IEOR 3106: Introduction to Operations Research: Stochastic Models

Final Exam, Thursday, December 16, 2010

There are 5 problems, each with multiple parts.

You need to show your work. Briefly explain your reasoning.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name.

You may keep the exam itself. Solutions will eventually be posted on line.

1. Is the Professor Cheerful? (15 points)

The professor's mood can either be cheerful or gloomy. The following observations have been made: If the professor has been gloomy for two consecutive days, then the professor will be gloomy again the following day with probability 2/3; and if the professor has been cheerful for two consecutive days, then professor will again be cheerful the following day with probability 1/2. If the professor was gloomy yesterday and is cheerful today, then he will be cheerful tomorrow with probability 5/6; if, instead the professor was cheerful yesterday and is gloomy today, then he will be cheerful tomorrow with probability 1/2.

(a) Let X_n be the professor's mood on day n. Is the stochastic process $\{X_n : n \geq 1\}$ a Markov chain? Why or why not?

(b) Suppose that the professor has been cheerful for 7 days in a row. What is the probability that his mood on the next four days will be first gloomy, then cheerful, then gloomy again, and finally cheerful again?

(c) What is the long-run proportion of all days on which the professor is cheerful? (Calculate the explicit value, if possible, but at least indicate how you would do the calculation.)

2. The Random Knight (15 points)

A knight (chess piece) is placed on one of the corner squares of an empty chessboard (having $8 \times 8 = 64$ squares) and then it is allowed to make a sequence of random moves, taking each of its legal moves in each step with equal probability, independent of the history of its moves up to that time. (Recall that the knight can move either (i) two squares horizontally (left or right) plus one square vertically (up or down) or (ii) one square horizontally (left or right) plus two squares vertically (up or down), provided of course it ends up at one of the other squares on the board.)

(a) What is the probability that the knight is back on its initial square after two moves?

(b) What is the probability that the knight is back on its initial square after five moves?

(c) Let $N(n)$ be a random variable counting the number of times that the knight visits its initial starting square among the first n random moves. What kind of stochastic process is $\{N(n) : n \geq 1\}$? Explain.

(d) What is the long-run proportion of moves after which the knight ends up at its initial square? Explain.

3. Replacing the Wang-Yang Bakery Delivery Vans (25 points)

Qiurui Wang and Zimu Yang have decided to open a bakery in Queens. They plan to deliver bread to stores in Manhattan every morning using a delivery van. Your assignment is to help them analyze some of the operational costs of operating the Wang-Yang Bakery.

Suppose that the Wang-Yang Bakery buys a new delivery van as soon as the current one breaks down or reaches the age of 5 years, whichever occurs first. Suppose that a new delivery van costs $\$30,000$ and that a 5-year old delivery van has a resale value of \$5000. Suppose that the delivery van has no resale value if it breaks down and even incurs a random cost with an expected value of \$3000 if it breaks down. Suppose that the lifetime of each delivery van is random, being uniformly distributed in the interval between 0 years and 8 years. (Ignore the time value of money; i.e., assume that money is worth the same throughout time.)

(a) In the long run, what proportion of the delivery vans break down?

(b) What is the expected interval between the times that the bakery gets a new delivery van?

(c) What is the long-run average cost of the delivery vans to the bakery (considering only the specified costs)?

(d) What is the long-run average age of the delivery van currently in use?

4. The Bad Investment Bank (25 points)

Alexander **B**oger, Kyle **A**rmington and Shazia **D**harssi have formed their own investment bank, the BAD Investment Bank. They have been so wildly successful that they have gone public, with a stock on the NASDAC exchange under the BAD label. Suppose that the initial price of a share of BAD stock is \$40. Suppose that the BAD share price evolves over time according to the stochastic process

$$S(t) = 40 + 5B(t),$$

where $B \equiv \{B(t) : t \geq 0\}$ is standard Brownian motion.

(a) What is the probability that the price of a share of BAD stock after $t = 4$ years exceeds 60?

(b) What is the probability that the price of a share of BAD stock reaches the level 60 at some time before 4 years?

(c) What is the probability that the price of a share of BAD stock drops to 30 before it reaches 60?

(d) What is expected time that a share of the BAD stock first hits either 30 or 60?

(e) What is $E[S(2)S(3)]$?

5. The Guo-Millet (GM) Tattoo Parlor (20 points)

Yuhan Guo and Alexandre Millet have opened the Guo-Millet (GM) Tattoo Parlor near campus, specializing in probability tattoos, including images of Pascal's triangle, Galton's quincunx and, their best seller - Gauss's bell curve. The two proprietors each work on one customer at a time. There is a waiting room, which can accommodate one person in addition to the two in service. Suppose that potential customers come to the GM Tattoo Parlor according to a Poisson process at constant rate of 2 per hour. Suppose that potential customers arriving when the waiting room is full (when there are two customers in service and one other waiting) leave without getting a tattoo, and without affecting future arrivals. Suppose that waiting customers have limited patience, so that they may leave without receiving service if they have waited too long. Suppose that the times successive customers are willing to wait before starting service are independent random variables, each with an exponential distribution having a mean of 1 hour. Yuhan is especially good at her work; it takes only an average of one half hour to give customers their desired tattoo. Alexandre is less experienced; it takes him an average of one hour to give customers their desired tattoo. However, the required times are random, since some tattoos are much more complicated and ornate than others. Suppose that these "service" times are mutually independent and exponentially distributed random variables. Some customers know about Yuhan's awesome skills, so that when both Yuhan and Alexandre are available, new arrivals select Yuhan with probability 2/3. However, neither Yuhan nor Alexandre is idle if there is a customer to be served.

(a) Suppose that the system starts empty. What is the probability that the first departure occurs before the second arrival?

(b) Suppose that new arrivals are not admitted after 7:00 pm, but otherwise the system operates as described above. Suppose that the GM Tattoo Parlor is full at 7:00 pm. What is the expected remaining length of time until all three customers present at 7:00 pm are gone?

(c) Give an expression (formula, not number) for the steady-state (or long-run limiting) probability that there are two customers in the GM Tattoo Parlor at some time, with two in service and nobody waiting.

(d) Give an expression (formula, not number) for the long-run proportion of all potential arrivals (including ones that are blocked) that enter, wait and abandonment.

There is a small bonus for exact numerical answers in parts (c) and (d).