Optimal Price Rebates for Demand Response under Power Flow Constraints

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***Abstract*—Demand side participation is essential for a real-time energy balance in today's electricity grid, especially in the presence of highly uncertain renewable sources of energy such as wind and solar. In this paper, we consider a price rebate approach for demand response where an electric utility company can offer real-time price rebates to the consumers to reduce the load consumption. We study the problem of computing near-optimal prices (or rebates) to offer to the customers to reduce the consumption in the presence of power flow constraints and transmission losses on the distribution grid. To the best of our knowledge, this is the first work that optimizes demand response under an AC power flow model.**

The main challenge in this problem arises from the non-convexity of AC power flow constraints and also the uncertainty in price elasticity of the demand. We formulate an SDP based convex relaxation of the power flow constraints and give an iterative procedure to compute the offer prices to minimize the total expected cost. We conduct numerical experiments to compare the performance of our heuristic with other optimization approaches including using DC power flow model or no power flow model at all. Our computational study shows that the performance of our AC power flow based heuristic is significantly better than the other approaches. Unlike the DC power flow constraints, the AC power flow constraints model transmission losses. Therefore, we can optimize the offer prices based on the topology of the grid and leverage both the actual load reduction as well as the reduction in the transmission losses.

I. Introduction

Due to an increasing integration of renewable sources such as wind and solar power on the grid, the supply uncertainty in the electricity market has increased significantly. Demand-side participation has, therefore, become essential to maintain a real-time energy balance in the grid. There are several ways to increase the demand-side participation for the real-time energy balance including time of use pricing, real-time pricing for smart appliances, interruptible demand-response contracts and real-time price rebates and incentives. In this paper, we consider a price rebate approach to demand-response where the electric utility company offers real-time price rebates or incentives to consumers to reduce their power consumption. Typically, an electric utility buys the forecast day-ahead load in the day-ahead market and pays the shortfall (if the actual demand turns out to be higher) on the real-time market. However, if the supply in the real-time market is scarce, the real-time prices can be very high and the utility is exposed to the high prices. Such a scenario can arise often if a significant fraction of power is generated by highly uncertain sources such as wind and solar plants. Since end customers, including residential and most commercial customers, do not pay the real-time prices, the demand does not adjust to the real-time prices and the utility is exposed to the price shocks. With price rebates or interruptible load contracts, the utility has the option of offering financial incentives to the customers to reduce their demand in such scenarios.

Interruptible load contracts have been studied extensively in the literature as an approach to demand response, both from the perspective of optimal execution of contracts (see Oren and Smith [1]. Caves et al. [2]) and also design and pricing (see Fahrioglu and Alvarado [3], Kamat and Oren [4], Tan and Varaiya [5], Oren [6], Bhattacharta et al. [7]) and more recently, Bitar and Low [8], Roozbehani et al. [9]. We refer readers to the survey by Baldick et al. [10] that provides a good overview of the literature. In this approach, the utility buys an option from consumers to reduce their load by a pre-specified amount at most a pre-specified number of times until the option expiration and the main operational problem is to decide on the optimal execution policy of the fixed number of contracts.

In this paper, we consider an alternative approach where the utility can offer real-time price incentives or rebates to consumers to reduce the power consumption. The goal is to compute the prices (or rebates) to offer to different consumers to reduce the power consumption such that the required reduction can be achieved in minimum possible cost. When the power consumption changes on the grid, the power flows and transmission losses also change and we need to consider them to accurately model the change in power consumption. Therefore, we study the price rebate optimization problem for demand response under an AC power flow model. The AC power flow model allows us to model the transmission losses in the distribution network.Therefore, we can optimize over *both* the reduction in demand and transmission losses in the network. To the best of our knowledge, this is the first work that models AC power flows in the context of demand response optimization. We show that the AC formulation leads to a

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significant reduction in the rebates that one needs to offer in order to shed a certain demand. Alternate formulations that do not model transmission losses, are not able to leverage losses to reduce rebate payments.

II. Problem Definition and Contributions

Let $\mathcal{K} := \{1, 2, ..., K\}$ denote the set of buses, $\mathcal{G} \subseteq \mathcal{K}$ denote the set of generator buses, $\mathcal{C} \subseteq \mathcal{K}$ denote the set of demand buses, and $\mathcal{N} \subseteq \mathcal{K} \times \mathcal{K}$ denote the set of transmission lines. Let $P_k^g + jQ_k^g$ denote the generation at bus $k \in \mathcal{G}$, and let $\bar{P}_i^c + j\bar{Q}_i^c$ denote the *nominal* load at demand bus $i \in \mathcal{C}$, i.e. the demand in the absence of any rebates, and. For each demand bus, we are given the response (or supply) function, $R_i(\gamma_i)$ that specifies the mean reduction in load at bus i at any given offer price (or rebate) γ_i. We assume that the actual demand reduction is random, and is given by

$$\tilde{R}_i(\gamma_i) = R_i(\gamma_i) + \epsilon_i,$$

where ϵ_i is a mean zero random variable with a known distribution. We assume that the distribution of ϵ does not depend on the rebate γ_i. We allow for the error distributions at different demand buses to be possibly different. The total expected payments at offer price $\mathbf{\Gamma}$ is given by $\mathbb{E}[\sum_i \gamma_i(R_i(\gamma_i) + \epsilon_i)] = \sum_i \gamma_i R(\gamma_i)$.

We are also given a power reduction target D and we pay a shortfall penalty λ per unit whenever we are not able to meet target reduction. Thus, the total expected cost of the DR program for an offer price $\mathbf{\Gamma}$ is given by

$$\sum_i \gamma_i R_i(\gamma_i) + \lambda \mathbb{E}_{\boldsymbol{\epsilon}}\big[\big(D - (P_{\mathsf{inj}}^0 - P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon}))\big)_+\big], \quad (1)$$

where $(y)_+ = \max\{y, 0\}$, $P_{\mathsf{inj}}^0 = \sum_{k \in \mathcal{G}} P_k^g$ denotes the total generation (or injection) without any price rebates, and $P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon}) = \sum_{k \in \mathcal{G}} P_k^g(\mathbf{\Gamma}, \boldsymbol{\epsilon})$ is the total generation when the price rebate is $\mathbf{\Gamma} = (\gamma_1, \ldots, \gamma_{|C|})$ and the load at each demand bus i is $(P_i^c - R_i(\gamma_i) - \epsilon_i + jQ_i^c)$. From (1) it follows that the offer price optimization problem can be formulated as the following stochastic optimization problem

$$\begin{array}{ll} \min_{\mathbf{\Gamma}} & \sum_i \gamma_i R_i(\gamma_i) + \lambda \mathbb{E}_{\boldsymbol{\epsilon}}\big[\big(D - (P_{\mathsf{inj}}^0 - P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon}))\big)_+\big] \\ \text{s.t.} & P_{\mathsf{inj}}^0, P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon}) \text{ satisfy power flow constraints.} \end{array} \quad (2)$$

Note that the power flow constraints for an AC power grid are non-convex and optimizing over these constraints is NP-hard in general [11]. Therefore, solving (2) to compute offer prices is computationally hard even for very simple supply functions R_i.

Our Contributions. In this paper, we propose an efficient iterative-heuristic to solve the offer price optimization problem under AC power flow constraints. Our heuristic is constructed using a sample average approximation (SAA) to the stochastic optimization problem (2), a semidefinite programming (SDP) based relaxation for the non-convex AC power flow constraints, and a linear approximation for $P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon})$.

In the SAA approach one approximates the expectation by an average over set a set of samples; thus, the SAA approach applied to (2) results in the optimization problem

$$\begin{array}{ll} \min_{\mathbf{\Gamma}} & \sum_i \gamma_i R_i(\gamma_i) + \frac{\lambda}{M} \sum_{k=1}^M \big(D - (P_{\mathsf{inj}}^0 - P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon}^k))\big)_+ \\ \text{s.t.} & P_{\mathsf{inj}}^0, P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon}^k) \text{ satisfy power flow constraints,} \end{array} \quad (3)$$

where $\boldsymbol{\epsilon}^1, \ldots, \boldsymbol{\epsilon}^M$, are M IID samples of stochastic error vector $\boldsymbol{\epsilon}$.

It is well known that the non-convex power flow constraints can be approximated by an SDP relaxation where the decision variable is one $2K$-dimensional symmetric semidefinite matrix [11] (see, also Section II). Thus, (3) can be approximated by an SDP with M $2K$-dimensional semidefinite matrices as variables – one for each of M samples of $\boldsymbol{\epsilon}$. Since $M \approx 100$ to 1000, this direct SDP formulation is intractable even for very small networks with $K = 30$ buses.

We construct a linear approximation $\hat{P}(\mathbf{\Gamma}, \epsilon)$ for the power generation $P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon})$ function in a small neighborhood of a given rebate vector $\mathbf{\Gamma}$. In particular, we define

$$P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon}) \approx \hat{P}(\mathbf{\Gamma}, \epsilon) = P_{\mathsf{inj}}(\mathbf{\Gamma}, \mathbf{0}) + \sum_i \pi_i^{\mathbf{\Gamma}} \epsilon_i, \quad (4)$$

where the sensitivity parameters $\boldsymbol{\pi}^{\mathbf{\Gamma}} = (\pi_i^{\mathbf{\Gamma}})_{\{i \in \mathcal{C}\}}$ is the vector of dual variables corresponding to the power balance constraint in the power flow SDP. We show that the optimal offer price $\mathbf{\Gamma}$ in the SAA approximation (3) with the linear approximation (4) for $P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon})$ can be computed by solving an SDP with only one $2K$-dimensional positive semidefinite decision variable. The dual variables for the power balance equations in this SDP are used to construct a new linear approximation for $P_{\mathsf{inj}}(\mathbf{\Gamma}, \boldsymbol{\epsilon})$.

Computational Study. We conduct detailed computational experiments to compare the performance of our AC power flow based heuristic. We compare our heuristic with the following alternative approaches for demand response:

i) offer price optimization under DC power flow constraints
ii) offer price optimization without any power flow constraints

Our computational experiments show that our proposed AC power flow based heuristic is efficient and performs significantly better than the other two approaches. While we consider an extensive set of instances for our experiments, due to space limitations, we only present our results for IEEE 57-bus test case in Section IV.

Outline. The rest of this paper is organized as follows. In Section II, we introduce the model notations and the SDP relaxation for the power flow constraints. We present the iterative heuristic for the offer price optimization problem (2) in Section III and present the computational study in Section IV.

III. SDP Formulation for Power Flow Constraints

In this section, we introduce the model notations and the SDP relaxation for the power flow constraints based on Lavaei and Low [11].

Let $Y \in \mathbb{C}^{K \times K}$ denote the admittance matrix of the distribution network where for each $(k, l) \in \mathcal{N}$, $Y_{kl} = -y_{kl}$

if $k \neq l$ and $y_{kk} + \sum_{m \in N_k} y_{km}$ otherwise (N_k denotes the set of all buses that are directly connected to bus k), and y_{km} denotes the effective admittance between bus k and bus l. Let $\boldsymbol{V}$ denote the complex voltage at bus $k \in \mathcal{K}$ and $\boldsymbol{I}$ denote the complex current injected at bus k. Then it follows that the power injected at bus k is given by

$$\begin{aligned} S_k &= \text{Re}(V_k I_k^*) + j\text{Im}(V_k I_k^*) \\ &= \text{Re}(\boldsymbol{V} e_k e_k Y \boldsymbol{V}^*) + j\text{Im}(\boldsymbol{V} e_k e_k Y \boldsymbol{V}^*) \\ &= \boldsymbol{X}^\top \boldsymbol{Y}_k \boldsymbol{X} + j\boldsymbol{X}^\top \bar{\boldsymbol{Y}}_k \boldsymbol{X} = \text{Tr}(\boldsymbol{Y}_k \boldsymbol{W}) + j\text{Tr}(\bar{\boldsymbol{Y}}_k \boldsymbol{W}) \end{aligned}$$

where

$$\begin{aligned} Y_k &:= e_k e_k^T \boldsymbol{Y} \\ \boldsymbol{Y}_k &:= \frac{1}{2}\begin{bmatrix} \text{Re}\left\{Y_k + Y_k^T\right\} & \text{Im}\left\{Y_k^T - Y_k\right\} \\ \text{Im}\left\{Y_k - Y_k^T\right\} & \text{Re}\left\{Y_k + Y_k^T\right\} \end{bmatrix} \\ \bar{\boldsymbol{Y}}_k &:= -\frac{1}{2}\begin{bmatrix} \text{Im}\left\{Y_k + Y_k^T\right\} & \text{Re}\left\{Y_k - Y_k^T\right\} \\ \text{Re}\left\{Y_k^T - Y_k\right\} & \text{Im}\left\{Y_k + Y_k^T\right\} \end{bmatrix} \\ \boldsymbol{X} &:= \begin{bmatrix} \text{Re}\left\{\mathbf{V}\right\}^T & \text{Im}\left\{\mathbf{V}\right\}^T \end{bmatrix}^T \\ \boldsymbol{W} &:= \boldsymbol{X}\boldsymbol{X}^T, \end{aligned}$$

and $e_1, e_2, ..., e_K$ standard unit vectors in $\mathbb{R}^K$. Thus, the power injection constraints are given by

$$\begin{aligned} \text{Tr}(\boldsymbol{Y}_k \boldsymbol{W}) &= \begin{cases} P_k^g & k \in \mathcal{G}, \\ -P_k^c & k \in \mathcal{C}, \end{cases} \\ \text{Tr}(\bar{\boldsymbol{Y}}_k \boldsymbol{W}) &= \begin{cases} Q_k^g & k \in \mathcal{G}, \\ -Q_k^c & k \in \mathcal{C}, \end{cases} \end{aligned} \tag{5}$$

And,

$$P_k^{min} \leq P_k^g \leq P_k^{max},\ Q_k^{min} \leq Q_k^g \leq Q_k^{max},\ k \in \mathcal{G} \tag{6}$$

In addition to the power injection constraints, there are upper and lower bounds on the magnitude $|V_k|^2 = \text{Re}(V_k)^2 + \text{Im}(V_k)^2$ at bus k, on the magnitude $|V_l - V_k|^2$ of the difference in voltage between bus l and m. reformulated as

$$\begin{aligned} &(V_k^{\min})^2 \leq \text{Tr}(\boldsymbol{M}_k \boldsymbol{W}) \leq (V_k^{\max})^2, \\ &\text{Tr}(\boldsymbol{M}_{lm} \boldsymbol{W}) \leq (\Delta V_{lm}^{\max})^2 \end{aligned} \tag{7}$$

where

$$\begin{aligned} \boldsymbol{M}_k &:= \begin{bmatrix} e_k e_k^\top & 0 \\ 0 & e_k e_k^\top \end{bmatrix}, \\ \boldsymbol{M}_{lm} &:= \begin{bmatrix} (e_l - e_m)(e_l - e_m)^T & 0 \\ 0 & (e_l - e_m)(e_l - e_m)^T \end{bmatrix} \end{aligned}$$

There are line constraints on the voltage, there are also constraints on the real and apparent power being carried on a line (l, m). The constraint that the real power $P_{lm} \leq P_{lm}^{\max}$ and the magnitude of the apparent power $|S_{lm}| \leq S_{lm}^{\max}$, can be re-formulated as

$$\begin{aligned} &\text{Tr}\{\mathbf{Y}_{lm} \boldsymbol{W}\} \leq P_{lm}^{\max}, \\ &\text{Tr}\{\mathbf{Y}_{lm} \boldsymbol{W}\}^2 + \text{Tr}\{\overline{\mathbf{Y}}_{lm} \boldsymbol{W}\}^2 \leq (S_{lm}^{\max})^2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} Y_{lm} &:= (\bar{y}_{lm} + y_{lm}) e_l e_l^\top - y_{lm} e_l e_m^\top \\ \mathbf{Y}_{lm} &:= \frac{1}{2}\begin{bmatrix} \text{Re}\left\{Y_{lm} + Y_{lm}^T\right\} & \text{Im}\left\{Y_{lm}^T - Y_{lm}\right\} \\ \text{Im}\left\{Y_{lm} - Y_{lm}^T\right\} & \text{Re}\left\{Y_{lm} + Y_{lm}^T\right\} \end{bmatrix} \\ \overline{\mathbf{Y}}_{lm} &:= -\frac{1}{2}\begin{bmatrix} \text{Im}\left\{Y_{lm} + Y_{lm}^T\right\} & \text{Re}\left\{Y_{lm} - Y_{lm}^T\right\} \\ \text{Re}\left\{Y_{lm}^T - Y_{lm}\right\} & \text{Im}\left\{Y_{lm} + Y_{lm}^T\right\} \end{bmatrix} \end{aligned}$$

The definition $\boldsymbol{W} = \boldsymbol{X}\boldsymbol{X}^\top$ is equivalent $\boldsymbol{W} \succeq 0$, i.e. $\boldsymbol{W}$ is symmetric positive semidefinite, and $\text{Rank}\{\boldsymbol{W}\} = 1$. Any rank-1 positive semidefinite matrix $\boldsymbol{W}$ that satisfied (5)-(8) represents a feasible power flow.

Relaxing the rank constraint on $\boldsymbol{W}$, we obtain an SDP relaxation of the power flow constraints. Lavaei and Low [11] show that the above relaxation is exact if the distribution network is a tree. Sojoudi and Lavaei [12] extend the results to several other classes of networks where the above SDP formulation is exact. However, in general, it is NP-hard to optimize over the power flow constraints.

IV. Offer Price Optimization Problem under AC Power flow constraints

In this section, we present our heuristic for the offer price optimization problem under AC power flow constraints. Recall that we are given a load reduction target D and the random load reduction at demand bus i as a function of the price rebate γ_i is $R_i(\gamma_i) + \epsilon_i$. We assume that we can generate samples of the random vector $\boldsymbol{\epsilon}$. We also assume that for each i, $R_i(\cdot)$ is linear, i.e., $R_i(\gamma_i) = a_i \gamma_i$ for some $a_i > 0$; as we will see this assumption can be relaxed.

Suppose $\boldsymbol{W}(\boldsymbol{\epsilon})$ denote any positive semidefinite solution for the constraints (5)-(8) when demand at bus $i \in \mathcal{C}$, $P_i^c + jQ_i^c = (\bar{P}_i^c - R_i(\gamma_i) - \epsilon_i) + j\bar{Q}_i^c$. Then the generation

$$P_i^g(\boldsymbol{\Gamma}, \boldsymbol{\epsilon}) = \text{Tr}(\mathbf{Y}_i \boldsymbol{W}(\boldsymbol{\epsilon})).$$

We would like to emphasize that the total reduction in generation

$$\sum_{i \in \mathcal{G}} (P_i^g(0) - P_i^g(\boldsymbol{\Gamma}, \boldsymbol{\epsilon})) = \sum_{i \in \mathcal{G}} (P_i^g(0) - \text{Tr}(\mathbf{Y}_i \boldsymbol{W}(\boldsymbol{\epsilon}))$$

is greater than the sum of load reductions at demand buses due to savings in transmission losses. Now, the offer price optimization problem with AC power flow constraints (AC-PF-DR) is given by the following stochastic optimization problem.

$$\begin{aligned} \min_{\boldsymbol{\Gamma}, \boldsymbol{W}(\boldsymbol{\epsilon})} \quad & \textstyle\sum_{i \in \mathcal{C}} \gamma_i R_i(\gamma_i) + \\ & \lambda \cdot \mathbb{E}_{\boldsymbol{\epsilon}}\Big[\big(D - \textstyle\sum_{i \in \mathcal{G}} (P_i^g(0) - \text{Tr}\{\mathbf{Y}_i W(\boldsymbol{\epsilon})\})\big)_+\Big] \\ \text{s.t.} \quad & \boldsymbol{W}(\boldsymbol{\epsilon}) \succeq 0 \text{ satisfies (5)-(8) with } P_i^c + jQ_i^c = \\ & \quad (\bar{P}_i^c - R_i(\gamma_i) - \epsilon_i) + j\bar{Q}_i^c \text{ for } i \in \mathcal{C} \end{aligned} \tag{9}$$

As mentioned earlier, a direct SAA approach applied to (9) is computationally intractable, since we need $W(\boldsymbol{\epsilon})$ to satisfy the constraints for each sample $\boldsymbol{\epsilon}$ and the number of samples is large even for small networks.

A. Linear Approximation for Injected Power

We approximate for the total random injected power $P_{\mathsf{inj}}(\boldsymbol{\Gamma}, \boldsymbol{\epsilon})$ by an affine function $\hat{P}_{\mathsf{inf}}(\boldsymbol{\Gamma}, \boldsymbol{\epsilon})$ of the random vector $\boldsymbol{\epsilon}$. In particular, we set

$$\hat{P}(\boldsymbol{\Gamma}, \boldsymbol{\epsilon}) = P_{\mathsf{inj}}(\boldsymbol{\Gamma}, \mathbf{0}) + \sum_{i \in \mathcal{C}} \pi_i^{\boldsymbol{\Gamma}} \epsilon_i, \tag{10}$$

where $P_{\mathsf{inj}}(\boldsymbol{\Gamma}, \mathbf{0})$ denotes the total generation when the offer price is $\boldsymbol{\Gamma}$ and the stochastic error $\boldsymbol{\epsilon} = \mathbf{0}$. and the sensitivity coefficients $\pi_i^{\boldsymbol{\Gamma}}$ depend on the offer price $\boldsymbol{\Gamma}$ and denote the change in total injected power per unit change in the demand at bus i. Thus, $\boldsymbol{\pi}^{\boldsymbol{\Gamma}}$ can be interpreted as the dual variables corresponding to the real power balance constraint in the optimal power flow formulation at offer price Γ. Algorithm 1 describes the computation of the linear approximation of the injected power at any offer price $\boldsymbol{\Gamma}$. We conduct numerical experiments to compare the error of the linear approximation for a set of test networks and observe that the difference between the true injected power and the linear approximation is at most, 2% across all test networks. Therefore, the linear approximation provides a good approximation under reasonable bounds on standard deviation of the stochastic errors.

Algorithm 1: Linear approximation of Injected Power

1: **Input**: Offer price $\boldsymbol{\Gamma}$
2: Compute a solution $\boldsymbol{W} \succeq 0$ to (5)-(8) that minimizes total injected power with demand at each node $k \in \mathcal{C}$, $P^c_k - R_k(\gamma_k)$.
3: $\pi^{\boldsymbol{\Gamma}}_k$: optimal dual variable for real power balance constraint at bus $k \in \mathcal{C}$.
4: **Return**:

$$\hat{P}(\boldsymbol{\Gamma}, \boldsymbol{\epsilon}) = P_{\mathsf{inj}}(\boldsymbol{\Gamma}, \boldsymbol{0}) + \sum_{k \in \mathcal{C}} \pi^{\boldsymbol{\Gamma}}_k \epsilon_k.$$

B. Offer Price Optimization Heuristic

Now we are ready to describe our iterative heuristic to compute the offer prices $\boldsymbol{\Gamma}$ using a linear approximation of the injected power (10). We start with an initial offer price, $\boldsymbol{\Gamma}^0$ and use Algorithm 1 to compute the linear approximation for the injected power,

$$\hat{P}(\boldsymbol{\Gamma}, \boldsymbol{\epsilon}) = P_{\mathsf{inj}}(\boldsymbol{\Gamma}, \boldsymbol{0}) + \sum_{i \in \mathcal{C}} \pi^0_i \epsilon_i.$$

We can then formulate the optimization problem to compute the next iterate for offer prices as follows.

$$\begin{array}{ll} \min\limits_{\boldsymbol{\Gamma}, \boldsymbol{W}} & \sum_{i \in \mathcal{C}} \gamma_i R_i(\gamma_i) \\ & + \lambda \mathbb{E}_{\boldsymbol{\epsilon}} \Big[\big(D - (P^0_{\mathsf{inj}} - (P_{\mathsf{inj}}(\boldsymbol{\Gamma}, \boldsymbol{0}) + \sum_{i \in \mathcal{C}} \pi^0_i \epsilon_i)) \big)_+ \Big] \\ \text{s.t.} & \boldsymbol{W} \succeq 0 \text{ satisfies (5)-(8) with } P^c_i + jQ^c_i = \\ & \bar{P}^c_i - R_i(\gamma_i) + j\bar{Q}^c_i \text{ for } i \in \mathcal{C} \end{array} \tag{11}$$

Note that the constraints in (11) no longer depend on $\boldsymbol{\epsilon}$; consequently, there is a single positive semidefinite variable $\boldsymbol{W}$. We approximate the expectation in the objective using M samples, i.e. we solve the following optimization problem

$$\begin{array}{ll} \min\limits_{\boldsymbol{\Gamma}, \boldsymbol{W}} & \sum_{i \in \mathcal{C}} \gamma_i R_i(\gamma_i) \\ & + \frac{\lambda}{M} \sum_{n=1}^{M} \big(D - P^0_{\mathsf{inj}} + P_{\mathsf{inj}}(\boldsymbol{\Gamma}, \boldsymbol{0}) + \sum_{i \in \mathcal{C}} \pi^0_i \epsilon^n_i \big)_+ \\ \text{s.t.} & \boldsymbol{W} \succeq 0 \text{ satisfies (5)-(8) with } P^c_i + jQ^c_i = \\ & \bar{P}^c_i - R_i(\gamma_i) + j\bar{Q}^c_i \text{ for } i \in \mathcal{C} \end{array}$$

where $\{\boldsymbol{\epsilon}^n : n = 1, \ldots, M\}$ denotes M samples of the random vector $\boldsymbol{\epsilon}$. Since we have a *single* SDP constraint in the above formulation as opposed to one SDP constraint for every sample of $\boldsymbol{\epsilon}$ in the original formulation (9), we can solve the above SAA problem efficiently. Note also that the coefficients π_i are constants in (11). We solve (11) to compute the offer prices in the next iterate. We then compute the linear approximation for the injected power at the new offer prices and continue this iterative procedure until it converges to a fixed point. We describe the details in Algorithm 2.

Algorithm 2: Offer Price optimization Heuristic, AC-PF-DR

1: **Initialize**: $t := 0$, $\delta := 1$, offer prices $\boldsymbol{\Gamma}^0$.
2: **while** $(\delta > 0.001)$ **do**
   Call **Algorithm 1** to compute coefficients $\pi^{\boldsymbol{\Gamma}^t}_i$
   Solve (11) to compute $\boldsymbol{\Gamma}^{t+1}$
   $t := t + 1$
   $\delta = \max\limits_i \dfrac{|\Gamma^t_i - \Gamma^{t-1}_i|}{\Gamma^{t-1}_i}$
   end
3: **Return:** Γ^t

V. Computational Study

In this section, we describe the results from our computational study that compares the performance of our iterative heuristic with two other approaches for offer price optimization $i)$ offer price optimization with DC power flow constraints, and $ii)$ offer price optimization without any power flow constraints. We begin by describing these two approaches.

A. Offer Price optimization with DC Power Flows

The DC power flow model is constructed by linearizing the AC power flow equations. Let $\boldsymbol{\theta}$ denote the vector of phase angles of voltage at all the buses. Under typical operating conditions, the angle difference $|\theta_l - \theta_m|$ for any transmission line $(l, m) \in \mathcal{N}$ is small ($\ll 10$ degrees). Therefore, $\sin(\theta_l - \theta_m) \approx (\theta_l - \theta_m)$ and $\cos(\theta_l - \theta_m) \approx 1$. For all transmission lines $(l, m) \in \mathcal{N}$ we assume that the resistance is nearly zero and also the magnitude of voltage is 1 p.u. at all buses. Furthermore, we can, without loss of generality, assume that $\theta_1 = 0$. With these approximations, we can formulate the power flow constraints as $\boldsymbol{P} = \boldsymbol{B}\boldsymbol{\theta}$, where $\boldsymbol{P} \in \mathbb{R}^{K-1}$ is the vector of power injections for buses $2, \ldots, K$, $\boldsymbol{\theta} \in \mathbb{R}^{K-1}$ is the vector of nodal voltage angles, and $\boldsymbol{B} \in \mathbb{R}^{(K-1)\times(K-1)}$ is the network admittance matrix. The constraint for bus 1 is linearly dependent on the other constraints and can therefore, be eliminated.

Let $\boldsymbol{N} \in \{0, -1, 1\}^{K \times K}$ be the bus-line incidence matrix for the distribution network, and let ρ denote the upper bound on allowed angle difference on any link. Then the DC offer price optimization problem (DC-PF-DR) can be formulated as follows.

$$\begin{array}{ll} \min\limits_{\boldsymbol{\Gamma}, \boldsymbol{\theta}(\boldsymbol{\epsilon}), \boldsymbol{P}(\boldsymbol{\epsilon})} & \sum_{i \in \mathcal{C}} \gamma_i R_i(\gamma_i) + \lambda \cdot \mathbb{E}_{\boldsymbol{\epsilon}} \left(D - P^0_{\mathsf{inj}} + P_{\mathsf{inj}}(\boldsymbol{\epsilon}) \right)_+ \\ \text{s.t.} & \boldsymbol{P}(\boldsymbol{\epsilon}) = \boldsymbol{B}\boldsymbol{\theta}(\boldsymbol{\epsilon}) \\ & P_{\mathsf{inj}}(\boldsymbol{\epsilon}) = \sum_{k \in \mathcal{G}} P_k(\boldsymbol{\epsilon}) \\ & P_k(\boldsymbol{\epsilon}) = P^c_k - R_k(\gamma_k) - \epsilon_k, \quad \forall k \in \mathcal{C} \\ & \left\| \boldsymbol{N}\boldsymbol{\theta}(\boldsymbol{\epsilon}) \right\|_\infty \leq \rho, \end{array} \tag{12}$$

where the notation $\boldsymbol{P}(\boldsymbol{\epsilon})$, $\boldsymbol{\theta}(\boldsymbol{\epsilon})$ and $P_{\mathsf{inj}}(\boldsymbol{\epsilon})$ emphasizes that the phase angles, the power on lines, and the overall power injection is a function of the stochastic error $\boldsymbol{\epsilon}$. Note that angle difference constraint can also be modeled as a penalty term $\eta\mathbb{E}_{\boldsymbol{\epsilon}}\left(N\theta(\epsilon)-\rho\right)_+$ in the objective. Unlike the AC-PF-DR, the DC-PF-DR problem solved efficiently using the SAA method. For a linear supply function $R_i(\gamma_i) = a_i\gamma_i$, The SAA approximation of the DC-PF-DR (12) is a quadratic program. Note that since we assume that the resistance on transmission lines is zero, the DC-PF-DR formulation is not able to model transmission losses.

B. Offer Price optimization without power flows

We also consider an offer price optimization approach without any power flow constraints. In this approach, we assume that for any given offer price $\boldsymbol{\Gamma}$ the total load reduction is $\sum_{i\in\mathcal{C}} R_i(\gamma_i)+\epsilon_i$, without taking power flow or transmission losses into account. The following offer price optimization problem without power flows (NO-PF-DR)

$$\min_{\boldsymbol{\Gamma}} \sum_{i\in\mathcal{C}} \gamma_i R_i(\gamma_i) + \lambda \cdot \mathbb{E}_{\boldsymbol{\epsilon}}\left[\Big(D - \big(\sum_{i\in\mathcal{C}} R_i(\gamma_i)+\epsilon_i\big)\Big)_+\right] \quad (13)$$

can be solved efficiently using the SAA method.

C. Experimental Setup

We compare the performance of our AC power flow based offer price optimization heuristic with the above two approaches. We compute the offer prices $\boldsymbol{\Gamma}^A$ for all three formulations $A \in \mathcal{A} = \{$AC-PF-DR, DC-PF-DR, NO-PF-DR$\}$. For each price vector $\boldsymbol{\Gamma}^A$ we compute the

$$\text{DR-cost}: \sum_{i\in\mathcal{C}} \gamma_i^A R_i(\gamma_i^A)$$
$$\text{Shortfall-penalty}: \lambda\cdot\mathbb{E}_{\boldsymbol{\epsilon}}\big(D - P^0_{\mathsf{inj}} + P_{\mathsf{inj}}(\boldsymbol{\Gamma}^{\boldsymbol{A}},\boldsymbol{\epsilon})\big)_+,$$

where $P_{\mathsf{inj}}(\boldsymbol{\Gamma}^{\boldsymbol{A}},\boldsymbol{\epsilon})$ is computed using the AC OPF equations. We approximate the expectation in the shortfall penalty by a sample average over $M = 100$ samples of the stochastic error $\boldsymbol{\epsilon}$. Since one has to solve an SDP for computing $P_{\mathsf{inj}}(\boldsymbol{\Gamma}^{\boldsymbol{A}},\boldsymbol{\epsilon}^n)$ for each sample $\boldsymbol{\epsilon}^n$, it is not computationally efficient to compute the sample average. Instead, we use the linear approximation $\hat{P}(\boldsymbol{\Gamma},\boldsymbol{\epsilon})$ defined in (10) and approximate the sample average using a single SDP. The experimental setup is described in Algorithm 3.

Remark. In Algorithm 3, we compute the total cost for a given set of offer prices by solving the SDP relaxation of the optimal power flow problem. Since the SDP relaxation is not tight in general, it only provides a lower bound on both the generated power and the total cost. However, for all the instances in our computational study, the SDP relaxation has a rank one optimal solution which implies that the relaxation is tight for our instances. Therefore, the comparison in Algorithm 3 is accurate. We would like to emphasize that this is not the case in general and the SDP objective value only provides a lower bound on the performance for a given set of offer prices. If the SDP relaxation is not tight, we can compute an upper bound on the total cost for a given set of offer prices by computing a feasible AC power flow solution using Matpower that computes only a local optimal solution. Computing the exact total cost for a given set of offer prices is as hard as computing the optimal power flow which is known to be NP-hard in general [11].

Algorithm 3: Computational Experiment

1: Compute

$$\boldsymbol{\Gamma}^{\mathsf{AC}} \leftarrow \text{offer-price using AC-PF-DR}$$
$$\boldsymbol{\Gamma}^{\mathsf{DC}} \leftarrow \text{offer-price using DC-PF-DR}$$
$$\boldsymbol{\Gamma}^{\mathsf{DR}} \leftarrow \text{offer-price using NO-PF-DR}$$

2: For $A \in \{\mathsf{AC},\mathsf{DC},\mathsf{DR}\}$, compute **DR-cost** as

$$\sum_{i\in\mathcal{C}} \gamma_i^A R_i(\gamma_i^{\mathcal{A}}).$$

3: For all $A \in \mathcal{A}$, compute the linear approximation $\hat{P}(\boldsymbol{\Gamma}^{\mathcal{A}},\boldsymbol{\epsilon})$ for injected AC power at $\boldsymbol{\Gamma}^{\mathcal{A}}$ using Algorithm 1.

$$\hat{P}(\boldsymbol{\Gamma}^{\mathcal{A}},\boldsymbol{\epsilon}) = P_{\mathsf{inj}}(\boldsymbol{\Gamma}^{\mathcal{A}},\mathbf{0}) + \sum_{i\in\mathcal{C}} \pi_i^{\mathcal{A}}\epsilon_i.$$

4: Sample $M = 100$ values: $\boldsymbol{\epsilon}^1,\ldots,\boldsymbol{\epsilon}^M$.

5: For each $\mathcal{A}$, compute **Shortfall-Penalty** as

$$\frac{\lambda}{M}\sum_{n=1}^{M}\left(D - P^0_{\mathsf{inj}} + P_{\mathsf{inj}}(\boldsymbol{\Gamma}^{\mathcal{A}},\mathbf{0}) + \sum_{i\in\mathcal{C}}\pi_i^{\mathcal{A}}\epsilon_i^n\right)_+ .$$

We conducted extensive numerical experiments on a large set of IEEE test instances; however, due to space limitations, we present our results for only the IEEE 57-bus network here. A comprehensive description of the experiments is deferred to the full version of the paper. We use the following values for the parameters in the experiments described in Tables I and II: $\lambda = 10$, $\rho = 10$ in degrees and target load reduction $D \in \{2\%, 5\%, 10\%, 15\%, 20\%, 25\%\}$ of the total active load.

The relative difference of total cost between the AC-PF-DR and DC-PF-DR and AC-PF-DR and NO-PF-DR are reported in the last two rows of both Tables. We observe that the cost of the AC power flow based heuristic is significantly lower than the other two for all values of the target demand reduction D. The DC-PF-DR and NO-PF-DR heuristics compute identical solutions where the total demand reduction at the demand buses is equal to the target D. Since these two approaches do not account for transmission losses, the actual reduction in injected power $P_{\mathsf{inj}}(\boldsymbol{\Gamma}) - P^0_{\mathsf{inj}} > D$; the DC-PF-DR and NO-PF-DR end up paying more rebate than is needed to meet the target. On the other hand, the AC power flow based heuristic achieves the target demand reduction through a combination of reduction in demand at demand buses and reduction in transmission losses (since lower cumulative power needs to be transmitted). This is because the AC-PF-DR models the transmission losses in the optimization phase. Therefore, while the shortfall penalty in all the heuristics is almost zero, the total payments for the AC power flow based heuristic are smaller, leading to a significant improvement in the total cost. We observe similar results in our extensive computational experiments over a large set of instances and different choices of parameter values. The details are deferred to the full version of the paper.

Demand	9.00	22.49	44.98
NO-PF-DR Model:			
Payment	2.23	13.92	55.69
Shortfall penalty	0	0	0
Total cost	2.23	13.92	55.69
Max angle diff.	4.57	4.17	3.92
Reduction at nodes	9.00	22.49	44.98
Total Reduction	9.57	23.88	47.62
Average Gamma	0.15	0.39	0.77
CPU time(s)	0.72	0.76	0.76
DC-PF-DR Model:			
Payment	2.23	13.92	55.69
Shortfall penalty	0	0	0
Total cost	2.23	13.92	55.69
Max angle diff.	4.57	4.17	3.92
Reduction at nodes	9.00	22.49	44.98
Total Reduction	9.57	23.88	47.62
Average Gamma	0.15	0.39	0.77
CPU time(s)	4.13	3.48	3.20
AC-PF-DR Model:			
Payment	2.00	12.41	49.70
Shortfall penalty	0.01	0.04	0.12
Total cost	2.01	12.45	49.82
Max angle diff.	4.57	4.18	3.94
Reduction at nodes	8.53	21.22	42.48
Total Reduction	9.08	22.56	45.02
Average Gamma	0.15	0.36	0.73
CPU time(s)	81.40	113.15	76.14
Iterations	3	4	3
(DC-AC)/DC	10.82%	11.86%	11.77%
(DR-AC)/DR	10.82%	11.86%	11.77%

TABLE I. COMPARISON OF AC-PF-DR, DC-PF-DR AND NO-PF-DR

Demand	67.47	89.96	112.45
NO-PF-DR Model:			
Payment	125.43	224.32	353.66
Shortfall penalty	0	0	0
Total cost	125.43	224.32	353.66
Max angle diff.	3.72	3.55	3.39
Reduction at nodes	67.47	89.96	112.45
Total Reduction	71.21	94.69	118.05
Average Gamma	1.16	1.55	1.93
CPU time(s)	0.62	0.76	0.48
DC-PF-DR Model:			
Payment	125.43	224.32	353.66
Shortfall penalty	0	0	0
Total cost	125.43	224.32	353.66
Max angle diff.	3.72	3.55	3.39
Reduction at nodes	67.47	89.96	112.45
Total Reduction	71.21	94.69	118.05
Average Gamma	1.16	1.55	1.93
CPU time(s)	3.43	3.26	2.64
AC-PF-DR Model:			
Payment	112.44	201.87	319.61
Shortfall penalty	0.15	0.23	0.36
Total cost	112.59	202.10	319.97
Max angle diff.	3.76	3.60	3.44
Reduction at nodes	63.88	85.39	107.00
Total Reduction	67.50	89.96	112.43
Average Gamma	1.10	1.47	1.84
CPU time(s)	132.15	105.02	73.69
Iterations	5	4	3
(DC-AC)/DC	11.40%	10.99%	10.53%
(DR-AC)/DR	11.40%	10.99%	10.53%

TABLE II. COMPARISON OF AC-PF-DR, DC-PF-DR AND NO-PF-DR

VI. CONCLUSIONS

In this paper, we consider a price rebate approach to demand-response under power flow constraints. We consider an AC-OPF model that allows us to model transmission losses and therefore, optimize the offer prices to achieve the target demand reduction through a combination of reduction in demand at demand nodes and reduction in transmission losses. Using an AC-OPF model is important since the DC-OPF does not model the transmission losses and cannot account for these reduction at the offer-price optimization stage. However, the AC-OPF based offer price optimization problem is non-convex, and therefore, hard to solve. We propose an iterative data-driven algorithm to compute offer prices (or price rebates) to achieve the required demand reduction with minimum possible cost. We conducted a computational study to compare the performance of our iterative algorithm with other demand response heuristics. Our results show that our iterative heuristic performs significantly better than other offer price optimization approaches based on DC-OPF or without any power flow constraints that are not able to account for the savings in transmission losses. Therefore, there is significant value in using an AC-OPF based model for demand-response optimization. It is important to note that our iterative algorithm only computes a local optimal solution and our computational study compares the performance of our local optimal solution to other heuristics. An optimal solution for the AC-OPF based optimization problem will perform even better and designing a provably near-optimal algorithm is an interesting open question.

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