IEOR 3106: Introduction to Operations Research: Stochastic Models

Fall 2011, Professor Whitt

SOLUTIONS to the First Midterm Exam

Open Book: but only the Ross textbook.

Honor Code: Students are expected to behave honorably when taking exams. After completing the test, please testify to your adherence by writing the following on your bluebook and signing your name: "I have neither given nor received help while taking this test."

Justify your answers; show your work.

1. The Return of Markov Mouse (35 **points**)

Just as we did in class, we consider Markov mouse, but now he moves randomly from room to room in a more elaborate maze, depicted below in Figure 1. This time there are twenty rooms in the maze, with an empty space in the middle, which he cannot reach. As before, Markov mouse can move to another room through a door, going either vertically or horizontally. From each room, independent of the past, Markov mouse moves through one of the available doors, with each available door chosen with equal probability.

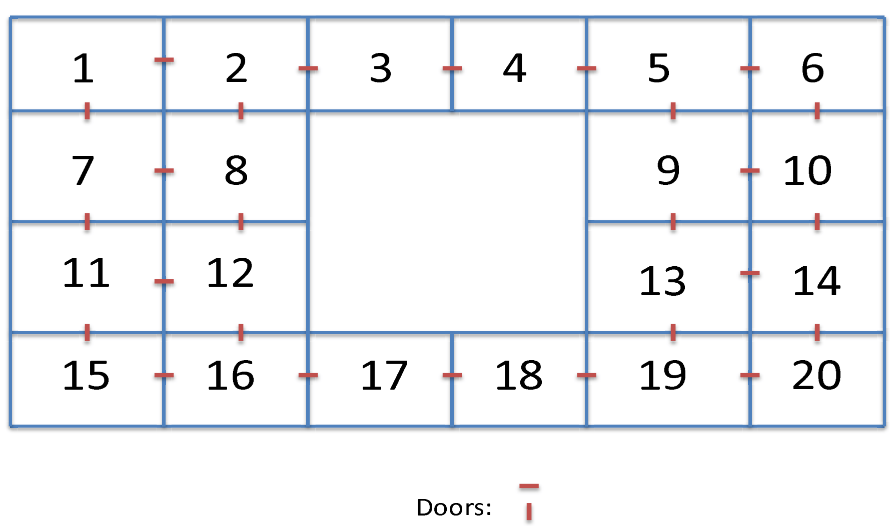

Figure 1: A more elaborate maze for Markov mouse.

(a) Suppose that Markov mouse starts in room 1. What is the probability that Markov mouse is in Room 2 after three moves? (4 points)

There are five ways to go from room 1 to room 2 in three moves. These paths and their probabilities are:

$$
\begin{array}{lll}
1 \to 2 \to 3 \to 2 & \text{with probability} & \frac{1}{2}\frac{1}{3}\frac{1}{2} = \frac{1}{12} = \frac{3}{36} \\
1 \to 2 \to 8 \to 2 & \text{with probability} & \frac{1}{2}\frac{1}{3}\frac{1}{3} = \frac{1}{18} = \frac{2}{36} \\
1 \to 2 \to 1 \to 2 & \text{with probability} & \frac{1}{2}\frac{1}{3}\frac{1}{2} = \frac{1}{12} = \frac{3}{36} \\
1 \to 7 \to 8 \to 2 & \text{with probability} & \frac{1}{2}\frac{1}{3}\frac{1}{3} = \frac{1}{18} = \frac{2}{36} \\
1 \to 7 \to 1 \to 2 & \text{with probability} & \frac{1}{2}\frac{1}{3}\frac{1}{2} = \frac{1}{12} = \frac{3}{36}
\end{array}
$$

Thus, **the total probability is** $13/36$.

(b) Suppose that Markov mouse starts in room 1. What is the conditional probability that Markov mouse made its first move to room 7, given that Markov mouse is in room 2 after 3 moves? (6 points)

$$
P(X_1 = 7 | X_0 = 1, X_3 = 2) = \frac{P(X_0 = 1, X_1 = 7, X_3 = 2)}{P(X_0 = 1, X_3 = 2)} = \frac{5/36}{13/36} = \frac{5}{13}.
$$

(c) What is the period of this Markov chain? Why? (3 points)

The period is 2, just as in the original Markov mouse example. Starting in any state, it is only possible to return to that state in an even number of steps. But it is always possible to return in two steps. That means that the period of the entire chain is 2. This property is easier to see if we renumber the states in the second and fourth row, so that all movement goes from an odd state to an even state, and from an even state to an odd state. The current way the rooms are numbered prevents this odd and even property from being true. But, without loss of generality, we can relabel the states so that this odd-even property holds.

(d) Suppose that Markov mouse starts in room 1. What is the long-run proportion of moves after which Markov mouse is in room 1? (7 points)

Here we use time reversibility, as in Section 4.8 of the book. In particular, this example, is a special case of a random walk on the vertices of a graph with weights on its edges. Here all the weights are 1. The nodes are the rooms and the arcs are the doors from any room to one of its neighbors. Thus the stationary probability of any room is the number of doors from that room divided by the sum over all rooms of the doors out of that room. The stationary probability coincides with the long-run proportion. (Recall that you cannot just count the number of doors. Each door has to be counted twice.)

The number of doors out of room 1 is 2. The sum of the numbers of doors over all rooms is 52. (There are eight rooms with 2 doors and 12 rooms with 3 doors.) Hence, **the long-run proportion of time Markov mouse is in room 1 is** $2/52 = 1/26$.

(e) Suppose that Markov mouse starts in room 1. What is the expected number of moves until Markov mouse first returns to room 1? (6 points)

The expected return time is the reciprocal of the steady state probability. Hence, **the expected number of moves is 26.**

(f) Suppose that Markov mouse starts in room 1. Does the probability that Markov mouse is in room 1 after n steps converge to a limit as $n \to \infty$? Why or why not? (3 points)

It does *not* converge, because the Markov chain is periodic. (See part (c) above.) The probability is 0 for every odd value of n, but it approaches 1/13, two times the stationary probability 1/26 found in part (d), as n increases among the even values; i.e., $P_{1,1}^{2n+1} = 0$ for all n, while

$$\lim_{n \to \infty} P_{1,1}^{2n} = \frac{1}{13}$$

Thus the long-run proportion of moves spent in state 1 is 1/26, as determined in part (d).

(g) Suppose that Markov mouse starts in room 1. Give an expression (a formula, defining the terms in the formula, but not a numerical answer) for the probability that Markov mouse reaches room 15 before room 14. (6 points)

Here we get an absorbing Markov chain by replacing states 14 and 15 by absorbing states. In the context of the theory for absorbing chains, the probability we want is $B_{1,15}$, where B is the 18×2 matrix of absorption probabilities, starting from one of the 18 transient states. Here B has the form $B = NR$, where N is the fundamental matrix, i.e., $N = (I - Q)^{-1}$, with I here being the 18×18 identity matrix and Q being the 18×18 square matrix representing the transition probabilities among the transient states, including all the original states except states 14 and 15. The matrix R is the 18×2 one-step transition matrix representing all the original one-step transition probabilities going from each of the new transient states to the states 14 and 15.

We do not ask for a numerical answer, because inverting the 18×18 square matrix $(I - Q)$, where is the identity matrix, is complicated.

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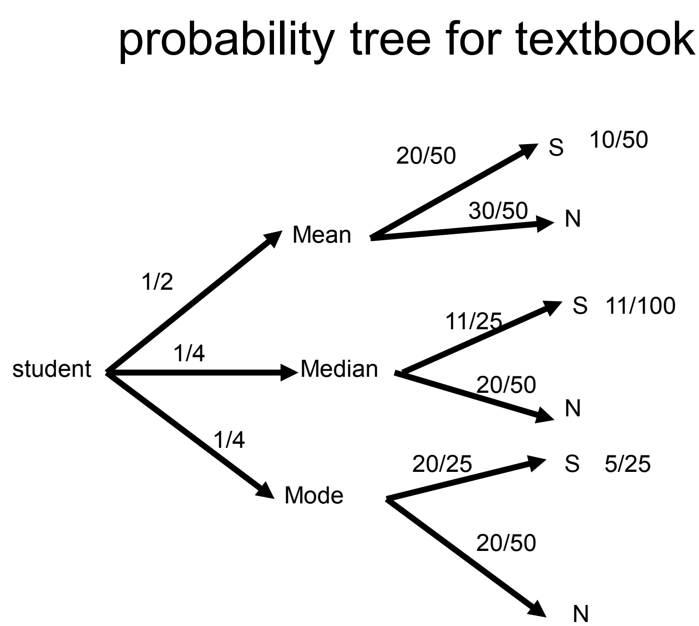
2. Which textbook is best? (25 points)

In its never-ending quest for educational excellence, the IEOR Department has tried a different probability-and-statistics textbook for its introductory probability-and-statistics course during each of the last three semesters. During the first semester, 50 students used the textbook by Professor Mean; during the second semester, 25 students used the textbook by Professor Median; and during the third semester, 25 students used the textbook by Professor Mode. Surveys were taken at the end of each course, asking all students their opinion. In the first survey, 20 of the 50 students admitted being satisfied with Mean's book; in the second survey, 11 of the 25 students admitted being satisfied with Median's book; and in the third survey, 20 of the 25 students admitted being satisfied with Mode's book.

(a) What is the probability that a student selected at random from all these students will have admitted being satisfied by his textbook? (3 points)

There were 100 students in all, of which 51 admitted to being satisfied. Thus, the probability is 0.51

We could reason in stages, by first considering the probability that the student is in each class, and then determine the probability for that class. That more complicated reasoning leads to using a probability tree. Let S denote satisfied and N not satisfied.



$$P(S) = P(S \cap Mean) + P(S \cap Median) + P(S \cap Mode) = 0.20 + 0.11 + 0.20 = 0.51$$

(b) Suppose that a student, selected at random from all these students, admitted having been satisfied by his textbook. What is the probability that the student used the textbook by Professor Mean? (7 points)

Use **Bayes' Theorem**, exploiting the definition of **conditional probability**:

$$P(Mean|S) = \frac{P(Mean \cap S)}{P(S)} = \frac{0.20}{0.20 + 0.11 + 0.20} = \frac{0.20}{0.51} \approx 0.39$$

The last numerical calculation is not required.

(c) By this experiment, which book is most likely to be best (assuming that we can judge by student opinion)? (3 points)

It is natural to pick the *book* with $P(S|book)$ being largest. These probabilities are obtained from the second set of branches on the tree. By that criterion, the textbook by Professor Mode is most likely to be best: $P(S|Mode) = 0.80$, larger than the other two $P(S|book)$.

(d) Looking at the results, one curmudgeon on the faculty (perhaps Professor Mean) said, "The experiment is inconclusive. An outcome like that could have occurred by chance, with each student tossing a coin to determine whether or not to say that he was satisfied with his textbook." How justified is that criticism? In particular, how well does the fair-coin model work for Professor Mode? (12 points)

We do a **statistical test**; we use a **normal approximation for the binomial**.

The question is: How likely could the observed outcome or something even more extreme occur by chance, assuming that the students make judgements randomly and independently, with each student's evaluation being positive or negative with probability 1/2? Clearly, the overall number of satisfied students – 51 out of 100 – is not inconsistent with the criticism. That clearly could occur by chance in exactly the way described.

But consider the results for Professor Mode. We can use a normal approximation for the binomial distribution.

Textbook by Professor Mode: Let Y be the random number of satisfied students in a class of 25 students, assuming that the decisions are random with probability 1/2. We observe 20. That value or more extreme would be 20 or larger. The question then is: what is $P(Y \geq 20)$?

Note that the mean is $EY = 25 \times 1/2 = 12.5$ and the variance is $Var(Y) = 25 \times 1/2 \times 1/2 = 25/4$, so the standard deviation is $SD(Y) = 5/2 = 2.5$. Reasoning quickly, 20 is precisely 3 standard deviations above the mean 12.5, so we conclude it is unlikely to occur by chance. It is not prohibitively unlikely, but it is quite unlikely. Reasoning more slowly, we get the normal approximation

$$P(Y \geq 20) = P\left(\frac{Y - 12.5}{2.50} \geq \frac{20 - 12.5}{2.50}\right) \approx P(N(0,1) \geq 3.00) \approx 0.0013$$

using the table on p. 81 of your textbook by Ross. A refined approximation, to account for the fact the Y is integer-valued is to consider

$$P(Y \geq 20) = P(Y \geq 19.5) = P\left(\frac{Y - 12.5}{2.50} \geq \frac{19.5 - 12.5}{2.50}\right) \approx P(N(0,1) \geq 2.80) \approx 0.0026$$

The probability that the outcome for Professor Mode would be that good or better is thus about 1/400. That probability is not extraordinarily small, but it is sufficiently small that we would reasonably conclude that the outcome did not occur by chance in the way contemplated.

We should perhaps consider both very small and very large values as extreme. Thus we should really look at the probability $P(|Y - EY| \geq 7) = P(Y \leq 5.5) + P(Y \geq 19.5)$, which is twice the probability above, or 0.0052. Again, we conclude that the outcome is unlikely to occur by chance in the way described. Thus, by this statistical test, we can fairly conclude that Professor Mode's textbook is indeed better.

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3. A Finite Markov Chain (30 **points)**

Consider a Markov chain on the eight states $\{1, 2, \cdots, 8\}$ with transition matrix P given by

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{pmatrix} 0.2 & 0.1 & 0.2 & 0.0 & 0.2 & 0.1 & 0.2 & 0.0 \\ 0.0 & 0.2 & 0.0 & 0.0 & 0.0 & 0.8 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.2 & 0.1 & 0.0 & 0.0 & 0.2 & 0.2 & 0.3 \\ 0.0 & 0.8 & 0.0 & 0.0 & 0.0 & 0.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.5 \end{pmatrix}$$

**Note that we are numbering the states 1, 2, ... , 8, with the columns labeled the same as the rows.

(a) Which states are accessible from state 1? (2 points)

All states are accessible from state 1. (That need not be in one step.) See the classification of states in Section 4.3.

(b) From which states is state 1 accessible? (2 points)

State 1 is only accessible from itself, which we take to be the definition, even if there is 0 transition probability. Here it is possible to go from state 1 back to itself.

(c) Put the transition matrix in canonical form (showing the original states in their new positions). (5 points)

We put the states in the closed communication classes first, grouped together. We put the states in the open communication classes below, grouped together. (See question (d).)

$$P = \begin{array}{c} 3 \\ 2 \\ 6 \\ 4 \\ 7 \\ 8 \\ 5 \\ 1 \end{array} \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.2 & 0.8 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.8 & 0.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.2 & 0.0 & 0.2 & 0.3 & 0.0 & 0.0 \\ 0.2 & 0.1 & 0.1 & 0.0 & 0.2 & 0.0 & 0.2 & 0.2 \end{pmatrix}$$

(d) Identify the open and closed communication classes for this Markov chain. Which states are transient? Which states are recurrent? (5 points)

The **closed communication classes** are the subsets of states: $\{3\}$, $\{2, 6\}$ and $\{4, 7, 8\}$. The **open communication classes** are the subsets of states: $\{1\}$, and $\{5\}$. The states in closed communication classes are **recurrent**; the states in open communication classes are **transient**. This chain is reducible since there is more than one communication class.

(e) Compute the two-step transition probability $P_{2,7}^{(2)}$ (2 points)

$P_{2,7}^{(2)} = 0$

(f) Compute the two-step transition probability $P_{4,3}^{(4)}$ (2 points)

$P_{4,3}^{(4)} = 0$

(g) Compute the two-step transition probability $P_{1,7}^{(2)}$ (2 points)

There are three ways to go from state 1 to state 7 in 2 steps: $1 \to 1 \to 7$, $1 \to 7 \to 7$ and $1 \to 5 \to 7$. The probability of the first path is $(0.2) \times (0.2) = 0.04$; the probability of the second path is $(0.2) \times (0.5) = 0.10$; the probability of the third path is $(0.2) \times (0.2) = 0.04$. **The total probability is** 0.18.

(h) Starting in state 2, what is the expected total number of visits to state 2? (2 points)

Since, 2 is in the closed communication class $\{2, 6\}$, the state 2 will be visited infinitely often, starting in state 2. The expected number is **infinite**.

(i) Starting in state 6, what is the long-proportion of time spent in state 2? (2 points)

It suffices to analyze the small 2×2 chain containing only the states 2 and 6. Since the transition matrix is doubly stochastic (all column sums are 1), the steady-state probability is $\pi = (0.5, 0.5)$. **The long-run of proportion of times spent in state 2 is 0.5.**

(j) Starting in state 1, what is the expected total number of visits to state 5? (3 points)

Since 1 and 5 are both transient states, we can apply absorbing chain theory. We want $N_{1,5}$, where $N = (I - Q)^{-1}$, i.e., N is a 2×2 matrix and Q is the 2×2 matrix of transition probabilities among the two transient states 1 and 5. However, since we can go from 1 to 5 with probability 0.2, but we cannot return to 1 from 5, this expected number is easy to compute directly: Since the chain can stay in state 1 for a while, we need to look at the conditional probability of going to state 5, given that the chain leaves state 1. It is 1 visit with probability

$0.2/0.8 = 0.25$. Hence, **the expected total number of visits to state 5 starting in state 1 is 0.25.**

(k) What is the approximate value of $P_{1,6}^{(25)}$? (3 points)

In general, this can be computed in two steps. In the first step we can apply absorbing chain theory in terms of $B = NR$, because we are asking for the absorption probability into the set of states $\{2, 6\}$ starting from the transient state 1. However, again this is not difficult to compute directly. We see that we can get to the set $\{2, 6\}$ in 4 ways (when we leave state 1): We can go: $1 \to 2$, $1 \to 6$ and $1 \to 5 \to 2$ and $1 \to 5 \to 6$. These have probabilities, respectively, of $0.1/0.8 = 0.125$, $0.1/0.8 = 0.125$, $(0.2/0.8) \times 0.2 = 0.05$ and $(0.2/0.8) \times 0.2 = 0.05$. (We again must condition on leaving 1 in the first step.) Thus the probability of reaching the set $\{2, 6\}$ starting in state 1 is 0.35. Once the chain is the set $\{2, 6\}$, we can apply the steady-state probabilities determined in part (g): $\pi_6 = 0.5$ within the set $\{2, 6\}$. Hence,

$$P_{1,6}^{(25)} \approx 0.35 \times 0.5 = 0.175.$$

Full credit on parts (j) and (k) for indicating the correct formula. A bonus of two points on each for the correct numerical answer.