historical sales data for a small number of assortments. The underlying customer preferences is latent and unobservable.

Many parametric choice models have been extensively studied in the literature in several areas including marketing, transportation, economics and operations management. Typically, the decision maker selects a parametric form for the choice model where the parameters are estimated from the data. The tractability of the parameter estimation and assortment optimization problems are important factors in the model selection. For these reasons, the multinomial logit (MNL) model is one of the most widely used parametric choice model in practice even though the model justifications (for instance, *Independence from Irrelevant alternative* (IIA) property) are not reasonable for many applications. A more complex choice model can capture a richer substitution behavior but leads to increased complexity of the assortment optimization problem and runs the risk of over-fitting the data.

1.1. Our Contributions

We present a new computationally tractable data-driven approach to choice modeling that is robust to model selection errors. Our approach is based on a new primitive to model the substitution behavior. In general, the substitution behavior of any customer is captured by his preference list over the products where he selects the most preferred product that is available (possibly the no-purchase alternative). This selection process can be interpreted as sequential transitions from one product to another in the order defined by the preference list until the customer finds an available product.

Markov Chain based Choice Model. Motivated by the above interpretation, we present a new choice model where substitution behavior is modeled as a sequence of state transitions of a Markov chain. We consider a Markov chain where there is a state for each product including the no-purchase alternative, and model the substitution behavior as follows: a customer arrives in the state corresponding to his most preferable product. If that product is not available, he/she transitions to other product states according to the transition probabilities of the Markov chain. Therefore, the sequential transitions based on the preference list are approximated by Markovian transitions in the Markov chain based choice model.

The Markov chain based choice model is completely specified by the arrival probabilities in each state and the transition probability matrix. We show that both the arrival probabilities to each state and the transition matrix can be estimated efficiently from choice probability data for a small number of assortments ($O(n)$ where n is the number of products). Furthermore, given the arrival probabilities and the transition probabilities, we can efficiently compute the choice probabilities for all assortments for the Markovian substitution model. For any assortment $S \subseteq \mathcal{N} = \{1, \ldots, n\}$, we modify the Markov chain to make all states corresponding to products $j \in S$ as absorbing. Then the stationary distribution over all absorbing states (including the no-purchase alternative) gives us the choice probabilities of all products in S. These can be computed efficiently by solving a system of linear equations.

Approximation Bounds. A natural question that arises is to study how accurately does the Markov chain model approximate the true underlying model given the data. We show that the Markov chain choice model provides a good approximation for all random utility discrete choice models under mild assumptions. The class of models arising from a random utility model is quite general and includes all models that can be expressed as distributions over permutations. This class includes MNL, Nested logit (NL) and mixture of MNL (MMNL) models (see [McFadden and Train 2000]). We present lower and upper bounds, related to the spectral properties of the Markov chain, on the ratio of the choice probability computed by the Markov chain model and the true underlying model. These bounds show that the Markov chain model provides a good approximation for all random utility based choice models under very mild assumptions.

Furthermore, we show that the Markov chain model is exact if the underlying model is MNL. In other words, if the Markov chain model parameters are estimated from data from an underlying

MNL model, then the choice probability computed by the Markov chain model coincides with the probability given by the MNL model for all products and all assortments.

We would like to emphasize that the estimation of the Markov chain is data-driven and does not require any knowledge about the type of underlying model that generates the data. Therefore, the Markov chain model circumvents the challenging model selection problem for choice modeling and provides a simultaneous approximation for all discrete choice models.

Computational Study and Asymptotic Bounds. In addition to the theoretical approximation bounds, we present a computational study to compare the choice probability estimates of the Markov chain model as compared with the choice probability of the true model. In particular, we consider random instances of mixture of MNL models and compare out of sample performance of the Markov chain model with respect to the true mixture of MNL model. The numerical experiments show that our model performs extremely well on random instances of mixture of MNLs. In particular, the maximum relative error of choice probabilities estimates as compared to the true choice probabilities is less than 3% on average over different offer sets.

Assortment Optimization. We show that the assortment optimization problem can be solved optimally in polynomial time for the Markov chain choice model. In an assortment optimization problem, the goal is to find an assortment (or offer set) that maximizes the total expected revenue, i.e.,

$$\max_{S \subseteq \{1,\ldots,n\}} r(S) = \sum_{j \in S} r_j \cdot \pi(j, S),$$

where r_j is the revenue per unit of product j and $\pi(j, S)$ denotes the choice probability of product j when the offer set is S. This result is quite surprising since in the Markov chain based choice model, we can not even express $\pi(j, S)$ as a simple functional form of the model parameters. Therefore, we are not able to even formulate the assortment optimization problem as a mathematical program directly. However, we present a policy iteration algorithm to compute an optimal assortment in polynomial time for the Markov chain based choice model. In particular, we show that we can compute an assortment $S \subseteq \mathcal{N}$ with revenue within $\epsilon > 0$ of the optimal assortment in $O(\log 1/\epsilon)$ policy iterations. By choosing a sufficiently small ϵ, we converge to an optimal solution in polynomial time. Moreover, our algorithm shows that the optimal assortment is independent of the arrival rates $\lambda_i, i \in \mathcal{N}$. This provides interesting insights about the structure of the optimal assortment.

Furthermore, we show under mild conditions that if Markov chain model is estimated from data generated by some underlying latent choice model, then the optimal assortment for the Markov chain model is also a good approximation for the assortment optimization problem over the underlying latent model.

1.2. Related Work

Discrete choice models have been studied very widely in the literature in a number of areas including Transportation, Economics, Marketing and Operations. There are two broad fundamental questions in this area: $i)$ learn the choice model or how people choose and substitute among products, and $ii)$ develop efficient algorithms to optimize assortment or other decisions for a given choice model. The literature in Transportation, Economics and Marketing is primarily focused on the choice model learning problem while the Operations literature is primarily focused on the optimization problem over a given choice model. Since this paper considers both these fundamental problems, we give a brief but broad review of the relevant literature.

A choice model, in the most general setting, can be thought of as a distribution over permutations that arise from preferences. In the random utility model of preferences, each customer has a utility $u_j + \epsilon_j$ for product j where u_j depends on the attributes of product j and ϵ_j is a random idiosyncratic component of the utility, i.i.d according to some distribution. The preference list of the customer is given by the decreasing order of utilities of products. Therefore, the distribution of ϵ_j completely specified the distribution over permutations, and thus, the choice model. This model was introduced by [Thurstone 1927] in the early 1900s. A special case of the model is obtained when ϵ_j's are i.i.d according to a normal distribution with mean 0 and variance 1. This is referred to as the *probit model*.

Another very important special case of the above model is obtained assuming ϵ_j's are i.i.d according to an extreme value distribution such as Gumbel. This model also referred to as the Plackett-Luce model and was proposed independently by [Luce 1959] and [Plackett 1975]. It came to be known as the Multinomial logit model (or the MNL model) after [McFadden 1973] referred to it as a conditional logit model. Before becoming popular in the Operations literature, the MNL model was extensively used in the areas of transportation (see [McFadden 1980], [Ben-Akiva and Lerman 1985]), and marketing (see [Guadagni and Little 1983] and surveys by [Wierenga 2008] and [Chandukala et al. 2008]). In the Operations literature, the MNL model is by far the most popular model as both the estimation as well as the optimization problems are tractable for this model. The assortment optimization for the MNL model can be solved efficiently and several algorithms including greedy, local search, and linear programming based methods are known (see [Talluri and Van Ryzin 2004], [Gallego et al. 2004] and [Farias et al. 2011]). However, the MNL model is not able to capture any heterogeneity in substitution behavior and also suffers from the Independence from Irrelevant Alternatives (IIA) property ([Ben-Akiva and Lerman 1985]).

More complex choice models such as the Nested logit model [Williams 1977], [McFadden 1978] and the mixture of MNL models have been studied in the literature to model a richer class of substitution behaviors. These generalizations avoid the IIA property but are still consistent with the random utility maximization principle. However, both the estimation and the resulting optimization problem become difficult when we use a richer class of parametric models. Moreover, we also run the risk of over-fitting to data in the estimation problem. [Rusmevichientong et al. 2010] show that the assortment optimization problem is NP-hard for a mixture of MNL model even for the case of mixture of only two MNL models. [Davis et al. 2011] show that the assortment optimization problem is NP-hard for the Nested logit model in general and give an optimal algorithm for a special case of the model parameters. We refer the readers to surveys ([Kök et al. 2009], [Lancaster 1990], [Ramdas 2003]) for a comprehensive review of the state-of-the-art in assortment optimization under general choice models.

In a recent paper, [Rusmevichientong and Topaloglu 2012] consider a model where the choice model is uncertain and could be any one of the given MNL models. They show that an optimal robust assortment can be computed in polynomial time. However, they consider uncertainty over model parameters and not over the class of choice models. It is quite challenging to select the appropriate parametric model and model mis-specification error can be costly in terms of performance. In this paper, we propose a data-driven approach to choice modeling that circumvents this model selection problem. The work by [Farias et al. 2012] and [van Ryzin and Vulcano 2011] are most closely related to this paper. [Farias et al. 2012] consider a non-parametric approach where they use the distribution over permutations with the sparsest support that is consistent with the data. Interestingly, they show that if a certain 'signature condition' is satisfied, the distribution with the sparsest support can be computed efficiently. However, the resulting assortment optimization problem can not be solved efficiently for the sparsest support distribution. [van Ryzin and Vulcano 2011] consider an iterative expectation maximization algorithm to learn a non-parametric choice model where in each iteration they add a new MNL to the mixture model. However, optimization over mixture of MNLs is NP-hard ([Rusmevichientong et al. 2010]).

Outline. The rest of the paper is organized as follows. In Section 2, we present our Markov chain based data-driven choice model. In Section 3, we show that our model is exact if the underlying choice model is MNL. In Section 4, we present approximations bounds for the choice probability estimates computed by the Markov chain model for general choice models. In Section 5, we consider the assortment optimization problem and present an optimal algorithm for our Markov chain based choice model. In Section 6, we present results from our computation study.

2. MARKOV CHAIN BASED CHOICE MODEL

In this section, we present the Markov chain based choice model. We denote the universe of n products by the set $\mathcal{N} = \{1, 2, \ldots, n\}$ and the outside or no-purchase alternative by product 0. For any $S \subseteq \mathcal{N}$, let S_+ denote the set of items including the no-purchase alternative, i.e., $S_+ = S \cup \{0\}$. And for any $j \in S_+$, let $\pi(j, S)$ denote the choice probability of item $j \in S_+$ for offer set, S.

We consider a Markov chain $\mathcal{M}$ to model the substitution behavior using Markovian transitions in $\mathcal{M}$. where there is a state corresponding to each product in $\mathcal{N}_+$ including a state for the no-purchase alternative 0. A customer with a random preference list is modeled to arrive in the state corresponding to the most preferable product. Therefore, for any $i \in \mathcal{N}_+$, a customer arrives in state i with probability $\lambda_i = \pi(i, \mathcal{N})$ and selects product i if it is available. Otherwise, the customer transitions to a different state $j \neq i$ (including the state corresponding to the no-purchase alternative) with probability ρ_{ij} that can be estimated from the data. After transitioning to state j, the customer behaves exactly like a customer whose most preferable product is j. He selects j if it is available and continues the transitions otherwise. Therefore, we approximate the linear substitution arising from a preference list by a Markovian transition model where transitions out of state i do not depend on the previous transitions. The model is completely specified by initial arrival probabilities λ_i for all states $i \in \mathcal{N}_+$ and the transition probabilities ρ_{ij} for all $i \in \mathcal{N}$, $j \in \mathcal{N}_+$. Note that for every state $i \in \mathcal{N}$, there is a probability of transitioning to state 0 corresponding to the no-purchase alternative in which case, the customer leaves the system. For any $j \in \mathcal{N}_+$, we use j to refer to both product j and the state corresponding to the product j in the Markov chain $\mathcal{M}$.

2.1. Estimation of Choice Model Parameters from Data

The arrival probabilities, λ_i for all $i \in \mathcal{N}_+$ can be interpreted as the arrival rate of customers who prefer i when everything is offered. The transition probability ρ_{ij}, for $i \in \mathcal{N}, j \in \mathcal{N}_+$ is the probability of substituting to j from i given that product i is the most preferable but is not available. We can estimate these probabilities from the choice probability data for a small number of assortments.

Suppose we are given the choice probabilities for the following $(n+1)$ assortments, $\mathcal{S} = \{\mathcal{N}, \mathcal{N} \setminus \{i\} \mid i = 1, \ldots, n\}$. We estimate arrival probabilities, λ_i and transition probabilities, ρ_{ij} for all $i \in \mathcal{N}, j \in \mathcal{N}_+$ as follows.

$$\lambda_i = \pi(i, \mathcal{N}), \quad \text{and } \rho_{ij} = \begin{cases} 1, & \text{if } i = 0, j = 0 \\ \dfrac{\pi(j, \mathcal{N} \setminus \{i\}) - \pi(j, \mathcal{N})}{\pi(i, \mathcal{N})}, & \text{if } i \in \mathcal{N}, j \in \mathcal{N}_+, i \neq j \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Note that $\pi(i, \mathcal{N})$ is exactly the fraction of customers whose most preferable product is i. For all $i \in \mathcal{N}, j \in \mathcal{N}_+$, $i \neq j$, the numerator in the definition of ρ_{ij}, $\delta_{ij} = \pi(j, \mathcal{N} \setminus \{i\}) - \pi(j, \mathcal{N})$, is the increase in probability of selecting j when i is removed from the assortment. Therefore, we can interpret the definition of ρ_{ij} as the conditional probability that a customer substitutes from product i to product j given that product i is the most preferable product but is not offered. This is consistent with the Markov chain interpretation.

In (1), we assume that accurate choice probability data for $(n+1)$ assortments $\mathcal{S}$ is available. We can estimate the transition probability parameters from other choice probability data as well. For instance, suppose we have the choice probability data for offer set S where $i, j \in S$ and offer set $S \setminus \{i\}$. We can estimate ρ_{ij} as follows.

$$\rho_{ij} = \tau \cdot \frac{\pi(j, S \setminus \{i\}) - \pi(j, S)}{\pi(i, S)}, \tag{2}$$

where τ is an appropriate normalizing factor. The above expression can be similarly interpreted as the probability of substituting to product j from i given that i is the most preferable product in S

and is not available. Therefore, if we are given a collection of all one and two product assortments, we can estimate the transition probabilities using (2).

We would like to mention that in practice, we typically have access to partial and noisy choice probability data. Therefore, it is important to study robust statistical estimation procedures for estimating the parameters of the Markov chain model from noisy partial data. However, our main focus in this paper is to introduce the Markov chain based choice model, analyze its performance in modeling random utility based choice models, and study the related assortment optimization problem. We leave the study of robust statistical estimation procedures from data for future work.

2.2. Computation of Choice Probabilities

Given the parameters λ_i and ρ_{ij} for all $i \in \mathcal{N}_+, j \in \mathcal{N}$, let us now describe how we can compute the choice probabilities for any $S \subseteq \mathcal{N}, j \in S_+$. Our primitive for the substitution behavior is that a customer arrives in state i with probability λ_i, and continues to transition according to probabilities ρ_{ij} until he reaches a state corresponding to a product in S. Therefore, for any offer set $S \subseteq \mathcal{N}$, we define a Markov chain, $\mathcal{M}(S)$ over the same state space, $\mathcal{N}_+$. We modify the transition probabilities such that all states $i \in S_+$ become absorbing, i.e., $\rho_{ii}(S) = 1$ and $\rho_{ij}(S) = 0$ for all $i \in S_+$ and $j \neq i$. Note that there are no transitions out of any state $i \in S_+$. The choice probability for any $j \in S_+$, $\hat{\pi}(j, S)$ can be computed as the probability of absorbing in state i in $\mathcal{M}(S)$. In particular, we have the following theorem.

THEOREM 2.1. *Suppose the parameters for the Markov chain model are given by λ_i, ρ_{ij} for all $i \in \mathcal{N}, j \in \mathcal{N}_+$. For any $S \subseteq \mathcal{N}$, let $\boldsymbol{B} = \boldsymbol{\rho}(\bar{S}, S_+)$ denote the transition probability sub-matrix from states $\bar{S} = \mathcal{N} \setminus S$ to S_+, and $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$ denote the transition sub-matrix from states in $\bar{S}$ to $\bar{S}$. Then for any $j \in S_+$,*

$$\hat{\pi}(j, S) = \lambda_j + (\boldsymbol{\lambda}(\bar{S}))^T (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{B} \boldsymbol{e}_j, \tag{3}$$

where $\boldsymbol{\lambda}(\bar{S})$ is the vector of arrival probabilities in $\bar{S}$ and e_j is the j^{th} unit vector.

PROOF. The transition probability matrix for Markov chain $\mathcal{M}(S)$ where states in S_+ are absorbing is given (after permuting the rows and columns appropriately) by

$$\mathcal{P}(S) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix}. \tag{4}$$

Here the first $|S| + 1$ rows and columns of $\mathcal{P}(S)$ correspond to states in S_+ and the remaining rows and columns correspond to states in $\bar{S}$. Let $p = |S|$. Then, $\boldsymbol{I}$ is a $(p+1) \times (p+1)$ identity matrix, $\boldsymbol{B} \in \mathbb{R}_+^{(n-p)\times(p+1)}$ and $\boldsymbol{C} \in \mathbb{R}_+^{(n-p)\times(n-p)}$. For any $j \in S_+$, the choice probability estimate $\hat{\pi}(j, S)$ can be computed as follows.

$$\hat{\pi}(j, S) = \lim_{q \to \infty} \boldsymbol{\lambda}^T \mathcal{P}^q(S) \boldsymbol{e}_j = \boldsymbol{\lambda}^T \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \boldsymbol{e}_j, \tag{5}$$

where the last equality follows from the fact that all row sums of $\boldsymbol{C}$ are strictly less than 1 (since $\rho_{i0} > 0$ for all $i \in \mathcal{N}$). Therefore, $\boldsymbol{C}$ is a M-matrix and $\sum_{j=0}^{\infty} \boldsymbol{C}^j = (\boldsymbol{I} - \boldsymbol{C})^{-1}, \ \lim_{q\to\infty} \boldsymbol{C}^q = \boldsymbol{0}$. Therefore,

$$\hat{\pi}(j, S) = \lambda_j + (\boldsymbol{\lambda}(\bar{S}))^T (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{B} \boldsymbol{e}_j,$$

where $\boldsymbol{e}_j$ is the j^{th} unit vector in $\mathbb{R}^{p+1}$. Note that $\boldsymbol{Y} = (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{B}$ is a $|\bar{S}| \times |S|$ matrix where for any $i \in \bar{S}$, $j \in S_+$, Y_{ij} denotes the resulting probability of transitioning to state j conditional on arrival in state i. In other words, Y_{ij} is the probability of substituting to product j given that product i is the most preferable product but is not available in offer set S. □

Therefore, for any $S \subseteq \mathcal{N}$, we can compute the choice probabilities of all products $j \in S_+$ by computing the stationary probabilities $\mathcal{M}(S)$. The parameters λ_i and ρ_{ij} for $i \in \mathcal{N}_+, j \in \mathcal{N}$, therefore give a compact representation of choice probabilities for all offer sets that can be computed efficiently. However, unlike several commonly used parametric choice models such as MNL, Nested logit and mixture of MNL, we do not have easy functional form for the choice probabilities since computing stationary probabilities requires a matrix inversion.

3. MULTINOMIAL LOGIT (MNL) AND MARKOV CHAIN MODEL

In this section, we show that the Markov chain model is exact if the underlying choice model is MNL. In other words, if the parameters λ_i, ρ_{ij} for all $i \in \mathcal{N}$, $j \in \mathcal{N}_+$ are computed using choice probability data from a MNL model, then for any assortment $S \subseteq \mathcal{N}$, the choice probability for any $j \in S_+$ computed using the Markov chain model is exactly equal to the choice probability given by the underlying MNL model. Suppose the parameters of the MNL model are given by $u_0, \ldots, u_n$ where $u_0 + \ldots + u_n = 1$. We would like to emphasize that the parameter estimation (1) is completely data-driven and does not require any structural information of the underlying model.

THEOREM 3.1. *Suppose the underlying model is Multinomial logit (MNL) for the given choice probability data, $\pi(j, S)$ for all $S \in \{\mathcal{N}, \mathcal{N} \setminus \{i\}, i = 1, \ldots, n\}$. Then for all $S \subseteq \mathcal{N}, j \in S_+$,*

$$\hat{\pi}(j, S) = \pi(j, S),$$

where $\hat{\pi}(j, S)$ is the choice probability computed by the Markov chain model (3) *and $\pi(j, S)$ is true choice probability given by the underlying MNL model.*

PROOF. Suppose the parameters for the underlying MNL model are $u_0, \ldots, u_n > 0$. We can assume wlog. that $u_0 + \ldots + u_n = 1$. Therefore, from (1), we can compute for any $i \in \mathcal{N}$, $j \in \mathcal{N}_+$, $j \neq i$, $\rho_{ij} = \frac{u_j}{1-u_i}$. We show that the Markov chain $\mathcal{M}$ is equivalent to another Markov chain $\hat{M}$ with the transition matrix $\hat{\rho}$ where for any $i \in \mathcal{N}$, $j \in \mathcal{N}_+$, $\hat{\rho}_{ij} = u_j$ that is independent of i. The probability of state transitions from i to j for all $i \neq j$ in $\hat{M}$ is

$$\sum_{q=0}^{\infty} (\hat{\rho}_{ii})^q \hat{\rho}_{ij} = \sum_{q=0}^{\infty} (u_i)^q u_j = \frac{u_j}{1 - u_i} = \rho_{ij}.$$

Note that the transition probability matrix of $\hat{M}$ has rank 1 as all the rows are identical.

Consider any offer set $S \subseteq \mathcal{N}$. For ease of notation, let $S = \{1, \ldots, p\}$ for some $p \leq n$. Now, $\hat{\pi}(j, S)$ is the stationary probability of state j in Markov chain $\mathcal{M}(S)$ where the states corresponding to S_+ are absorbing states. From the equivalence of Markov chains, $\mathcal{M}$ and $\hat{M}$, we can equivalently compute the stationary probability of state j in Markov chain $\hat{M}(S)$ where states in S_+ are absorbing, and the transition probabilities $\hat{\rho}_{ij}(S)$ are defined as described earlier where

$$\hat{\rho}_{ij}(S) = \begin{cases} 1 & \text{if } j = i, i \in S_+, \\ 0 & \text{if } j \neq i, i \in S_+ \\ u_j & \text{if } i \notin S. \end{cases}$$

For all $i \notin S$, $\hat{\rho}_{ij}(S) = u_j$ that does not depend on i. Therefore, transition to any state $j \in S_+$ from states in $\bar{S}$ is proportional to u_j which implies that the stationary probability,

$$\hat{\pi}(j, S) = \pi(j, \mathcal{N}) + \alpha \cdot u_j = u_j(1 + \alpha),$$

where the second equality follows as $\pi(j, \mathcal{N}) = u_j$ for all $j \in \mathcal{N}_+$. Also, $\sum_{j \in S_+} \hat{\pi}(j, S) = 1$ which imples $(1 + \alpha) = 1/(\sum_{j \in S_+} u_j)$. □

Therefore, if the underlying choice model is a multinomial logit model, then for any offer set $S \subseteq N$, the Markov chain based model exactly computes the choice probabilities for all the products.

The above proof shows that the MNL model can be represented by an equivalent Markov chain, $\hat{M}$, whose transition matrix has rank one. Conversely, we can also show that if the transition matrix has rank one, then the Markov chain model is equivalent to an MNL model using similar arguments as above.

4. GENERAL CHOICE MODELS

In this section, we discuss the performance of the Markov chain based model for general choice models. From Theorem 3.1, we know that the model is exact if the underlying choice model is MNL and Algorithm 1 computes an optimal assortment for the underlying MNL model (Corollary 5.3). However, Markov chain is not an exact approximation for general discrete choice models, i.e., if the Markov chain model is estimated from data arising from a general choice model such as mixture of two MNL models, then the choice probability estimates computed by the Markov chain model can be different from the true probabilities of the underlying model. In fact, we can construct pathological instances where the two probabilities are significantly different (see Section 4.2). However, under a fairly general assumption, we show that the Markov chain model is a good approximation for all random utility based discrete choice model Furthermore, we also show that we can compute a good approximation to the optimal assortment problem for the true model using the Markov chain model.

4.1. Bounds for Markov chain choice probability approximation

Here, we show that the choice probabilities from the Markov chain approximation for any discrete choice model are a good approximation to the true probabilities of the underlying model. To prove such an approximation result, we need to show that for every discrete choice model, the Markov chain model computes good choice probability estimates if the data arises from that choice model. The following theorem from McFadden and Train [McFadden and Train 2000] allows us to restrict to proving our result for the mixture of multinomial logit (MMNL) model.

THEOREM 4.1 (MCFADDEN AND TRAIN [MCFADDEN AND TRAIN 2000]). *Any random utility based discrete choice model can be approximated as closely as required by a mixture of multinomial logit models (MMNL).*

Therefore, it suffices for us to prove that the Markov chain model is a good approximation for the mixture of multinomial logit (MMNL) model with an arbitrary number of segments. Consider a MMNL model given by a mixture of K multinomial logit models. Let θ_k, $k = 1, \ldots, K$ denote the probability that a random customer belongs to the MNL model k. We also refer to the MNL model k as segment k. For segment $k = 1, \ldots, K$, let $\boldsymbol{u}_{jk}, j = 0, \ldots, n$ denote the utility parameters for the corresponding MNL model. We assume wlog that for all $k = 1, \ldots, K$, $\sum_{j=0}^{n} u_{jk} = 1,$ and $u_{0k} > 0$. Also, for any $k = 1, \ldots, K$ and any $S \subseteq \mathcal{N}_+$, let $u_k(S) = \sum_{j \in S} u_{jk}$. The choice probability $\pi(j, S)$ for any offer set S and $j \in S_+$ for the mixture of MNLs model can be expressed as follows.

$$\pi(j, S) = \sum_{k=1}^{K} \theta_k \cdot \frac{u_{jk}}{1 - \sum_{i \in \bar{S}} u_{ik}} = \sum_{k=1}^{K} \theta_k u_{jk} \cdot \left(1 + \sum_{q=1}^{\infty} \left(u_k(\bar{S})\right)^q\right), \tag{6}$$

where the last equality follows from the fact that $u_k(\bar{S}) < 1$ since $u_{0k} > 0$ for all $k = 1, \ldots, K$. Here $\bar{S} = \mathcal{N} \setminus S$.

We show that for any $S \subseteq \mathcal{N}$ and $j \in S_+$, the choice probability $\hat{\pi}(j, S)$ computed by the Markov chain model is a good approximation of $\pi(j, S)$. For any $S \subseteq \mathcal{N}$, let

$$\alpha(S) = \max_{k=1}^{K} \sum_{i \in S} \pi_k(i, \mathcal{N}) = \max_{k=1}^{K} \sum_{i \in S} u_{ik}, \tag{7}$$

where $\pi_k(i, \mathcal{N})$ is the probability that a random customer from segment k selects $i \in S$ when the offer set is $\mathcal{N}$. Therefore, $\alpha(S)$ is the maximum probability that the most preferable product for a random customer from any segment $k = 1, \ldots, K$ belongs to S. We prove lower and upper bounds on the relative error between $\hat{\pi}(j, S)$ and $\pi(j, S)$ that depend on $\alpha(\bar{S})$. In particular, we prove the following theorem.

THEOREM 4.2. *For any $S \subseteq \mathcal{N}$, $j \in S_+$, let $\hat{\pi}(j, S)$ be the choice probability computed by the Markov chain model, and $\pi(j, S)$ be the true choice probability given by the mixture of MNL model. Then,*

$$\left(1 - (\alpha(\bar{S}))^2\right) \cdot \pi(j, S) \leq \hat{\pi}(j, S) \leq \left(1 + \frac{(\alpha(\bar{S}))^2}{1 - \alpha(\bar{S})}\right) \cdot \pi(j, S).$$

If the offer set S is sufficiently large, then $\alpha(\bar{S})$ would typically be small and we get sharp upper and lower bounds for $\hat{\pi}(j, S)$. However, the bounds get worse as $\alpha(\bar{S})$ increases or the size of offer set S decreases. It is reasonable to expect this degradation as we estimate the transition probability parameters from choice probability data for offer sets $\mathcal{N}$ and $\mathcal{N} \setminus \{i\}$ for $i \in \mathcal{N}$ and make a Markovian assumption for the substitution behavior. For significantly smaller offer sets, the number of state transitions for a random demand before reaching an absorbing state is large with high probability and the error due to Markovian assumption gets worse.

To prove the above theorem, we first compute upper and lower bounds on the choice probability $\hat{\pi}(j, S)$ for any $S \subseteq \mathcal{N}$, $j \in S_+$ in the following two lemmas.

LEMMA 4.3. *For any $S \subseteq \mathcal{N}$, $j \in S_+$, let $\hat{\pi}(j, S)$ denote the choice probability of product j for offer set S computed by the Markov chain model. Then*

$$\hat{\pi}(j, S) \geq \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{i \in \bar{S}} \left(\sum_{q=1}^{\infty} (u_{ik})^q\right)\right).$$

The above lower bound on the choice probability $\hat{\pi}(j, S)$ for $j \in S_+$ is computed by considering only a one-step substitution to another product if the first choice product is not available. According to our Markov chain model, a customer with product i as the first choice product transitions to another state if product i is not available. The transitions continue according to the transition matrix $\mathcal{P}(S)$ until the customer ends up in an absorbing state. Therefore, by considering only a single transition in the Markov chain $\mathcal{M}(S)$, we obtain a lower bound on $\hat{\pi}(j, S)$ for any $j \in S_+$. We present the proof of Lemma 4.3 in the Appendix.

In the following lemma, we prove an upper bound on $\hat{\pi}(j, S)$ for any $S \subseteq \mathcal{N}$, $j \in S_+$. The bound depends on the spectral radius of the transition sub-matrix $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$ of transition probabilities from $\bar{S}$ to $\bar{S}$.

LEMMA 4.4. *For any $S \subseteq \mathcal{N}$, $j \in S_+$, let $\hat{\pi}(j, S)$ be the choice probability of product $j \in S_+$ for offer set S computed by the Markov chain model. Let $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$ denote the sub-matrix of transition probabilities from states $\bar{S} = \mathcal{N} \setminus S$ to $\bar{S}$, and let γ be the maximum eigenvalue of $\boldsymbol{C}$. Then*

$$\hat{\pi}_j(S) \leq \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{1}{1 - \gamma} \cdot \left(\sum_{i \in \bar{S}} \sum_{q=1}^{\infty} (u_{ik})^q\right)\right).$$

In the following lemma, we show the spectral radius of the transition sub-matrix $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$ is related to the parameter $\alpha(\bar{S})$ defined in Theorem 4.2.

LEMMA 4.5. *Consider any $S \subseteq \mathcal{N}$ and let $\alpha = \alpha(\bar{S})$. Let $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$ be the probability transition sub-matrix of $\mathcal{P}(S)$ from states $\bar{S}$ to $\bar{S}$. Then the maximum eigenvalue of $\boldsymbol{C}$, γ is at most α.*

We present the proofs of Lemmas 4.4 and 4.5 in the Appendix. Now, we are ready to prove the main theorem.

Proof of Theorem 4.2 Let $\alpha = \alpha(\bar{S})$. From (6), we know that

$$\pi(j,S) = \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{q=1}^{\infty} \left(u_k(\bar{S})\right)^q\right),$$

and from Lemma 4.3, we have that

$$\hat{\pi}(j,S) \;\geq\; \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{i \in \bar{S}} \left(\sum_{q=1}^{\infty} (u_{ik})^q\right)\right) \geq \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + u_k(\bar{S})\right),$$

where the second inequality follows as $u_{ik} > 0$ for all $i \in \mathcal{N}_+, k = 1, \ldots, K$. Therefore,

$$\begin{aligned} \pi(j,S) - \hat{\pi}(j,S) \;&\leq\; \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{q=1}^{\infty} \left(u_k(\bar{S})\right)^q\right) - \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + u_k(\bar{S})\right) \\ &=\; \sum_{k=1}^{K} \theta_k u_{jk} \cdot \left(u_k(\bar{S})\right)^2 \cdot \left(\sum_{q=0}^{\infty} u_k(\bar{S})^q\right) \\ &\leq\; \alpha^2 \cdot \left(\sum_{k=1}^{K} \theta_k u_{jk} \cdot \left(\sum_{q=0}^{\infty} u_k(\bar{S})^q\right)\right) = \alpha^2 \pi(j,S), \end{aligned} \tag{8}$$

where (8) follows from the fact that $u_k(\bar{S}) \leq \alpha$ for all $k = 1, \ldots, K$.

Let γ denote the maximum eigenvalue of $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$, the transition sub-matrix from $\bar{S}$ to $\bar{S}$. From Lemma 4.5, we know that $\gamma \leq \alpha$. Therefore, from Lemma 4.4, we have that

$$\hat{\pi}_j(S) \;\leq\; \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{1}{1-\alpha} \cdot \sum_{i \in \bar{S}} \sum_{q=1}^{\infty} u_{ik}^q\right) \leq \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{1}{1-\alpha} \cdot \sum_{q=1}^{\infty} u_k(\bar{S})^q\right),$$

where the second inequality follows as $u_{ik} > 0$ for all $i \in \mathcal{N}_+, k = 1, \ldots, K$. Therefore,

$$\begin{aligned} \pi(j,S) - \hat{\pi}(j,S) \;&\geq\; \sum_{k=1}^{K} \theta_k u_{jk} \left(1 - \frac{1}{1-\alpha}\right) \cdot \left(\sum_{q=1}^{\infty} \left(u_k(\bar{S})\right)^q\right) \\ &\geq\; \frac{-\alpha^2}{1-\alpha} \cdot \sum_{k=1}^{K} \theta_k u_{jk} \cdot \left(\sum_{q=0}^{\infty} \left(u_k(\bar{S})\right)^q\right) \\ &=\; \frac{-\alpha^2}{1-\alpha} \cdot \pi(j,S), \end{aligned} \tag{9}$$

where the second inequality follows as $u_k(\bar{S}) \leq \alpha$ for all $k = 1, \ldots, K$. ■

We would like to emphasize that the lower and upper bounds in Theorem 4.2 are approximate and can be quite conservative in practice. For instance, in computing the lower bound on $\hat{\pi}(j,S)$ in Lemma 4.3, we only considers a one-step substitution in the Markov chain to approximation the stationary probability of state $j \in S_+$ in Markov chain $\mathcal{M}(S)$. To further investigate this gap between theory and practice, we do a computational study to compare the performance of the Markov chain model with the true underlying model. In the computational results, we observe that the performance of our model is significantly better than the theoretical bounds. The results are presented in Section 6.

4.2. A Tight Example

We show that the bounds in Theorem 4.2 are tight. In particular, we prove the following theorem.

THEOREM 4.6. *For any $\epsilon > 0$, there is a MMNL choice model over $\mathcal{N}$, $S \subseteq \mathcal{N}, j \in S$ such that the approximation bound in Theorem 4.2 is tight up to a factor of n^{ϵ}.*

PROOF. Consider the following MMNL choice model that is a mixture of two MNL models each with probability $1/2$, i.e., $\theta_1 = \theta_2 = 1/2$. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_+^{n+1}$ denote the utility parameters of the two MNL models. Let $n_0 = (n - n^{1-\epsilon})$. For all $j = 0, \ldots, n$, let

$$u_j = \left\{ \begin{array}{ll} n^{2-\epsilon}, & \text{if } j = 0 \\ 1, & \text{if } j = 1 \text{ or } j > n_0 \\ n - j + 1, & \text{otherwise} \end{array} \right\} \text{ and } v_j = \left\{ \begin{array}{ll} 1, & \text{if } j = 0 \text{ or } j \geq n_0 \\ j, & \text{otherwise.} \end{array} \right\}$$

Let $S = \{1\}, j = 1$. The true choice probability, $\pi(1, S) > \frac{1}{4}$. Let $s_1 = \sum_{j=0}^{n} u_j = \Theta(n^2),\ s_2 = \sum_{j=0}^{n} v_j = \Theta(n^2)$. For all $j = 2, \ldots, n$, $\rho_{j1} = O\left(\frac{1}{n^2}\right)$, and

$$\lambda_j = \left\{ \begin{array}{ll} \Theta\left(\dfrac{1}{n}\right), & \text{if } 2 \leq j \leq n_0 \\ \Theta\left(\dfrac{1}{n^2}\right), & \text{otherwise} \end{array} \right\} \text{ and } \quad \rho_{j0} = \left\{ \begin{array}{ll} \Omega\left(\dfrac{1}{n^{2\epsilon}}\right) & \text{if } j \leq n_0 \\ \Omega\left(\dfrac{1}{n^{1+\epsilon}}\right) & \text{otherwise} \end{array} \right\}$$

Let c, c_1, c_2 be some constants. Therefore, we can bound $\hat{\pi}(1, S)$ as follows.

$$\begin{aligned} \hat{\pi}(1, S) \ &\leq \ \pi(1, \mathcal{N}) + \left(\sum_{i=2}^{n_0} \lambda_i \right) \cdot \left(\sum_{q=0}^{\infty} \left(1 - \frac{c_1}{n^{2\epsilon}} \right)^q \right) \cdot \frac{c}{n^2} + \sum_{i=n_0}^{n} \lambda_i \cdot \left(\sum_{q=0}^{\infty} \left(1 - \frac{c_2}{n^{1+\epsilon}} \right)^q \right) \cdot \frac{c}{n^2} \\ &\leq \ O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n^{2-2\epsilon}}\right) + O\left(\frac{1}{n^2}\right) = O\left(\frac{1}{n^{2-2\epsilon}}\right) \cdot \pi(1, S), \end{aligned} \tag{10}$$

where the first inequality follows as $\rho_{j1} = O(1/n^2)$ for all $j = 2, \ldots, n$, $\rho_{j0} = \Omega(1/n^{2\epsilon})$ for all $j = 2, \ldots, n_0$ and $\rho_{j0} = \Omega(1/n^{1+\epsilon})$ for $j \geq n_0$. Inequality (10) follows as $\pi(1, \mathcal{N}) = O(1/n^2)$, and $\sum_{j=2}^{n_0} \lambda_j \leq 1, \sum_{j=n_0+1}^{n} \lambda_j \leq O\left(\frac{1}{n^{1+\epsilon}}\right)$. Also, $(1 - \alpha(\bar{S})^2) = \Theta\left(\frac{1}{n^2}\right)$. From Theorem 4.2, we have

$$\hat{\pi}(1, S) \geq (1 - \alpha^2) \cdot \pi(1, S) = \Theta\left(\frac{1}{n^2}\right) \cdot \pi(1, S),$$

which implies that the lower bound is almost tight up to a factor of $n^{2\epsilon}$. □

It is important to note that the family of instances in Theorem 4.6 are pathological cases where the parameters are carefully chosen to show that the bound is tight. The choice model is a mixture of two MNLs where the MNL parameters of one class are increasing and the second class are decreasing for almost all products. If we change the example slightly, we can observe that the bounds in Theorem 4.2 are conservative.

5. ASSORTMENT OPTIMIZATION FOR MARKOV CHAIN MODEL

In this section, we consider the problem of finding the optimal revenue assortment for the Markov chain based choice model. For any $j \in \mathcal{N}$, let r_j denote the revenue of product j. The goal in an assortment optimization problem is to select an offer set $S \subseteq \mathcal{N}$ such that the total expected revenue is maximized, i.e.,

$$\max_{S \subseteq \mathcal{N}} \sum_{j \in S} r_j \cdot \hat{\pi}(j, S), \tag{11}$$

where for all $j \in S$, $\hat{\pi}(j, S)$ is the choice probability of product j given by the Markov chain model (3). [Rusmevichientong et al. 2010] show that the assortment optimization problem is NP-hard for a general choice model, in fact even for a mixture of two MNL models.

We present a polynomial time algorithm for the assortment optimization problem (11) for the Markov chain based choice model. The result is quite surprising as we can not even express the choice probability $\hat{\pi}(j, S)$ for any $S \subseteq \mathcal{N}$, $j \in S_+$ using a simple functional form of the model parameters. The computation of $\hat{\pi}(j, S)$ in (3) requires a matrix inversion where the coefficients of the matrix depend on the assortment decisions.

We give an iterative algorithm that computes an optimal solution in a polynomial number of iterations. The algorithm crucially exploits the fact that the choice model is specified by a Markov chain. Therefore, if product i is the most preferable product for any customer, and it is not offered in the assortment, then the customer substitutes to product $j \in \mathcal{N}_+$ (including the no-purchase alternative) with probability ρ_{ij}.

For all $i \in \mathcal{N}$ and $S \subseteq \mathcal{N}$, let $g_i(S)$ denote the expected revenue from a customer that arrives in state i (i.e. the most preferable product is i) when the offer set is S. If product $i \in S$, then the customer selects product i and $g_i(S) = r_i$. Otherwise, the customer substitutes according to the transitions in the Markov chain and $g_i(S) = \sum_{j \in \mathcal{N}} P_{ij} g_j(S)$ and the total expected revenue for offer set S is given by $\sum_{j \in \mathcal{N}} \lambda_j \cdot g_j(S)$. Therefore, we can reformulate the assortment optimization (11) as follows.

$$\max_{S \subseteq \mathcal{N}} \sum_{j \in \mathcal{N}} \lambda_j \cdot g_j(S), \tag{12}$$

where $g_j(S)$ denotes the expected revenue from a customer with most preferable product is j when the offer set is S. This optimization problem is equivalent to selecting an optimal set of stopping (or absorbing) states in the Markov chain, $\mathcal{M}$.

Motivated by the reformulation (12), we consider the following approach. For all $i \in \mathcal{N}$, let g_i be the maximum expected revenue that can be obtained from a customer whose first choice is product i, where the maximization is taken over all offer sets. We can compute g_i using an iterative procedure. For $t \in Z_+, i \in \mathcal{N}$, let g_i^t denote the maximum expected revenue starting from state i in at most t state transitions where we can stop at any state. Stopping at any state j corresponds to selecting product j resulting in revenue r_j. Therefore, for all $i \in \mathcal{N}$, $g_i^0 = r_i$ since the only possibility is to stop at state i when no state transitions are allowed. Algorithm 1 describes an iterative procedure to compute g_j for all $j \in \mathcal{N}$, by computing g_j^t for $t \geq 1$ until they converge.

We show that Algorithm 1 computes an optimal assortment for the Markov chain based choice model in a polynomial number of iterations. In particular, we prove the following theorem.

THEOREM 5.1. *Algorithm 1 computes an optimal assortment for the assortment optimization problem* (12)*. Furthermore, the number of iterations is* $O\big(1/\delta \cdot \log(r_{\max}/r_{\min})\big)$*, where* $\delta = \min_j \ \rho_{j0} > 0$*,* $r_{\max} = \max_j \ r_j$ *and* $r_{\min} = \min_j \ r_j$*.*

To prove the above theorem, we first show that Algorithm 1 converges in a polynomial number of iterations and correctly computes g_j for all $j \in \mathcal{N}$.

LEMMA 5.2. *Consider* $\boldsymbol{g} \in \mathbb{R}^n$ *as computed by Algorithm 1. Then for all* $j \in \mathcal{N}$*,* g_j *is the maximum possible expected revenue from a customer arriving in state* j*.*

Algorithm 1 Compute optimal assortment for Markov chain Choice Model

```
Initialize: g_j^0 := r_j for all j ∈ N, Δ := 1, t := 0
while (Δ > 0) do
    t := t + 1
    for i = 1 → n do
        g_i^t := max( r_i, Σ_{j≠i} ρ_ij · g_j^{t−1} )
    end for
    Δ := ||g^t − g^{t−1}||_∞
end while
Return: (g := g^t, S := {j ∈ N | g_j = r_j})
```

PROOF. We first prove that for all $t \in \mathbb{Z}_+$, for all $j \in \mathcal{N}$, g_j^t computed in Algorithm 1 is the maximum expected revenue starting from state j after at most t state transitions, and g_j^t is increasing in t. We prove this by induction.

Base Case ($t = 0$**,** $t = 1$). Clearly, $g_j^0 = r_j$ for all $j \in \mathcal{N}$ when no state transitions are allowed. For $t = 1$, we can either stop in state j or stop after exactly one transition. Therefore, for all $j \in \mathcal{N}$,

$$g_j^1 = \max\left(r_j, \sum_{i \in \mathcal{N}} \rho_{ji} \cdot r_i\right) = \max\left(r_j, \sum_{i \in \mathcal{N}} \rho_{ji} \cdot g_i^0\right) \geq r_j = g_j^0.$$

Induction Step ($t = T$). For any $j \in \mathcal{N}$, the maximum expected revenue in at most T transitions starting from state j is either obtained by zero state transition (i.e. stopping at state j) or at least one transition. In the former case, the revenue is r_j. For the latter, we transition to state i with probability ρ_{ji}. From state i, we can make at most $T-1$ transitions and by induction hypothesis, g_i^{T-1} is the maximum expected revenue that can be obtained starting from state i in at most $T-1$ transitions. Therefore, g_j^T for all $j \in \mathcal{N}$ computed in Algorithm 1 as

$$g_j^T = \max\left(r_j, \sum_{i \in \mathcal{N}} \rho_{ji} \cdot g_i^{T-1}\right),$$

is the maximum expected revenue in at most T state transitions starting from state j. Also, by induction hypothesis, $g_i^{T-1} \geq g_i^{T-2}$ for all $i \in \mathcal{N}$. Therefore, for all $j \in \mathcal{N}$.

$$g_j^T = \max\left(r_j, \sum_{i \in \mathcal{N}} \rho_{ji} \cdot g_i^{T-1}\right) \geq \max\left(r_j, \sum_{i \in \mathcal{N}} \rho_{ji} \cdot g_i^{T-2}\right) = g_j^{T-1}.$$

Therefore, for all $j \in \mathcal{N}$, g_j^t is an increasing sequence upper bounded by $r_{\max} = \max_{i \in \mathcal{N}} r_i$ and it converges to g_j. This implies that Algorithm 1 correctly computes g_j for all $j \in \mathcal{N}$.

To prove that the algorithm converges in a small number of iterations, let $\delta = \min_{i \in \mathcal{N}} \rho_{i0}$, and $r_{\min} = \min_{i \in \mathcal{N}} r_i$. We assume that δ is polynomially bounded away from zero. Therefore, in each transition, there is at least a probability of δ to transition to state 0 which corresponds to zero revenue. If starting from any state j, we stop after t transitions, the maximum possible expected revenue is $(1-\delta)^t \cdot r_{\max}$. Therefore, if $(1-\delta)^t \cdot r_{\max} \leq r_j$, it is better to stop at state j itself as compared to continuing the transitions. This implies that Algorithm 1 converges in at most $\frac{1}{\delta} \cdot \log\left(\frac{r_{\max}}{r_{\min}}\right)$ iterations. Since δ is polynomially bounded away from zero, the algorithm converges in polynomial number of iterations. □

Proof of Theorem 5.1 Suppose Algorithm 1 returns $(\boldsymbol{g}, S)$ where $\boldsymbol{g} = \lim_{t\to\infty} \boldsymbol{g}^t$, and $S = \{j \in \mathcal{N} \mid g_j = r_j\}$. From Lemma 5.2, we know that for all $j \in \mathcal{N}$, g_j is the maximum expected revenue starting from state j when we can stop at any state after any number of iterations. Recall that for any $Q \subseteq \mathcal{N}$, $g_j(Q)$ denotes the expected revenue starting from state j in Markov chain $\mathcal{M}(Q)$ where all states in Q are absorbing states.

We claim that $g_j(S) = g_j$ for all $j \in \mathcal{N}$. Clearly, for any $j \in S$, $g_j = r_j = g_j(S)$. For all $j \notin S$, $g_j > r_j$. Therefore,

$$g_j = \sum_{i\in\mathcal{N}} \rho_{ji} \cdot g_i = \sum_{i\in S} \rho_{ji} \cdot r_i + \sum_{i\notin S} \rho_{ji} g_i, \ \forall j \notin S. \tag{13}$$

Let $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$ be the probability transition sub-matrix from states in $\bar{S}$ to $\bar{S}$, and $\boldsymbol{B} = \boldsymbol{\rho}(\bar{S}, S)$. Then, the above equation can be formulated as

$$(\boldsymbol{I} - \boldsymbol{C})\boldsymbol{g}_{\bar{S}} = \boldsymbol{B}\boldsymbol{r}_S, \tag{14}$$

where $\boldsymbol{g}_{\bar{S}}$ is the restriction of $\boldsymbol{g}$ to $\bar{S}$ and similarly, $\boldsymbol{r}_S$ is the restriction of $\boldsymbol{r}$ on S. Note that for all $j \notin S$, $g_j(S)$ also satisfy (13) and consequently (14). Since $(\boldsymbol{I} - \boldsymbol{C})$ is an M-matrix, (14) has a unique non-negative solution which implies $g_j = g_j(S)$ for all $j \in \mathcal{N}$.

Now, consider an optimal assortment, S^* for (12). Therefore, $\mathsf{OPT} = r(S^*) = \sum_{j\in\mathcal{N}} \lambda_j \cdot g_j(S^*)$ where $g_j(S^*)$ is the expected revenue starting from state j and stopping on the set of states in S^*. Clearly, for all $j \in \mathcal{N}$, $g_j(S^*) \leq g_j$. Therefore,

$$r(S) = \sum_{j\in\mathcal{N}} \lambda_j \cdot g_j(S) = \sum_{j\in\mathcal{N}} \lambda_j \cdot g_j \geq \sum_{j\in\mathcal{N}} \lambda_j \cdot g_j(S^*) = \mathsf{OPT},$$

where the second equality follows as $g_j(S) = g_j$ for all $j \in \mathcal{N}$, and the third inequality follows as $g_j \geq g_j(S^*)$ for all j. ∎

Therefore, Algorithm 1 computes an optimal assortment for the Markov chain choice model. If the Markov chain model is estimated from data arising from an underlying MNL model, then the Markov chain model is exact (Theorem 3.1). Therefore, optimal assortment for the Markov chain model also corresponds to an optimal assortment for the underlying MNL model and we have the following corollary.

COROLLARY 5.3. *Suppose the parameters of the Markov chain based choice model, λ_j, ρ_{ij} for all $i \in \mathcal{N}, j \in \mathcal{N}_+$ are estimated from data (using* (1)*) arising from an underlying MNL model. Then Algorithm 1 computes an optimal assortment for the underlying MNL choice model.*

5.1. Assortment Optimization for General Choice Models

In this section, we show that if the Markov chain model parameters are estimated from data arising from a general random utility choice model, then the optimal assortment for the resulting Markov chain model is a good approximation for the underlying choice model as well. Since a general random utility based choice model can be approximated as closely as required by a mixture of MNL model as [McFadden and Train 2000], we can assume wlog. that the underlying choice model is a mixture of MNL model. In Theorem 4.2, we show that a mixture of MNL model can be well approximated by a Markov chain model. Therefore, an optimal assortment for the corresponding Markov chain model can be a good approximation for the underlying mixture of MNL model. In particular, we prove the following theorem.

THEOREM 5.4. *Suppose the parameters of the Markov chain model are estimated from data arising from an underlying mixture of MNL model, $\pi(\cdot, \cdot)$. Let S^* be an optimal assortment for the mixture of MNL model and let S be an optimal assortment for the Markov chain model computed* *using Algorithm 1. Then*

$$\sum_{j\in S}\pi(j,S)\cdot r_j \geq \frac{(1-(\alpha(\bar{S}^*))^2)(1-\alpha(\bar{S}))}{1-\alpha(\bar{S})+(\alpha(\bar{S}))^2}\cdot\left(\sum_{j\in S^*}\pi(j,S^*)\cdot r_j\right).$$

where $\alpha(\cdot)$ *is as defined in* (7).

PROOF. Suppose the underlying mixture model is a mixture of K MNL segments with parameters u_{jk} for $j=0,\ldots,n$ and $k=1,\ldots,K$. We can assume wlog. that $u_{0k}+\ldots+u_{nk}=1$ for all $k=1,\ldots,K$. Let the Markov chain parameters λ_j, ρ_{ij} for all $i=1,\ldots,n$ and $j=0,\ldots,n$ be estimated as in (1) and let $\hat{\pi}(\cdot,\cdot)$ denote the choice probabilities computed from the Markov chain model (3). Therefore, we know that the assortment S computed by Algorithm 1 maximizes $\sum_{j\in S}\hat{\pi}(j,S)r_j$. Let $\alpha_1=\left(1+\frac{\alpha(\bar{S})^2}{1-\alpha(\bar{S})}\right)$, $\alpha_2=(1-\alpha(\bar{S}^*)^2)$. From Theorem 4.2, we know that

$$\hat{\pi}(j,S)\leq\alpha_1\cdot\pi(j,S),\ \forall j\in S,\ \text{and}\ \hat{\pi}(j,S^*)\geq\alpha_2\cdot\pi(j,S^*),\ \forall j\in S^*. \tag{15}$$

Then

$$\sum_{j\in S}\pi(j,S)\cdot r_j\geq\frac{1}{\alpha_1}\cdot\sum_{j\in S}\hat{\pi}(j,S)\cdot r_j\geq\frac{1}{\alpha_1}\cdot\sum_{j\in S^*}\hat{\pi}(j,S^*)\cdot r_j\geq\frac{\alpha_2}{\alpha_1}\cdot\sum_{j\in S^*}\pi(j,S^*)\cdot r_j,$$

where the first inequality follows from (15). The second inequality follows as S is an optimal assortment for the Markov chain choice model, $\hat{\pi}$ and the last inequality follows from (15). □

The performance bound of the assortment, S computed from the Markov chain choice model depends on $\alpha(\bar{S}^*)$ and $\alpha(\bar{S})$ where S^* is the optimal assortment for the true model. If size of either S or S^* is $o(n)$ (say less than $\sqrt{n}$), the performance bound in Theorem 5.4 is not good. In this case, we can recalibrate the Markov chain model by estimating the transition probability matrix only for the states in S and ignoring the other states. Then, we can recompute an optimal assortment for the new Markov chain choice model.

6. COMPUTATIONAL RESULTS

In this section, we present a computational study on the performance of the Markov chain choice model in modeling random utility based discrete choice models. In Theorem 4.2, we present theoretical bounds on the relative error between the choice probabilities computed by the Markov chain model and the true choice probability. While in Theorem 4.6, we present a family of instances where the bound is tight but these can be conservative in general. In this computational study, we compare the performance of the Markov chain choice model for random instances of the mixture of MNLs model.

We generate random instances of the mixture of MNLs model and estimate the Markov chain parameters, $\lambda_i, i\in\mathcal{N}$ and transition probabilities, $\rho_{ij}, i\in\mathcal{N}, j\in\mathcal{N}_+$ using the choice probability data for only assortments $\mathcal{S}=\{\mathcal{N},\mathcal{N}\setminus\{i\}\ |; i=1,\ldots,n\}$ using (1). We then compare the choice probability compute by the Markov chain model with the true choice probability for out of sample offer sets, $S\subseteq\mathcal{N}$ where $S\notin\mathcal{S}$.

Random MMNL instances. We generate the random instances of the MMNL model as follows. Let n denote the number of products and K denote the number of customer segments in the MMNL model. For each $k=1,\ldots,K$, the MNL parameters of segment k, $u_{0k},\ldots,u_{nk}$ are i.i.d samples of the uniform distribution in $[0,1]$. Also, for all $k=1,\ldots,K$, the probability of segment k, $\theta_k=1/K$.

For a random instance, we only use the choice probability data for assortments $\mathcal{S}=\{\mathcal{N},\mathcal{N}\setminus\{i\}\ |; i=1,\ldots,n\}$ to compute the Markov chain choice model parameters as described in (1). We would like to reemphasize that in estimating the parameters for the Markov chain model, we do not require any knowledge of the MMNL model parameters or even the number of segments

Table I. Relative Error in choice probabilities of the Markov chain model and the MNL model with respect to true MMNL choice model with random parameters. Here n is the number of products and K is the number of segments in the MMNL model

	n	K	errMNL(%)	errMC(%)
1.	10	3	12.53	3.10
2.	20	3	11.07	2.42
3.	30	4	8.62	2.52
4.	40	4	4.87	2.42
5.	60	5	4.24	1.96
6.	80	5	6.66	1.60
7.	100	5	4.78	1.63
8.	150	6	4.09	1.26
9.	200	6	3.70	1.14
10.	500	7	2.21	0.81
11.	1000	7	1.43	0.65

Table II. Relative Error in choice probabilities of the Markov chain and the MNL models with respect to the random permutation MMNL choice model

	n	K	errMNL(%)	errMC(%)
1.	10	3	7.38	3.15
2.	20	3	6.07	2.74
3.	30	4	6.71	3.02
4.	40	4	5.48	2.51
5.	60	5	4.01	1.81
6.	80	5	3.96	1.82
7.	100	5	3.20	1.51
8.	150	6	2.92	1.39
9.	200	6	2.78	1.33
10.	500	7	1.70	0.82
11.	1000	7	1.28	0.62

K in the MMNL model. In addition, we also consider the MNL approximation of the MMNL model where we approximate the MMNL model by a single MNL model with parameters $v_j = \sum_{k=1}^{K} \theta_k u_{jk},\ \forall j = 0, \ldots, n$. In the computational experiments, we also compare the performance of the MNL approximation with the true MMNL choice probability.

Experimental Setup. To compare the out of sample performance of the Markov chain model and MNL approximation, we generate T random offer sets of size between $n/3$ and $2n/3$ and compare the choice probability computed by the Markov chain model with the true choice probability. We use $n \geq 10$, $K = \lceil \log n \rceil$ and $T = 500$. For all assortments $S_1, \ldots, S_T$, we compute the maximum relative errors of the choice probability of the Markov chain model and MNL approximation with respect to the true choice probability. For any $S \subseteq \mathcal{N}, j \in S_+$, let $\pi^{\mathsf{MC}}(j, S)$, $\pi^{\mathsf{MNL}}(j, S)$, and $\pi(j, S)$ denote the choice probability of the Markov chain model, approximate MNL model, and the true MMNL model respectively. Then for all $t = 1, \ldots, T$,

$$\mathsf{errMC}(t) = 100 \cdot \max_{j \in S_t} \frac{|\pi^{\mathsf{MC}}(j, S_t) - \pi(j, S_t)|}{\pi(j, S_t)},\ \mathsf{errMNL}(t) = 100 \cdot \max_{j \in S_t} \frac{|\pi^{\mathsf{MNL}}(j, S_t) - \pi(j, S_t)|}{\pi(j, S_t)},$$

and let errMC and errMNL denote the average maximum relative error over T subsets.

We present the computational results for the average maximum relative error in Table I. We observe that the relative error of the choice probability computed by the Markov chain model with respect to the true MMNL choice probability is less than 3.2% for all values of n, K presented in the table. Moreover, the Markov chain model performs significantly better than the MNL approximation for the MMNL model; for all value of n and K in our computational experiments, the relative error for the MNL approximation is more than twice the relative error for the Markov chain model. We would also like to note that the average size of the offer sets, $S_1, \ldots, S_T$ is approximately $n/2$. Therefore, $|\bar{S}_t|$ and $\alpha(\bar{S}_t)$ is large on average and the approximation bounds in Theorem 4.2 are quite conservative as compared to the computational experiments.

Alternate Family of Random MMNL Instances. We consider another random family of instances of the mixture of MNLs model to compare the performance of the Markov chain choice model. Motivated by the bad example in Theorem 4.6 that shows that the tightness of approximation bounds for the Markov chain model, we consider the following family of MMNL instances. As before, let K denote the number of customer segments each occurring with probability $1/K$. For the first two segments, the parameters are given by

$$u_{j1} = j + 1,\ u_{j2} = n + 1 - j,\ \forall j = 0, \ldots, n,$$

i.e. the MNL parameters belong to $\{1, \ldots, n+1\}$ and are in increasing order for segment 1 and in decreasing for segment 2. For segments $k = 3, \ldots, K$, the $u_{jk}, j = 0, \ldots, n$ are a random permutation of $\{1, \ldots, n+1\}$. This construction is similar to the structure of the bad example in Theorem 4.6 that is a mixture of two MNL models with increasing parameters for one model and decreasing for another for almost all the products.

As before, we use $n \geq 10$, $K = \lceil \log n \rceil$, and generate $T = 500$ random offer sets of size between $n/3$ and $2n/3$. For each offer set $S_t, t = 1, \ldots, T$, we compute the maximum relative error in $\pi^{\mathsf{MC}}(j, S_t)$ and $\pi^{\mathsf{MNL}}(j, S_t)$ with respect to the true MMNL choice probability, $\pi(j, S_t)$. We present our results in Table II. The computational results are similar to the other family of random MMNL instances with relative error less than 3.2% for the Markov chain model for all values of n, K. Furthermore, as in the other case, the relative error of the Markov chain model is significantly lower (less than a factor half) than that of the MNL approximation.

Therefore, on these set of instances, the Markov chain model is an extremely good approximation of the choice probabilities of the MMNL model. Since we get a very good approximation of the choice probabilities using the Markov chain model, Algorithm 1 can be used to also obtain a good approximation for the assortment optimization problem for the MMNL model.

7. CONCLUDING REMARKS

In this paper, we address the problem of selecting the "right" choice model from the data and introduce a tractable Markov chain based model. This model is based on a new primitive for substitution behavior where substitutions from one product to another are modeled as Markovian transitions between the states corresponding to the two products. We give a data-driven procedure to estimate the parameters for the Markov chain choice model that does not require any knowledge of the underlying choice model except the choice probability data for certain collection of assortments. We also show that if the data comes from an underlying MNL model, we show that the Markov chain model is exact, i.e., the choice probabilities computed by the Markov chain model are equal to the true choice probabilities for all offer sets. Furthermore, we show that under mild assumptions, the Markov chain model is a good approximation for general random utility based choice models and we give approximation bounds and a family of instances that show that the bound is tight. We also consider the assortment optimization problem for the Markov chain choice model and present a policy iteration based algorithm that computes the optimal assortment in polynomial time.

In addition to the theoretical bounds, we also present computational results to compare the performance of the Markov chain model. Our results show that for random instances of the MMNL model, the Markov chain model performs extremely well and the relative error for all values of n, K in our experiments is less than 3.2%. The empirical performance is significantly better than the theoretical bounds. The theoretical and computational results presented in this paper make the Markov chain model a promising practical data-driven approach to modeling choice. In this paper, we present estimation procedures for the Markov chain model assuming we have noiseless and complete data for certain collection of assortments. An important future step would be to study statistical estimation methods to compute Markov chain model parameters from partial noisy data that is typical in most practical applications.

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APPENDIX

Proof of Lemma 4.3 Let $\boldsymbol{\lambda}(S)$ denote the vector of choice probabilities for $j \in S_+$ and $\boldsymbol{\lambda}(\bar{S})$ denote the choice probabilities for $j \notin S$. Therefore, $\boldsymbol{\lambda}^T = [\boldsymbol{\lambda}(S)^T \ \boldsymbol{\lambda}(\bar{S})^T]$. From Theorem 2.1, we know that

$$\begin{aligned}\hat{\pi}(j,S) &= \lambda_j + \left(\boldsymbol{\lambda}(\bar{S})\right)^T (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{B}\boldsymbol{e}_j \\ &= \lambda_j + \boldsymbol{\lambda}(\bar{S})^T \left(\sum_{t=0}^{\infty} \boldsymbol{C}^t\right) \boldsymbol{B}\boldsymbol{e}_j \\ &\geq \lambda_j + \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{B}\boldsymbol{e}_j = \lambda_j + \sum_{i \in \bar{S}} \lambda_i \rho_{ij},\end{aligned}$$

where the second last inequality follows as $C_{ij} \geq 0$ for all $i, j \in \bar{S}$ and $\boldsymbol{\lambda} \geq \mathbf{0}$. Using (1), for any $i \in \bar{S}$, we can compute

$$\rho_{ij} = \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i,\mathcal{N})} \cdot \frac{u_{jk}}{1 - u_{ik}}.$$

Therefore,

$$\begin{aligned}\hat{\pi}(j,S) \ &\geq \ \lambda_j + \sum_{i \in \bar{S}} \lambda_i \cdot \left(\sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i,\mathcal{N})} \cdot \frac{u_{jk}}{1 - u_{ik}}\right) \\ &= \ \pi(j,\mathcal{N}) + \sum_{i \in \bar{S}} \pi(i,\mathcal{N}) \left(\sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i,\mathcal{N})} \cdot \frac{u_{jk}}{1 - u_{ik}}\right) \\ &= \ \sum_{k=1}^{K} \theta_k u_{jk} + \sum_{i \in \bar{S}} \sum_{k=1}^{K} \theta_k \cdot u_{ik} \cdot \frac{u_{jk}}{1 - u_{ik}} \\ &= \ \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{i \in \bar{S}} \frac{u_{ik}}{1 - u_{ik}}\right) \\ &= \ \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \sum_{i \in \bar{S}} \left(\sum_{q=1}^{\infty} (u_{ik})^q\right)\right),\end{aligned}$$

where the second equality follows as $\lambda_j = \pi(j,\mathcal{N}$ for all $j \in \mathcal{N}_+$ and (16) follows from the definition of $\pi(j,\mathcal{N})$ for the mixture of MNL model. The last equality follows as $u_{ik} < 1$ for all $i \in \mathcal{N}_+, k = 1, \ldots, K$. ■

Proof of Lemma 4.4 From Theorem 2.1, we know that

$$\begin{aligned}\hat{\pi}(j,S) &= \lambda_j + \left(\boldsymbol{\lambda}(\bar{S})\right)^T (\boldsymbol{I} - \boldsymbol{C})^{-1} \boldsymbol{B}\boldsymbol{e}_j \\ &= \lambda_j + \sum_{q=0}^{\infty} \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{C}^q \boldsymbol{B}\boldsymbol{e}_j.\end{aligned}$$

We claim that for any $q \in \mathbb{Z}_+$,

$$\boldsymbol{\lambda}(\bar{S})^T \boldsymbol{C}^q \boldsymbol{B}\boldsymbol{e}_j \leq \gamma^q \cdot \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{B}\boldsymbol{e}_j, \tag{16}$$

where γ is the maximum eigenvalue of $\boldsymbol{C} = \boldsymbol{\rho}(\bar{S}, \bar{S})$. Note that $\boldsymbol{C}$ is a M-matrix with all row sums strictly less than one. Therefore, γ is also strictly less than one. Now,

$$
\begin{aligned}
\hat{\pi}(j, S) &= \lambda_j + \sum_{q=0}^{\infty} \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{C}^q \boldsymbol{B} \boldsymbol{e}_j \\
&\leq \lambda_j + \sum_{q=0}^{\infty} \gamma^q \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{B} \boldsymbol{e}_j \qquad (17) \\
&= \lambda_j + \frac{1}{1-\gamma} \cdot \boldsymbol{\lambda}(\bar{S})^T \boldsymbol{B} \boldsymbol{e}_j \\
&= \pi(j, \mathcal{N}) + \frac{1}{1-\gamma} \cdot \sum_{i \in \bar{S}} \pi(i, \mathcal{N}) \rho_{ij} \qquad (18) \\
&= \sum_{k=1}^{K} \theta_k u_{jk} + \frac{1}{1-\gamma} \cdot \sum_{i \in \bar{S}} \pi(i, \mathcal{N}) \left(\sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} \cdot \frac{u_{jk}}{1 - u_{ik}} \right) \qquad (19) \\
&= \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{1}{1-\gamma} \cdot \sum_{i \in \bar{S}} \frac{u_{ik}}{1 - u_{ik}} \right) \\
&= \sum_{k=1}^{K} \theta_k u_{jk} \left(1 + \frac{1}{1-\gamma} \cdot \left(\sum_{i \in \bar{S}} \sum_{q=1}^{\infty} (u_{ik})^q \right) \right),
\end{aligned}
$$

where (17) follows from (16). Equations(18) and (19) follows from substituting the values of the parameters λ_j and ρ_{ij} for all $i \in \bar{S}$. ■

Proof of Lemma 4.5 Note that $\alpha < 1$ since $u_{0k} > 0$ for all $k = 1, \ldots, K$. For any $i, j \in \bar{S}, i \neq j$, $C_{ij} = \rho_{ij} > 0$ and $C_{ii} = 0$. Therefore, $\boldsymbol{C}$ is irreducible since $C_{ij}^2 > 0$ for all $i, j \in \bar{S}$. From Perron-Frobenius theorem ([Perron 1907]), this implies that the maximum eigenvalue of $\boldsymbol{C}$ is at most

$$
\begin{aligned}
\max_{i \in \bar{S}} \sum_{j \in \bar{S}} C_{ij} &= \max_{i \in \bar{S}} \sum_{j \in \bar{S}} \rho_{ij} \\
&= \max_{i \in \bar{S}} \sum_{j \in \bar{S}, j \neq i} \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} \cdot \frac{u_{jk}}{1 - u_{ik}} \\
&= \max_{i \in \bar{S}} \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} \cdot \frac{\sum_{j \in \bar{S}, j \neq i} u_{jk}}{1 - u_{ik}} \\
&\leq \max_{i \in \bar{S}} \sum_{k=1}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} \cdot \frac{\alpha - u_{ik}}{1 - u_{ik}} \qquad (20) \\
&< \alpha \max_{i \in \bar{S}} \sum_{k=1\gamma}^{K} \frac{\theta_k u_{ik}}{\pi(i, \mathcal{N})} \cdot \frac{1 - u_{ik}/\alpha}{1 - u_{ik}} \leq \alpha,
\end{aligned}
$$

where (20) follows as $u_k(\bar{S}) \leq \alpha < 1$ for all $k = 1, \ldots, K$, and the last inequality follows as $\theta_k u_{ik} \leq \pi(i, \mathcal{N})$ ■