IEOR 3106: Introduction to Operations Research: Stochastic Models

SOLUTIONS to Final Exam, December 22, 2011

There are 4 problems, each with multiple parts.

You need to show your work. Briefly explain your reasoning.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name.

You may keep the exam itself. Solutions will eventually be posted on line.

1. Oatpower, Inc. (30 points)

Ever since the 2003 power outage in the northeastern United States, there has been growing investor enthusiasm for the company Oatpower, Inc., which is developing a new way to efficiently generate vast power from ordinary oats. Oatpower claims that it will be possible to generate sufficient power from a single cup of oats to run a subway train for ten years. If Oatpower is successful, subways and elevators will no longer have to depend on America's aging electric power grid. The power generation method is highly secret, but there is a rumor that it is based on a surprising chemical reaction between oats and Raspberry Snapple.

The current price of Oatpower stock is $100 per share. Suppose that the Oatpower stock price over time (measured in years) can be modelled as the stochastic process $\{S(t) : t \geq 0\}$, where

$$S(t) \equiv 100 + 5B(t), \quad t \geq 0,$$

and $\{B(t) : t \geq 0\}$ is standard (drift zero, unit variance) Brownian motion.

(a) (4 points) Calculate $E[S(4)]$ and $E[S(4)^2]$.

This problem is about Chapter 10.

$$E[S(4)] = E[100 + 5B(4)] = 100 + 5E[B(4)] = 100$$

because $E[B(t)] = 0$ for all t. Since $Var(a + bX) = b^2 Var(X)$ for any random variable X,

$$Var(S(4)) = Var(5B(4)) = 25Var(B(4)) = 25 \times 4 = 100.$$

Then the second moment is

$$E[S(4)^2] = Var(S(4)) + E[S(4)]^2 = 100 + (100)^2 = 10,100$$

(b) (4 points) Calculate $P(S(4) > 110)$.

From part (a), $S(4)$ is distributed as $N(100, 100)$. Hence,

$$\begin{aligned} P(S(4) > 110) &= P(N(100, 100) > 110) = P(100 + 10N(0, 1) > 110) \\ &= P(N(0, 1) > 1) \approx 0.16 \end{aligned}$$

by the table on p. 82.

(c) (5 points) Let T_s be the first time that the stock price reaches the level s. Calculate $P(T_{110} \leq 4)$.

$$P(T_{110} \leq 4) = P(\max_{0 \leq t \leq 4} \{S(t)\} > 110) = 2P(S(4) > 110) = 2(0.16) = 0.32$$

by §10.2 of the book and then by part (a).

(d) (5 points) Let $T \equiv \min\{T_{90}, T_{140}\}$. Calculate $E[S(T)]$ and $E[T]$.

It is helpful to rephrase the question in terms of ordinary Brownian motion. The hitting time T is distributed the same as $T \equiv \min\{T_{-2}, T_8\}$ for ordinary Brownian motion. That is, the original stochastic process $S(t)$ hits either 90 or 140 the same time that the component $B(t)$ hits either -2 or $+8$.

We use the optional stopping theorem with martingales to obtain

$$E[B(T)] = E[B(0)] = 0, \quad \text{so that} \quad E[S(T)] = E[S(0)] = 100.$$

For the second part we can However, by Exercise 10.18 and by the lecture notes, $B(t)^2 - t$ is a martingale so that

$$E[T] = 2 \times 8 = 16$$

(e) (5 points) Calculate $P(T_{90} < T_{140} < T_{80})$.

This is just like Exercise 10.5. First, since we start at $S(0) = 100$, we use $E[S(T)] = 0$ to obtain

$$P(T_{90} < T_{140}) = \frac{4}{4+1} = \frac{4}{5}.$$

After we hit 90, we have a second independent problem of hitting 140 before 80.

$$P(T_{140} < T_{80} | T_{90} < T_{140}) = \frac{1}{1+5} = \frac{1}{6}.$$

Since these two events are independent,

$$P(T_{90} < T_{140} < T_{80}) = \left(\frac{4}{5}\right) \times \left(\frac{1}{6}\right) = \frac{2}{15}$$

(f) (3 points) Calculate $E[S(1)|S(4) = 120]$.

Here you should use the first equation in display (10.4) in §10.1.

$$E[S(1)|S(4) = 120] = 100 + \frac{1}{4}20 = 105$$

Again, it may be helpful to rephrase the question in terms of ordinary Brownian motion.

$$E[S(1)|S(4) = 120] = 100 + 5E[B(1)|B(4) = 4] = 100 + 5 \times 1 = 105.$$

(g) (4 points) Calculate $E[S(1)^2|S(4) = 120]$.

Here we should use the first equation in display (10.4) in §10.1.

$$Var(S(1)|S(4) = 120) = \left(\frac{1 \times 3}{4}\right) 25 = \frac{75}{4}.$$

Again, it may be helpful to rephrase the question in terms of ordinary Brownian motion.

$$Var(S(1)|S(4) = 120) = 25Var(B(1)|B(4) = 4) = 25\left(\frac{1 \times 3}{4}\right) = \frac{75}{4}.$$

Hence, the second moment is

$$E[S(1)^2|S(4) = 120] = Var(S(1)|S(4) = 120) + (E[S(1)|S(4) = 120])^2 = \frac{75}{4} + (105)^2 = 11,043.75$$

It would be OK to omit the final calculation.

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2. Random Walk on a Graph (25 points)

Consider the graph shown in Figure 2 on top of the next page. There are 7 nodes, labelled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight. Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc, independent of the history prior to reaching the current node. Thus the probability of going from node C to node A in one step is $1/(1+3+5) = 1/9 = 1/9$, while the probability of moving from node C to node B in one step is $3/9 = 1/3$. Let X_n be the node occupied after the n^{th} step of the random walk.

(a) (3 points) What is the probability of going from node A back to to node A in two steps?

There are two possible paths: $A - C - A$ and $A - B - A$. Thus,

$$P_{A,A}^{(2)} = \left(\frac{1}{2} \times \frac{1}{9}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) = \frac{13}{72}$$

(b) (3 points) What is the probability of going from node A back to to node A in three steps?

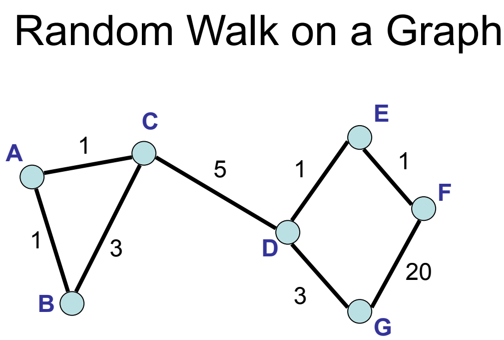

Figure 1: A random walk on a graph.

Again there are two possible paths: $A - C - B - A$ and $A - B - C - A$. Thus,

$$P_{A,A}^{(3)} = \left(\frac{1}{2} \times \frac{3}{9} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{3}{4} \times \frac{1}{9}\right) = \frac{1}{12}$$

(c) (2 points) Is the stochastic process $\{X_n : n \geq 0\}$ a periodic irreducible discrete-time Markov chain? Why or why not?

No. The stochastic process $\{X_n : n \geq 0\}$ *is* an irreducible discrete-time Markov chain (every state can be reached from every other state in some finite number of steps), but it is not periodic. From parts (a) and (b), we see that it can return in either 2 or 3 steps. So the period is necessarily 1. See the first page of §4.4 in the book for discussion.

(d) (5 points) What is the long-run proportion of moves ending in the node A?

Here we use the time reversibility, as discussed in §4.8. See Example 4.36. We need to solve the equation $\pi = \pi P$. However, here we have an explicit formula: The steady-state probability of being at A is the sum of the weights out of A, divided by the sum over all nodes. Hence, the long-run proportion of steps spent in A (which equals the steady state probability) is

$$\pi_A = \frac{2}{2+4+9+9+2+21+23} = \frac{2}{70} = \frac{1}{35}$$

(e) (4 points) Starting from node A, what is the expected number of steps required to return to node A?

We use the fact that the expected time to return is the reciprocal of the steady-state probability. Hence the answer is 35.

(f) (4 points) Give an expression (not the numerical value) for the expected number of visits to node G, starting from node A, before coming to either node B or node F.

These last two parts involve an absorbing Markov chain, as in liberating Makov mouse. See Section 4.6. As discussed in the lecture notes, here the absorbing DTMC has the general form:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} ,$$

where I is an identity matrix (1's on the diagonal and 0's elsewhere) and 0 (zero) is a matrix of zeros. In this case, I would be 2×2, R is 5×2 and Q is 5×5. The matrix Q describes the probabilities of motion among the transient states, as discussed. The matrix R gives the probabilities of absorption in one step (going from one of the transient states to one of the absorbing states in a single step). Here the absorbing states are nodes B and F. In general Q would be square, say m by m, while R would be m by k, and I would be k by k. In particular,

$$Q = \begin{array}{c} A \\ C \\ D \\ E \\ G \end{array} \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1/9 & 0 & 5/9 & 0 & 0 \\ 0 & 5/9 & 0 & 1/9 & 3/9 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 3/23 & 0 & 0 \end{pmatrix} ,$$

and

$$R = \begin{array}{c} A \\ C \\ D \\ E \\ G \end{array} \begin{pmatrix} 1/2 & 0 \\ 3/9 & 0 \\ 0 & 0 \\ 0 & 1/2 \\ 0 & 20/23 \end{pmatrix} ,$$

The columns of Q are the same as the rows of Q, whereas the columns of R are B and F, in that order.

$$I - Q = \begin{array}{c} A \\ C \\ D \\ E \\ G \end{array} \begin{pmatrix} 1 & -1/2 & 0 & 0 & 0 \\ -1/9 & 1 & -5/9 & 0 & 0 \\ 0 & -5/9 & 1 & -1/9 & -3/9 \\ 0 & 0 & -1/2 & 1 & 0 \\ 0 & 0 & -3/23 & 0 & 1 \end{pmatrix} .$$

Then the matrix N is the inverse of the matrix $I - Q$ above. The answer then is $N_{A,G}$, where $N = (I - Q)^{-1}$, where A and G are the appropriate states. That is the element in the first row and the last column.

(g) (4 points) Give an expression (not the numerical value) for the probability of reaching node B before node F, starting from node A.

Here the answer is $B_{A,B}$, where $B = NR$, and A and B are the state labels. It is the row of N corresponding to the initial state A multiplied by the column of R corresponding to the ending absorbing state B.

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3. Back and Forth to Campus (20 points)

Professor Prhab Hubilliti lives at the bottom of the hill on the corner of 117th Street and 7th Avenue. Going each way - up hill to to teach his class at Columbia or down hill back home - Prhab either runs or walks. Going up the hill, Prhab either walks at 2 miles per hour or runs at 4 miles per hour. Going down the hill, Prhab either walks at 3 miles per hour or runs at 6 miles per hour. In each direction, he always runs the entire way or walks the entire way. Since Prhab often works late into the night, he often gets up late, and has to run up hill to get to his class. On any given day, Prhab runs up hill with probability 3/4 and walks up hill with probability 1/4. On the other hand, Prab is less likely to run going back home. On any given day, he runs down hill with probability 1/3 and walks down hill with probability 2/3. The distance in each direction is 1 mile.

(a) (3 points) What is the average speed Prhab goes up the hill to campus when he is going up hill?

$$\text{average speed} \quad = \frac{3}{4} \times 4 + \frac{1}{4} \times 2 = \frac{14}{4} = 3.5 \quad \text{mile per hour}$$

(b) (5 points) What is the average time required for Prhab to go up the hill to campus on each trip?

We need to use the formula $D = RT$, i.e., "distance equals rate multiplied by time." Hence, $T = D/R$. Since here $D = 1$, the time is simply the reciprocal of the speed. hence

$$\text{Expected time} \quad = \left(\frac{3}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{2}\right) = \frac{5}{16} \quad \text{hour}$$

(c) (7 points) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends going up hill to campus?

Let T_U equal the time to go up and let T_D equal the time to go down. Use the renewal reward formula

$$\begin{aligned}
\text{Long run average reward} \quad &= \frac{\text{average reward per cycle}}{\text{average length of cycle}} \\
&= \frac{E[T_U]}{E[T_U] + E[T_D]} = \frac{(5/16)}{(5/16) + (5/18)} = \frac{18}{34} = \frac{9}{17} \approx 0.529
\end{aligned}$$

(d) (5 points) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends walking up hill to campus?

The general approach is the same. The mean cycle length is the same, but now there is a reward only if he is walking up hill.

$$\begin{aligned}
\text{Long run average reward} &= \frac{\text{average reward per cycle}}{\text{average length of cycle}} \\
&= \frac{E[T_U; walking]}{E[T_U] + E[T_D]} = \frac{(1/8)}{(5/16) + (5/18)} = \frac{36}{170} = \frac{18}{85} \approx 0.21
\end{aligned}$$

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4. The IEOR Printers (25 points)

The IEOR Department has three printers that are maintained by two repairmen. Each printer is working for an exponential length of time with mean 1 week. One repairman works on each failed printer until it is repaired, but the last printer to fail must wait for a repairman to become free whenever all three machines are not working. The repair times are exponential random variables with a mean of 1/2 week. All the failure and repair times are mutually independent. Suppose that all three printers are initially working.

(a) (2 points) If all printers are initially working, then what is the expected time until the first failure?

The time to the first failure is the minimum of three exponential random variables, , each with mean 1. Hence, the time to the first failure is again exponential with a rate equal to the sum of the rates. Hence the rate is 3. Since the mean is the reciprocal of the rate, the mean time until the first failure is 1/3 week (too often).

(b) (5 points) Let $X(t)$ be the number of working printers at time t. Characterize the stochastic process $\{X(t) : t \geq 0\}$, based on the assumptions above.

The random variables $X(t)$ can assume one of the values 0, 1, 2 or 3. Thus there are 4 states. Just like Example 3.2 in the CTMC notes, the system can be analyzed using a CTMC. However, unlike Example 3.2, the stochastic process $\{X(t) : t \geq 0\}$ is itself directly a Markov process. It can be directly represented as a CTMC, in fact as a birth-and-death (BD) process, because the failure and repair rates do not depend on the specific machine or the specific repairman. Instead of the complex rate diagram in Figure 1 in the CTMC notes, we have the simple BD rate diagram in Figure 2 below.

Since the mean repair time is 0.5 days, the rate of repair for each machine, when worked on by a repairman, is 2. In states 0 and 1, when both repairmen are working, the birth rate is $\lambda_0 = \lambda_1 = 2 \times 2 = 4$. In state 2, only one machine is under repair, so the birth rate is $\lambda_2 = 2$. There is no repair going in in state 3 when all machines are working.

Since the mean time for a machine to fail is 1 day, the failure rate of each machine is 1 per day. In state 3 there are three working machines, each of which can fail. Thus the death rate in state 3 is $\mu_3 = 3 \times 1 = 3$. In state 2, there are two working machines, so that the death rate is $\mu_2 = 2 \times 1 = 2$. Overall, the rates are given on the rate diagram.

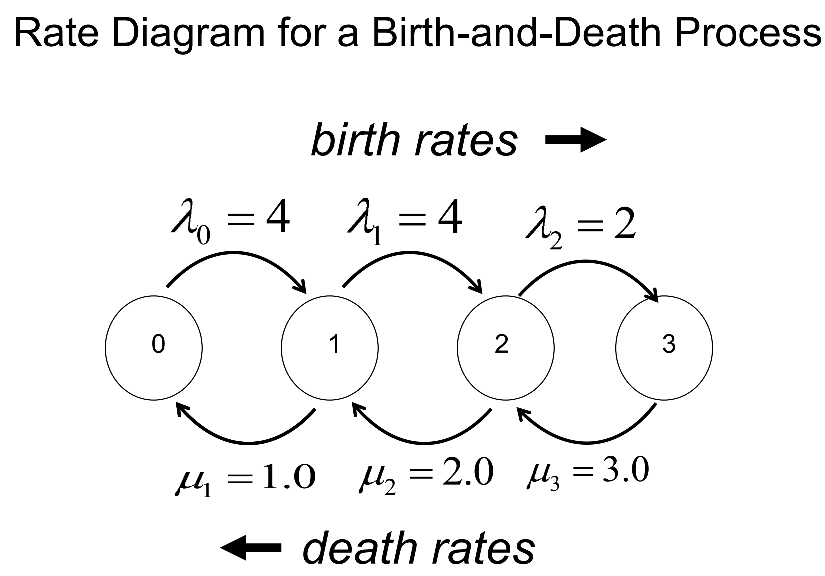

Figure 2: A rate diagram showing the transition rates for the birth-and-death process.

We can also specify the CTMC model by specifying the transition **rate matrix** Q. Here there are 4 states: $0, 1, 2, 3$, representing the number of working printers at any time. For this problem, all transitions are either up 1 or down 1, so that this is a BD stochastic process. That means the Q matrix has positive elements going up 1 or down 1 and negative elements on the diagonal. That is, the birth rates are $Q_{i,i+1} = \lambda_i$, the death rates are $Q_{i,i-1} = \mu_i$ and the diagonal elements are minus the off-diagonal row sum; i.e. $Q_{i,i} = -(\lambda_i + \mu_i)$. Here the rate matrix, with rates expressed per hour, is:

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} -4 & 4 & 0 & 0 \\ 1 & -5 & 4 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 3 & -3 \end{pmatrix}$$

(c) (3 points) If only one printer is initially working, then what is the probability that one of the other printers is repaired before this working printer fails?

The probability that one of the other printers is repaired before this working printer fails is the birth rate divided by the sum of the birth and death rates. The minimum of two independent exponential random variables is the first with probability equal to the rate of the first divided by the sum of the two rates, i.e.,

$$\frac{\lambda_1}{\lambda_1 + \mu_1} = \frac{4}{4+1} = \frac{4}{5}$$

(d) (5 points) What is the long run proportion of time that no printer is working.

Now we need to solve for the steady state probabilities, $P(X(\infty) = k)$, i.e., the limit of $P(X(t) = k)$ as $t \to \infty$. That is given by α_k, where $\alpha Q = 0$ and $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$. However, by Theorem 5.2 of the CTMC notes, we have

$$\alpha_k = \frac{r_k}{\sum_{i=0}^{i=3} r_i},$$

where $r_0 = 1$, $r_1 = \lambda_0/\mu_1 = 8$, $r_2 = \lambda_0\lambda_1/\mu_1\mu_2 = 32$ and $r_3 = \lambda_0\lambda_1\lambda_2/\mu_1\mu_2\mu_3 = 128/3$. Hence,

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \left(\frac{3}{55}, \frac{12}{55}, \frac{24}{55}, \frac{16}{55}\right).$$

The long run proportion of time that no printer is working is

$$\alpha_0 = 3/55.$$

(e) (3 points) What is the expected proportion of time each repairman is working on one of the printers?

$$\alpha_0 + \alpha_1 + \frac{\alpha_2}{2} = \frac{3}{55} + \frac{12}{55} + \frac{12}{55} = \frac{27}{55} \approx 0.491$$

(f) (3 points) Let $N(t)$ count the number of instants in the interval $[0, t]$ that a printer fails when all three were working. What kind of stochastic process is $\{N(t) : t \geq 0\}$?

The stochastic process $\{N(t) : t \geq 0\}$ is a renewal counting process, because the successive intervals are independent and identically distributed.

(g) (4 points) Let T be the random time between successive instants that a printer fails when all three were working. What is the expected value $E[T]$?

The last part was a hint. The idea is to represent the long-run proportion of time that all three printers are working in two different ways. One way is as $\alpha_3 = 16/55$. The other way is to exploit the renewal reward theorem, letting the cycles be successive intervals between failures when all printers are working. The desired $E[T]$ appears in the denominator as the expected length of the renewal cycle. The numerator is the expected time that all printers are simulatneously working. IN part (a) we found that to be 1/3. Hence, we have the relation

$$\alpha_3 = \frac{(1/3)}{E[T]},$$

so that

$$E[T] = \frac{1}{3\alpha_3} = \frac{1}{(48/55)} = \frac{55}{48} \approx 1.146$$
