**IEOR 3106: Introduction to Operations Research: Stochastic Models**

**First Midterm Exam, Chapters 1-4, October 6, 2011**

**There are 3 problems, each with multiple parts.**

**This exam is open book, but only the Ross textbook.**

**You need to show your work. Briefly explain your reasoning.**

**Honor Code:** Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.

**1. The Return of Markov Mouse (35 points)**

Just as we did in class, we consider Markov mouse, but now he moves randomly from room to room in a more elaborate maze, depicted below in Figure 1. This time there are twenty rooms in the maze, with an empty space in the middle, which he cannot reach. As before, Markov mouse can move to another room through a door, going either vertically or horizontally. From each room, independent of the past, Markov mouse moves through one of the available doors, with each available door chosen with equal probability.

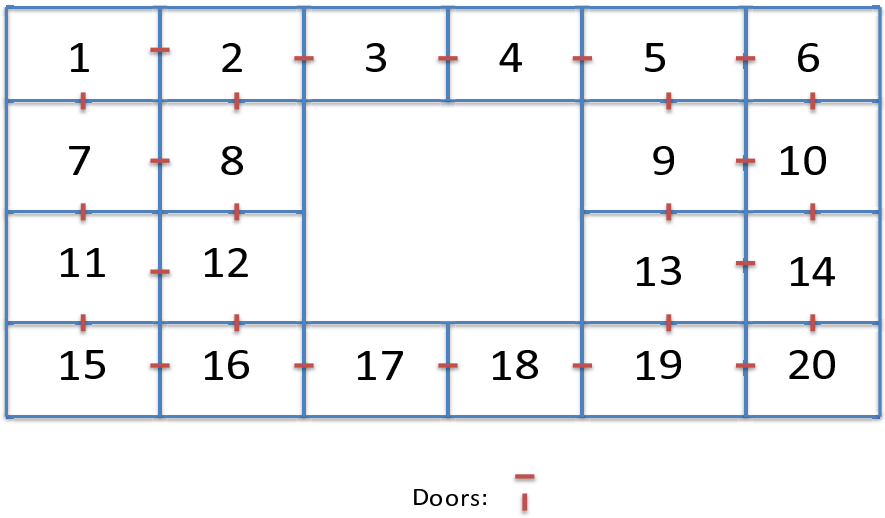


Figure 1: A more elaborate maze for Markov mouse.

(a) Suppose that Markov mouse starts in room 1. What is the probability that Markov mouse is in Room 2 after three moves? (4 points)

(b) Suppose that Markov mouse starts in room 1. What is the conditional probability that

Markov mouse made its first move to room 7, given that Markov mouse is in room 2 after 3 moves? (6 points)

(c) What is the period of this Markov chain? Why? (3 points)

(d) Suppose that Markov mouse starts in room 1. What is the long-run proportion of moves after which Markov mouse is in room 1? (7 points)

(e) Suppose that Markov mouse starts in room 1. What is the expected number of moves until Markov mouse first returns to room 1? (6 points)

(f) Suppose that Markov mouse starts in room 1. Does the probability that Markov mouse is in room 1 after $n$ steps converge to a limit as $n \to \infty$? Why or why not? (3 points)

(g) Suppose that Markov mouse starts in room 1. Give an expression (a formula, defining the terms in the formula, but not a numerical answer) for the probability that Markov mouse reaches room 15 before room 14. (6 points)

**2. Which Textbook Is Best? (25 points)**

In its never-ending quest for educational excellence, the IEOR Department has tried a different probability-and-statistics textbook for its introductory probability-and-statistics course during each of the last three semesters. During the first semester, 50 students used the textbook by Professor Mean; during the second semester, 25 students used the textbook by Professor Median; and during the third semester, 25 students used the textbook by Professor Mode. Surveys were taken at the end of each course, asking all the students their opinion. In the first survey, 20 of the 50 students admitted being satisfied with Mean's book; in the second survey, 11 of the 25 students admitted being satisfied with Median's book; and in the third survey, 20 of the 25 students admitted being satisfied with Mode's book.

(a) What is the probability that a student selected at random from all these students will have admitted being satisfied by his (or her) textbook? (3 points)

(b) Suppose that a student, selected at random from all these students, admitted having been satisfied by the textbook. What is the probability that the student used the textbook by Professor Mean? (7 points)

(c) By this experiment, which book is most likely to be best (assuming that we can judge by student opinion)? (3 points)

(d) Looking at the results, one curmudgeon on the faculty (perhaps Professor Mean) said, "The experiment is inconclusive. An outcome like that could have occurred by chance, with each student tossing a fair coin to determine whether or not to say that he was satisfied with his textbook." How justified is that criticism? In particular, how well does the fair-coin model work for Professor Mode? How likely is it that an outcome that favorable could occur by chance in that way? (12 points)

**3. A Finite Markov Chain (30 points)**

Consider a Markov chain on the eight states $\{1, 2, \cdots, 8\}$ with transition matrix $P$ given by

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0.2 & 0.1 & 0.2 & 0.0 & 0.2 & 0.1 & 0.2 & 0.0 \\ 0.0 & 0.2 & 0.0 & 0.0 & 0.0 & 0.8 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.2 & 0.1 & 0.0 & 0.0 & 0.2 & 0.2 & 0.3 \\ 0.0 & 0.8 & 0.0 & 0.0 & 0.0 & 0.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.5 \end{pmatrix} \end{array}$$

**Note that we are numbering the states 1, 2, ... , 8, with the columns labeled the same as the rows.

(a) Which states are accessible from state 1? (2 points)

(b) From which states is state 1 accessible? (2 points)

(c) Put the transition matrix in canonical form (showing the original states in their new positions). (5 points)

(d) Identify the open and closed communication classes for this Markov chain. Which states are transient? Which states are recurrent? (5 points)

(e) Compute the two-step transition probability $P_{2,7}^{(2)}$ (2 points)

(f) Compute the two-step transition probability $P_{4,3}^{(4)}$ (2 points)

(g) Compute the two-step transition probability $P_{1,7}^{(2)}$ (2 points)

(h) Starting in state 2, what is the expected total number of visits to state 2? (2 points)

(i) Starting in state 6, what is the long-proportion of time spent in state 2? (2 points)

(j) Starting in state 1, what is the expected total number of visits to state 5? (3 points)

(k) What is the approximate value of $P_{1,6}^{(25)}$? (3 points)

Full credit on parts (j) and (k) for indicating the correct procedure (formulas). A bonus of two points on each for the correct numerical answer. Therefore, the maximum total score is 104.