IEOR 3106: Final Exam, December 15, 2013

This exam is closed book. YOU NEED TO SHOW YOUR WORK.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. You may keep the exam itself. Solutions will eventually be posted on line.

1. Forecasting the Weather (20 points)

Consider the following probability model of the weather over successive days. First, suppose that on each day we can specify if the weather is rainy or dry. Suppose that the probability that it will be rainy on any given day is a function of the weather on the previous two days. If it was rainy both yesterday and today, then the probability that it will be rainy tomorrow is 0.7. If it was dry yesterday, but rainy today, then the probability that it will be rainy tomorrow is 0.5. If it was rainy yesterday, but dry today, then the probability that it will be rainy tomorrow is 0.4. If it was dry both yesterday and today, then the probability that it will be rainy tomorrow is 0.2. Let X_n be the weather on day n.

(a) Calculate the conditional probability that it rains tomorrow but is dry on the next two days, given that it rained both yesterday and today.

(b) Is the stochastic process $\{X_n : n \geq 0\}$ a Markov chain? Why or why not? If not, construct an alternative finite-state stochastic process for this problem that is a Markov chain.

(c) With your Markov chain model in part (b), calculate the long-run proportion of days that are rainy. (The proper equation is good, but a numerical answer is better.)

(d) Which is larger: (i) the long-run proportion of days that it was rainy yesterday and dry today or (ii) the long-run proportion of days that it was dry yesterday and rainy today?

2. A Computer with Three parts (20 points)

A computer has three critical parts, each of which is needed for the computer to work. The computer runs continuously as long as the three required parts are working. The three parts have mutually independent exponential lifetimes before they fail. The expected lifetime of parts 1, 2 and 3 are 10 weeks, 20 weeks and 30 weeks, respectively. When a part fails, the computer is shut down and an order is made for a new part of that type. When the computer is shut down (to order a replacement part), the remaining two working parts are not subject to failure. The time to replace part 1 is exponentially distributed with mean 1 week; the time to replace part 2 is uniformly distributed between 1 week and 3 weeks; and the time to replace part 3 has a gamma distribution with mean 3 weeks and standard deviation 10 weeks.

(a) Assuming that all parts are initially working, what is the expected time until the first part fails?

(b) What is the probability that part 1 is the first part to fail?

(c) What is the long-run proportion of time that the computer is working?

(d) Suppose that new parts of type 1 each cost \$50; new parts of type 2 each cost \$100; and new parts of type 3 each cost \$400. What is the long-run average cost of replacement parts per week?

3. Investment Strategies for Brownian Stocks (25 points)

You have decided to invest \$4000 by buying 40 shares of stocks at time 0, each initially priced at \$100 per share.

Part a. Four Independent Stocks (16 points)

Suppose that you are considering the 4 different stocks, whose prices evolve independently over time, with the price of stock j evolving according to the model

$$S_j(t) \equiv 100 + 2.5t + 10B_j(t), \quad t \geq 0, \tag{1}$$

where $\{B_j(t) : t \geq 0\}$ is a standard (drift zero, unit variance) Brownian motion (BM) for each j, with the four different BM's being stochastically independent.

(a) (2 points) Suppose that you employ a **focused investment strategy** and buy 40 shares of stock 1 at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[40S_1(4)]$ and $Var(40S_1(4))$?

(b) (2 points) With the focused investment strategy in part (a), what is the probability that you will have made a profit? That is, what is the probability that $P(40S_1(4) > 4000)$?

(c) (4 points) Let T be the first time t that the share price of stock 1 either exceeds its expected value at time t by \$20 or falls below its expected value by \$10; i.e., let

$$T \equiv \inf\{t > 0 : S_1(t) - E[S_1(t)] \geq 20 \quad \text{or} \quad S_1(t) - E[S_1(t)] \leq -10\}.$$

What is the probability $P(S_1(T) = E[S_1(T)] + 20)$ and what is $E[T]$? Briefly explain.

(d) (2 points) Suppose that, instead, you decide to employ a **diversified investment strategy** and buy 10 shares of each of the 4 different stocks at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[10S_1(4) + \cdots + 10S_4(4)]$ and $Var(10S_1(4) + \cdots + 10S_4(4))$?

(e) (2 points) With the alternative investment scheme in part (c), what is the probability that you will have made a profit? That is, what is the probability that $P(10S_1(4) + \cdots + 10S_4(4) > 4000)$?

(f) (4 points) True or false: Indicate whether each of the following five statements is true or false, explaining briefly:

(i) The total value of the stock at time $t = 4$ has a probability distribution that is the same for both investment strategies.

(ii) An investor whose sole goal is to maximize his expected return should strongly prefer the diversified investment strategy.

(iii) An investor who wants to achieve the largest possible expected return but also minimize his risk, as defined by the probability of suffering a loss over the investment period $[0, 4]$, should strongly prefer the diversified investment strategy.

(iv) An investor who wants to maximize the probability that he achieves at least 20% more than the expected value should strongly prefer the focused investment strategy.

Part b. Three Other Stocks Dependent on the Initial Group (9 points)

Suppose that, in addition to the four Brownian stocks specified in (1), we also have three other stocks with

$$S_j(t) \equiv 100 + 2.5t + 10X_j(t), \quad t \geq 0, \tag{2}$$

for $j = 5$, 6 and 7, where

$$\begin{aligned} X_5(t) &\equiv B_1(t) - B_3(t), \quad t \geq 0, \\ X_6(t) &\equiv B_2(t) - B_1(t), \quad t \geq 0, \\ X_7(t) &\equiv 2B_3(t) - 2B_2(t), \quad t \geq 0. \end{aligned}$$

(g) (3 points) Suppose that you employ a **focused investment strategy** and buy 40 shares of stock 5 at time 0. What are the mean and variance of your investment at time $t = 4$? That is, what are $E[40S_5(4)]$ and $Var(40S_5(4))$?

(h) (3 points) Find an investment strategy for investing your \$4000 in these 7 stocks in (1) and (2) that maximizes the probability of your making at least \$1000 profit at $t = 4$ (i.e. of having final value above \$5000).

(i) (3 points) Find an investment strategy for investing your \$4000 in these 7 stocks that achieves *both* the highest expected profit and the lowest risk, as specified by the probability of not making a profit at time $t = 4$ (i.e., the probability of not making a profit at time $t = 4$ is the probability that the value at time $t = 4$ is less than \$4000).

4. An Airport Security Check (35 points)

An airport security check for departing passengers has been designed with two inspection stations. At each inspection station, passengers are processed one at a time in order of arrival at that station. There is ample waiting space at each station. All departing passengers go through the first (standard) stage of inspection. However, only those passengers that fail the first stage of inspection go to the second stage. There is a more elaborate inspection at the second station.

Suppose that passengers arrive at station 1 according to a Poisson process with rate 4 per minute; Suppose that the processing times at the stations are independent exponentially distributed random variables. Let the mean processing times be 10 *seconds* at station 1 and 10 *minutes* at station 2. Suppose that each successive passenger fails first-stage inspection, and thus requires second stage inspection, with probability p (independent of the history prior to the passenger).

The First Station Starting Empty (9 points)

Consider arrivals to the first inspection station at the beginning of the day, starting empty.

(a) Let T_k be the the arrival time of the k^{th} passenger at the first inspection station. What are the mean and variance: $E[T_k]$ and $Var(T_k)$?

(b) What is the probability that three passengers arrive at the first inspection station before any inspections have been completed?

(c) Suppose that 30 passengers arrive at the first inspection station during the first 4 minutes. What is the probability distribution of the number of these passengers that arrived during the first minute?

The Number of Passengers at the First Station (10 points)

(d) Let $Q_1(t)$ be the number of passengers at station 1 at time t. True or false: Indicate whether each of the following statements is true or false (and briefly explain):

(i) The stochastic process $\{Q_1(t) : t \geq 0\}$ is a Poisson process.
(ii) The stochastic process $\{Q_1(t) : t \geq 0\}$ is a Markov process.
(iii) The stochastic process $\{Q_1(t) : t \geq 0\}$ is a birth and death process.
(iv) There exists a probability vector $\alpha \equiv (\alpha_k : k \geq 0)$ such that $P(Q_1(t) = k) = \alpha_k$ for all $k \geq 0$ and $t \geq 0$ if we set $P(Q_1(0) = k) = \alpha_k$ for $k \geq 0$.
(v) If there exists a stochastic process $\{Q_1(t) : t \geq 0\}$ with $P(Q_1(t) = k) = \alpha_k$ for all $k \geq 0$ and $t \geq 0$, then that stochastic process is a time-reversible Markov process.

(e) What is the probability that at some fixed time in steady state there are 4 passengers at station 1, either being inspected or waiting to be inspected?

(f) What is the expected steady-state number of passengers at station 1, either being inspected or waiting to be inspected?

The Numbers of Passengers at the Two Stations (16 points)

(g) For $j = 1, 2$, let $Q_j(t)$ be the number of passengers at station j at time t. True or false: Indicate whether each of the following statements is true or false (and briefly explain):

(i) The stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ is a Poisson random measure or Poisson process on the plane.
(ii) The stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ is a continuous-time Markov chain.
(iii) The stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ is a birth and death process.
(iv) For all strictly positive values of the parameter p (the probability of a passenger requiring second-stage inspection), there is stationary version of the stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ with $P(Q_1(t) = k_1, Q_2(t) = k_2) = \alpha_{k_1,k_2}$ for all $t \geq 0$, $k_1 \geq 0$ and $k_2 \geq 0$, where $\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \alpha_{k_1,k_2} = 1$.
(v) If there does exist a stationary version of the stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ as described in part (iv), then that stationary version with the parameters above is a time-reversible Markov process.
(vi) If there does exist a stationary version of the stochastic process $\{(Q_1(t), Q_2(t)) : t \geq 0\}$ as described in part (iv), then that stationary version has the product form, i.e.,

$$P(Q_1(t) = k_1, Q_2(t) = k_2) = P(Q_1(t) = k_1)P(Q_2(t) = k_2) \quad \text{for all} \quad t, k_1, k_2 \geq 0,$$

where $Q_j(t)$ has the stationary distribution of a birth and death process.

(h) In steady state, what is the probability that exactly 8 passengers complete the first-stage inspection during a given 2 minute interval? Justify your answer.

(i) For what values of p is the second station stable (the number of passengers at the second station does *not* grow without bound as $t \to \infty$), so that the number of passengers waiting at the second station has a proper (finite) steady-state distribution?

(j) Suppose that $p = 0.02$. What is the probability that exactly 2 passengers complete the second-stage inspection during a 60 minute interval in steady state?

(k) Again suppose that $p = 0.02$. What is the probability that simultaneously, at some fixed time in steady state, there are 3 passengers at the first station, either being inspected or waiting to be inspected, and 4 passengers at the second station, either being inspected or waiting to be inspected?