IEOR 3106: Introduction to Operations Research: Stochastic Models

Final Exam, Thursday, December 16, 2010

SOLUTIONS

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name.

You may keep the exam itself. Solutions will eventually be posted on line.

1. Is the Professor Cheerful? (15 points)

The professor's mood can either be cheerful or gloomy. The following observations have been made: If the professor has been gloomy for two consecutive days, then the professor will be gloomy again the following day with probability 2/3; and if the professor has been cheerful for two consecutive days, then professor will again be cheerful the following day with probability 1/2. If the professor was gloomy yesterday and is cheerful today, then he will be cheerful tomorrow with probability 5/6; if, instead the professor was cheerful yesterday and is gloomy today, then he will be cheerful tomorrow with probability 1/2.

(a) Let X_n be the professor's mood on day n. Is the stochastic process $\{X_n : n \geq 1\}$ a Markov chain? Why or why not?

This example is just like Example 4.4 in the book. NO. The stochastic process $\{X_n : n \geq 1\}$ is *not* a Markov chain. To be a Markov chain, we need the Markov property, i.e.,

$$P(X_{n+1} = k_{n+1} | X_1 = k_1, \ldots, X_n = k_n) = P(X_{n+1} = k_{n+1} | X_n = k_n)$$

for all n and for all states $k_1, \ldots k_{n+1}$. In other words, we need

$$P(\text{future event}|\text{present states and past states}) = P(\text{future event}|\text{present states})$$

But that is not satisfied here. Here the weather on day $n+1$ depends on the weather on *both* days n and $n-1$.

(b) Suppose that the professor has been cheerful for 7 days in a row. What is the probability that his mood on the next four days will be first gloomy, then cheerful, then gloomy again, and finally cheerful again?

For this part and the next part, it is helpful to create a Markov chain, depending on the professor's mood on two successive days. Let the origin state be the mood on days $n-1$ and n; let the destination state be the weather on days n and $n+1$. In both cases, put the earlier day first. Thus we get the 4×4 transition matrix

$$P = \begin{matrix} CC \\ GC \\ CG \\ GG \end{matrix} \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 5/6 & 0 & 1/6 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 0 & 2/3 \end{pmatrix} .$$

The columns are labelled the same as the row, with the same pairs in the same order. For this part, we have the product of four transition probabilities, i.e.,

$$P_{(C,C),(C,G)}P_{(C,G),(G,C)}P_{(G,C),(C,G)}P_{(C,G),(G,C)}$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \frac{1}{48}.$$

(c) What is the long-run proportion of all days on which the professor is cheerful? (Calculate the explicit value, if possible, but at least indicate how you would do the calculation.)

First, we see that the transition matrix is irreducible and aperiodic. Positive diagonal elements ensure that it is aperiodic. We see that we can get from any state to any other state in a finite number steps (even if not necessarily one step). Thus the long-run probability is given by the stationary vector.

We need to solve $\pi = \pi P$ for the transition matrix above, where the elements of the vector π sum to 1, and then add the probabilities for two states; i.e., the answer is $\pi_{(C,C)} + \pi_{(G,C)} = \pi_{(C,C)} + \pi_{(C,G)}$. In the long run, the probability of being cheerful yesterday is the same as the probability of being cheerful today. From that relation we see that, necessarily, $\pi_{(G,C)} = \pi_{(C,G)}$. That means that we can write

$$\pi = (x, y, y, 1 - x - 2y) \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < y < 1.$$

The first equation is

$$\frac{x}{2} + \frac{5y}{6} = x \quad \text{or} \quad y = \frac{3x}{5}.$$

Substituting that into π gives in terms of x alone,

$$\pi = (x, 3x/5, 3x/5, 1 - x - 6x/5) = (x, 3x/5, 3x/5, 1 - 11x/5) \quad \text{for} \quad 0 < x < 1.$$

The fourth equation, using this form for π, gives

$$\frac{3x/5}{2} + \frac{(1 - (11x/5)2}{3} = 1 - (11x/5) \quad \text{or} \quad x = \frac{10}{31}.$$

That makes

$$\pi = (10/31, 6/31, 6/31, 9/31).$$

There is plenty of opportunity of mistakes in such a computation, so we check our answer by verifying that the alleged stationary vector does indeed satisfy $\pi = \pi P$. Remember there is one and only one solution that is a probability vector. The long run probability of being cheerful is thus

$$\pi_{(C,C)} + \pi_{(G,C)} = \pi_{(C,C)} + \pi_{(C,G)} = 10/31 + 6/31 = 16/31.$$

Good for him/her! The professor is more likely to be cheerful than not.

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2. The Random Knight (15 **points)**

A knight (chess piece) is placed on one of the corner squares of an empty chessboard (having $8 \times 8 = 64$ squares) and then it is allowed to make a sequence of random moves, taking each of its legal moves in each step with equal probability, independent of the history of its moves up to that time. (Recall that the knight can move either (i) two squares horizontally (left or right) plus one square vertically (up or down) or (ii) one square horizontally (left or right) plus two squares vertically (up or down), provided of course it ends up at one of the other squares on the board.)

(a) What is the probability that the knight is back on its initial square after two moves?

This problem is problem 3 in the sample entertainment page and Exercise 4.76 in the book, assigned in homework 4. The square occupied by the knight can be represented by a discrete-time Markov chain with a 64×64 transition matrix P. However, we do not want to write it down, let alone directly solve $\pi = \pi P$, 64 equations in 64 unknowns.

For part (a) a, we can directly calculate:

$$P_{1,1}^{(2)} = 2 \times (1/2)(1/6) = 1/6.$$

(b) What is the probability that the knight is back on its initial square after five moves?

We observe that the knight moves from black squares to red squares and from red squares to black squares, so that the Markov chain transition matrix is **periodic** with period 2. Hence, it only can occupy its initial square after an even number of steps. Hence the probability is

$$P_{1,1}^{(5)} = P_{1,1}^{(2n+1)} = 0.$$

(c) Let $N(n)$ be a random variable counting the number of times that the knight visits its initial starting square among the first n random moves. What kind of stochastic process is $\{N(n) : n \geq 1\}$? Explain.

The stochastic process $\{N(n) : n \geq 1\}$ is a discrete-time **renewal process**. We could also write it as $\{N(t) : t \geq 0\}$, a renewal process, where the times between renewals are all integer values, and thus have a lattice distribution. The times (number of steps) between successive visits to any state in a Markov chain are independent and identically distributed random variables. Thus these visits constitutes renewals.

(d) What is the long-run proportion of moves after which the knight ends up at its initial square? Explain.

Here we avoid the tedious task of solving $\pi = \pi P$ by exploiting reversibility, as in Section 4.8. This problem has the same structure as a random walk on the nodes of a weighted graph, where the weights on all arcs here are 1. There is an arc between two squares if the knight can move from one square to the other. Thus,

$$\pi_1 = \frac{\text{arcs out of square}\, 1}{\sum_{i=1}^{i=64} \text{arcs out of square}\, i} = \frac{2}{336} = \frac{1}{168}$$

See the solutions to homework 4.

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3. Replacing the Wang-Yang Bakery Delivery Vans (25 points)

Qiurui Wang and Zimu Yang have decided to open a bakery in Queens. They plan to deliver bread to stores in Manhattan every morning using a delivery van. Your assignment is to help them analyze some of the operational costs of operating the Wang-Yang Bakery.

Suppose that the Wang-Yang Bakery buys a new delivery van as soon as the current one breaks down or reaches the age of 5 years, whichever occurs first. Suppose that a new delivery van costs $\$30,000$ and that a 5-year old delivery van has a resale value of \$5000. Suppose that the delivery van has no resale value if it breaks down and even incurs a random cost with an expected value of \$3000 if it breaks down. Suppose that the lifetime of each delivery van is random, being uniformly distributed in the interval between 0 years and 8 years. (Ignore the time value of money; i.e., assume that money is worth the same throughout time.)

(a) In the long run, what proportion of the delivery vans break down?

This problem is about renewal theory, Chapter 7 of the book. This problem is like Example 7.12, on a car buying model. The car breaks down if its random life is less than the planned replacement time of 5 years. Since the lifetime of each car is assumed to be uniformly distributed on the interval $[0, 8]$, the probability that each car breaks down before it is replaced is $5/8$. Hence **the long-run proportion of delivery vans that break down is** $5/8$**.**

(b) What is the expected interval between the times that the bakery gets a new delivery van?

Let T be the time until we buy a new delivery van, given that we buy a new delivery van at time 0. Then

$$E[T] = \int_0^5 x\frac{1}{8}\,dx + \left(\frac{3}{8}\right)5 = \frac{25}{16} + \frac{15}{8} \quad = \quad \frac{55}{16} \approx 3.44\,\text{years}$$

(c) What is the long-run average cost of the delivery vans to the bakery (considering only the specified costs)?

Use the renewal-reward theorem, Proposition 7.3 in the book. The long-run average cost of delivery vans is the average cost per cycle divided by the expected length of a cycle, where a cycle is the interval between buying a new delivery van. We computed the expected length of a cycle in part (b). The expected cost per cycle is

$$E[\text{cost per cycle}] = 30 + (5/8)3 + (3/8)(-5) = 30\,\text{thousand dollars}.$$

Hence the long run average cost per year is

$$\frac{E[\text{cost per cycle}]}{E[\text{length of cycle}]} = \frac{30}{55/16} = \frac{96}{11} = 8.7272727...$$

The long-run average cost of the delivery vans is $\$8,727.27$ per year.

(d) What is the long-run average age of the delivery van currently in use?

I had a hard time in selling this in the last class. This is Example 7.16 in the book. The answer is given in formula (7.14). Let $A(t)$ be the age at time t. First, the long-run average age is directly defined as

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T A(t)\,dt.$$

The age itself is a sawtooth function with slope +1. It starts at 0 whenever a new van is bought and rises up at rate 1 until that van is replaced by a new van, when it starts over. The long-run average age is

$$\text{long-run average age} = \frac{E[T^2]}{2E[T]},$$

where $E[T]$ has been computed in part (b). Hence we need $E[T^2]$, but that can be computed in the same way:

$$E[T^2] = \int_0^5 x^2 \frac{1}{8}\,dx + \left(\frac{3}{8}\right)5^2 = \frac{125}{24} + \frac{75}{8} \quad = \quad \frac{350}{24} = \frac{175}{12} \approx 14.583$$

Hence,

$$\text{long-run average age} = \frac{E[T^2]}{2E[T]} = \frac{175/12}{55/8} = \frac{70}{33} = 2.12\,\text{years}.$$

We could also ask a related question. We could ask about the expected age at time t and then let $t \to \infty$. Then we do not take an average. It turns out that, in addition, the expected age at time t after t has become very large has the same formula. That is a consequence of a theorem called the "key renewal theorem."

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4. The Bad Investment Bank (25 points)

Alexander **B**oger, Kyle **A**rmington and Shazia **D**harssi have formed their own investment bank, the BAD Investment Bank. They have been so wildly successful that they have gone public, with a stock on the NASDAC exchange under the BAD label. Suppose that the initial price of a share of BAD stock is $40. Suppose that the BAD share price evolves over time according to the stochastic process

$$S(t) = 40 + 5B(t),$$

where $B \equiv \{B(t) : t \geq 0\}$ is standard Brownian motion.

(a) What is the probability that the price of a share of BAD stock after $t = 4$ years exceeds 60?

Since $B(t)$ is distributed as $N(0,t)$, $S(t)$ is distributed as $N(40, 25t)$. Hence, $S(4)$ is distributed as $N(40, 100)$ or as $40 + 10N(0,1)$. Thus,

$$\begin{aligned} P(S(4) \geq 60) &= P(40 + 10N(0,1) \geq 60) \\ &= P(N(0,1) \geq 2) = 1 - P(N(0,1) \leq 2) = 0.0228 \end{aligned}$$

(b) What is the probability that the price of a share of BAD stock reaches the level 60 at some time before 4 years?

This is the maximum variable involving the reflection principle. See Section 10.2.

$$\begin{aligned} P(\max_{0 \le s \le 4} S(s) > 60) &= 2P(S(4) > 60) \\ &= 2P(N(0,1) > 2) = 2 \times 0.0228 = 0.046 \end{aligned}$$

(c) What is the probability that the price of a share of BAD stock drops to 30 before it reaches 60?

This is given at the end of Section 10.2. We get this from the Optional Sampling Theorem for martingales. From $E[B(T)] = E[B(0)] = 0$, we get that the probability is

$$p = \frac{20}{20+10} = \frac{2}{3}.$$

(d) What is expected time that a share of the BAD stock first hits either 30 or 60?

Recall that, again by the Optional Stopping Theorem for martingales applied to various Brownian motion processes,

$$E[T] = \frac{ab}{\sigma^2} = \frac{(10)(20)}{25} = 8.$$

Here a is the distance down, while b is the distance up, e.g., $a = 40 - 30 = 10$,

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(e) What is $E[S(2)S(3)]$?

Just proceed step by step, reducing it to a relation involving standard Brownian motion, being careful in each step:

$$\begin{aligned} E[S(2)S(3)] &= E[(40+5B(2))(40+5B(3))] = 1600 + 200E[B(2)] + 200E[B(3)] + 25E[B(2)B(3)] \\ &= 1600 + 25E[B(2)B(3)] \quad \text{because} \quad E[B(t)] = 0 \\ &= 1600 + 25E[B(2)(B(3) - B(2) + B(2))] \quad \text{adding and subtracting} \quad B(2) \\ &= 1600 + 25E[B(2)(B(3) - B(2))] + 25E[B(2)^2] \quad \text{by linearity of expectation} \\ &= 1600 + 25E[B(2)^2] \quad \text{by independent increments, each with mean zero} \\ &= 1600 + 50 = 1650 \quad \text{because} \quad E[B(2)^2] = Var(B(2)) = 2. \end{aligned}$$

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5. The Guo-Millet (GM) Tattoo Parlor (20 points)

Yuhan Guo and Alexandre Millet have opened the Guo-Millet (GM) Tattoo Parlor near campus, specializing in probability tattoos, including images of Pascal's triangle, Galton's quincunx and, their best seller - Gauss's bell curve. The two proprietors each work on one customer at a time. There is a waiting room, which can accommodate one person in addition to the two in service. Suppose that potential customers come to the GM Tattoo Parlor according to a Poisson process at constant rate of 2 per hour. Suppose that potential customers arriving when the waiting room is full (when there are two customers in service and one other waiting) leave without getting a tattoo, and without affecting future arrivals. Suppose that waiting customers have limited patience, so that they may leave without receiving service if they have waited too long. Suppose that the times successive customers are willing to wait before starting service are independent random variables, each with an exponential distribution having a mean of 1 hour. Yuhan is especially good at her work; it takes only an average of one half hour to give customers their desired tattoo. Alexandre is less experienced; it takes him an average of one hour to give customers their desired tattoo. However, the required times are random, since some tattoos are much more complicated and ornate than others. Suppose that these "service" times are mutually independent and exponentially distributed random variables. Some customers know about Yuhan's awesome skills, so that when both Yuhan and Alexandre are available, new arrivals select Yuhan with probability 2/3. However, neither Yuhan nor Alexandre is idle if there is a customer to be served.

(a) Suppose that the system starts empty. What is the probability that the first departure occurs before the second arrival?

It is important to note that the appropriate model is *not* a birth and death process, because the service rates of the two servers are different. We must know which one is busy when only one customer is in the system being served. Yuhan's service rate is 2 per hour, while Alexandre's service rate is only 1 per hour. (Sorry, Alexandre!!)

First, we have an arrival. This arrival goes to Yuhan with probability 2/3 and to Alexandre with probability 1/3. Given that the first arrival goes to Yuhan, we have two exponential random variables, the time until the second arrival and the time until the first service completion. The probability that the departure occurs first is

$$P(\text{departure occurs first}) = \frac{\mu_1}{\lambda_1 + \mu_1} = \frac{2}{2+2} = \frac{1}{2}.$$

On the other hand, given that the arrival goes to Alexandre, we have two exponential random variables, the time until the second arrival and the time until the first service completion. The probability that the departure occurs first is

$$P(\text{departure occurs first}) = \frac{\mu_1}{\lambda_1 + \mu_1} = \frac{1}{1+2} = \frac{1}{3}.$$

Combining these two cases, by the probabilities of their occurrences, we get

$$P(\text{departure occurs first}) = (2/3)(1/2) + (1/3)(1/3) = (1/3) + (1/9) = 4/9.$$

(b) Suppose that new arrivals are not admitted after 7:00 pm, but otherwise the system operates as described above. Suppose that the GM Tattoo Parlor is full at 7:00 pm. What is the expected remaining length of time until all three customers present at 7:00 pm are gone?

Initially the rate out of the system is the sum of the abandonment rate, which has been assumed to be 1, plus the sum of the two service rates, 1 and 2. The total rate out is thus 4. The mean time required to reduce the number from 3 to 2 is 1/4 hour. There is lack of memory for the exponential variables, so the future is independent of how we get to 2 in the system. Then both servers are busy. Now we have to wait the *maximum* of these two service times. We can analyze this maximum in two ways. In both ways we exploit that the service times are two independent exponential random variables, but with different mean values.

First, observe that if X_1 and X_2 are two independent exponential random variables with rates λ_1 and λ_2, respectively, then

$$\max\{X_1, X_2\} + \min\{X_1, X_2\} = X_1 + X_2,$$

so that

$$\max\{X_1, X_2\} = X_1 + X_2 - \min\{X_1, X_2\}$$

and

$$E[\max\{X_1, X_2\}] = E[X_1] + E[X_2] - E[\min\{X_1, X_2\}] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

So, in our case, the expected value is $1 + 1/2 - 1/3 = 7/6$. Hence the total expected time is $1/4 + 7/6 = 17/12$ hours. The expected time is 1 hour and 25 minutes. The tatoo parlor can close its doors on average at $8:25$ pm.

We can also reason step by step. With two busy servers, the time until the first to finish is the minimum. We then use the fact that the time value of the minimum is iindependent of the identiy of the variable yielding the minimum. The expected time until the first completed service is thus $1/(1+2) = 1/3$ hour. Then, with probability 2/3 the Yuhan finishes first, making the remaining expected service time 1 hour. And, with probability 1/3 the Alexandre finishes first, making the remaining expected service time 1/2 hour. This analysis makes the total expected time for the last two to finish service equal to $(1/3) + (2/3)1 + (1/3)(1/2) = (2/6) + (5/6) = 7/6$ hour, just as computed above. By this reasoning, the total expected time is also $1/4 + 7/6 = 17/12$ hours.

(c) Give an expression (formula, not number) for the steady-state (or long-run limiting) probability that there are two customers in the GM Tattoo Parlor at some time, with two in service and nobody waiting.

We have a CTMC, but not a BD process. Hence, we can solve $\alpha Q = 0$, where now the CTMC must have 5 states: 0 nobody there, Y Yuhan busy, A Alexandre busy, 2 both busy but nobody waiting, and 3 the system full, with two in service and one waiting. We also require that $\alpha = (\alpha_0, \alpha_Y, \alpha_A, \alpha_2, \alpha_3)$ is a probability vector on the five states $(0, Y, A, 2, 3)$. To give a complete answer here, you should give an explicit representation for the rate matrix Q. It is

$$Q = \begin{matrix} 0 \\ Y \\ A \\ 2 \\ 3 \end{matrix} \begin{pmatrix} -2 & 4/3 & 2/3 & 0 & 0 \\ 2 & -4 & 0 & 2 & 0 \\ 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -5 & 2 \\ 0 & 0 & 0 & 4 & -4 \end{pmatrix}.$$

Although it was not asked for, we continue to actually solve for the probability vector α. First, we can write the equation

$$2\alpha_2 = 4\alpha_3,$$

so that we can conclude that $\alpha_3 = \alpha_2/2$. Hence, we can reduce the number of unknowns from 5 to 4. Since the probabilities sum to 1, that reduces it to three unknowns. We can write

$$\alpha = (1 - 3x - y - z, y, z, 2x, x).$$

The fourth equation gives

$$2y + 2z - 10x + 4x = 0 \quad \text{or} \quad x = (y + z)/3.$$

That reduces the number of unknowns to 2: y and z. We now can write

$$\alpha = (1 - 2(y + z), y, z, 2(y + z)/3, (y + z)/3).$$

Now the first equation reduces to

$$-2 + 6y + 5z = 0 \quad \text{or} \quad 6y + 5z = 2,$$

while the second equation reduces to

$$-6y - 2z + 4/3 = 0 \quad \text{or} \quad 18y + 6z = 4.$$

Multiplying the first equation by 3 and subtracting gives $z = 2/9$. From either of the previous equations, we then get $y = 4/27$. Hence, $x = 10/81$. Hence, we get

$$\alpha = (\alpha_0, \alpha_Y, \alpha_A, \alpha_2, \alpha_3) = (21/81, 12/81, 18/81, 20/81, 10/81).$$

There are plenty of opportunities for mistakes, but we can verify that we have the right answer by checking that indeed $\alpha Q = 0$.

(d) Give an expression (formula, not number) for the long-run proportion of all potential arrivals (including ones that are blocked) that enter, wait and abandonment.

The abandonment rate is $\alpha_3\theta$, where $\theta = 1$ is the abandonment rate when there is somebody waiting. The external arrival rate, including customers that are blocked is simply 2. Hence, the long-run proportion of all arivals that abandon is $\alpha_3/2$. By our detailed calculations in part (c), that is $10/81 \times (1/2) = 5/81$.

There is a small bonus for exact numerical answers in parts (c) and (d). A demonstration of awesomeness.